

The Use of Concessions in Forestalling War*

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Abstract

We determine whether and how concessions to an aggressive country by its non-aggressive rival can be used to forestall war in a dynamic environment. In every period, the aggressive country can seize some of non-aggressive country's resources by war. Alternatively, it can wait for the non-aggressive country to concede these resources peacefully. With some probability, making concessions is too costly, but this is not observed by the aggressive country. Both countries suffer from limited commitment, and war is the static Nash equilibrium. In a dynamic environment, the realization of war in the future can sustain concessions along the equilibrium path, and the two countries can fluctuate between periods of war and periods of peace. However, if the cost of war to the non-aggressive country is low, the two countries must converge to permanent war. While permanent war minimizes the welfare of the two countries in the long run, it maximizes their welfare along the equilibrium path by providing incentives for concession-making. In contrast, if the cost of war to the non-aggressive country is high, the two countries can avoid permanent war and can fluctuate between periods of war and periods of peace forever.

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JEL Classification: C73, F51, P48

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1 Introduction

Destructive wars are an inefficient means of distributing resources across groups of people. They are often the result of the aggressive action of one country with military power which is disproportionately large relative to its economic and political influence.¹ The purpose of this paper is to determine whether and how concessions to such an aggressive country can be used to forestall war.

As an example, consider the interaction between Germany and France in the events leading up to World War I. Germany's aspirations for political and economic power commensurate with its military might led it to request concessions from France, and this caused multiple diplomatic standoffs such as the Moroccan Crisis.² The crisis illustrates three important points which can guide our analysis: concessions are zero-sum, countries suffer from limited commitment, and countries cannot distinguish between the intentional and unintentional failure of concessions. First, France had to harm its commercial and political interests in Morocco in order to appease Germany. Second, once Germany and France decided to peacefully negotiate, they temporarily forestalled the possibility of war. Nonetheless, Germany could only imperfectly commit to restraining its military power in the future, and France could only imperfectly commit to honoring the terms of peace in the future. Third, on multiple occasions, France broke the terms of peace in order to control domestic strife in Morocco. However, Germany could never tell whether these emergencies were real so that concessions were too costly for France, or whether these emergencies were artificially fomented by France as an excuse to forgo concession-making.

In this paper, we develop a dynamic model which incorporates these three features in a setting in which one country (country 1) seeks resources from its rival (country 2). In every period, country 1 can seize these resources by war at a cost, and if war occurs, the period ends (e.g., Germany can fight over Morocco today). Alternatively, if war does not occur, then country 2 can concede some resources to country 1 (e.g., France can reduce its commercial and political influence in Morocco). With some exogenous probability, country 2 cannot make any concessions because they are too costly (e.g., France experiences a real domestic emergency in Morocco which prohibits it from honoring the terms of peace). Because country 1 does not observe the cost of making concessions, it cannot distinguish between the intentional and unintentional failure of concessions (e.g., Germany does not know whether this emergency is real or fomented). Country 1's military power is disproportionately large relative to its economic and political power, and

¹See Fearon (1995 and Powell (2002) for a discussion of this view.

²See Kissinger (1994) and Richardson (1994) for an interesting discussion of this episode.

therefore it prefers war to receiving zero concessions with certainty. In a static setting, country 1 cannot refrain from war, and country 2 cannot commit to making concessions. Consequently, war is the unique static Nash equilibrium.

We analyze the efficient sequential equilibrium in which reputational mechanisms can sustain equilibrium actions and can potentially improve upon the static Nash equilibrium. In the efficient sequential equilibrium, country 1 must sometimes refrain from war and country 2 must sometimes make a concession. If either country publicly deviates from this implicit agreement, then the two countries revert to the repeated static Nash equilibrium. If country 2 privately deviates from this implicit agreement by making zero concessions though concessions are not prohibitively costly, then country 2 is punished with a reduction in future welfare. We explicitly characterize the dynamic path of war and concessions in the efficient sequential equilibrium. In our analysis, we distinguish between *limited war* and *total war*. Total war starting from some date is defined as the repeated static Nash equilibrium in which country 1 chooses war in every period, and country 2 never has the opportunity to make a concession. Limited war starting from some date is defined as an equilibrium in which war occurs for some periods, but with positive probability, country 1 refrains from war in the future and allows country 2 the opportunity to make a concession. We ask whether concessions today can avert either total war or limited war in the future.

There are four main results in our paper. First, wars (total or limited) are necessary along the equilibrium path. This is because the positive probability of war in the future enforces concessions by country 2 today. Under permanent peace, country 2 would make zero concessions and pretend that concessions are too costly in every period, but this would not be acceptable to country 1 which would prefer to go to war. Note that whenever country 1 punishes country 2's failed concession with war, it does so with very little information since information in our environment is *coarse*. Though country 1 is always certain that country 2 is cooperating whenever concessions succeed, country 1 receives no information if concessions fail, and it cannot deduce the likelihood that country 2 is genuinely unable to make a concession. Therefore, there is a chance that country 1 is making a mistake by going to war.

Second, limited war as opposed to total war can occur after the first few failures of concessions. Country 1 can effectively forgive country 2's first few failed concessions by refraining from total war, and this is a consequence of the coarseness of information. Since no information is revealed whenever concessions fail, the realization of the extreme threat of total war does not necessarily improve efficiency as it would if sufficiently rich information about the cost of concessions allowed country 1 to deduce the likelihood that country 2 is genuinely unable to make a concession whenever concessions fail. Because

information is coarse, the prospect for error in punishing failed concessions with total war is large, and thus limited war is efficient along the equilibrium path.

Third, while limited war can occur along the equilibrium path, the efficient equilibrium necessarily converges to total war if the discounted cost of war to country 2 is low relative to the maximal feasible concession by country 2. While total war minimizes the welfare of the two countries in the long run, it maximizes their welfare along the equilibrium path by providing incentives for concession-making. With positive probability, a very long sequence of concessions fails, and country 1 must provide incentives for making these concessions by punishing their failure with total war. Because country 2's discounted suffering under war is not very high relative to the size of extractable concessions, the potential welfare cost of making an error and going to total war relative to the added benefit of using extreme threats to induce concessions is small. Therefore, even though limited war can occur along the equilibrium path, concessions cannot avert total war.

Fourth, the efficient equilibrium can converge to limited war forever if the discounted cost of war to country 2 is high relative to the maximal feasible concession by country 2. In this case, the realization of total war in the long run is not necessary to enforce concession-making since the possibility of temporary war is sufficiently distasteful to country 2. Even if a very long sequence of concessions fails, country 1 need not provide incentives for making these concessions by punishing their failure with total war. Because country 2's discounted suffering under war is high relative to the size of extractable concessions, the potential welfare cost of making an error and going to total war relative to the added benefit of using extreme threats to induce concessions is large. Consequently, concessions can avert total war, and the two countries can fluctuate between periods of war and periods of peace forever.

Our paper makes two contributions. First, it is an application of a dynamic incomplete information game with history dependent strategies to war. This is important since the study of war is a dynamic issue in which countries have long memories—particularly in long lasting conflicts—and since the literature on war has recognized the importance of limited commitment and imperfect information (see Powell 1999, 2002, and Fearon, 1995)). In contrast to the current work on war, we provide an explanation for war which combines these two frictions in a dynamic setting in which countries follow history dependent strategies, and our model can account for escalation to total war and as well as limited wars.³

Second, our paper is an application of a dynamic incomplete information game in an en-

³For related work on war, see also Alesina and Spolaore (2005), Caselli and Coleman (2006), Dixit (1987), Esteban and Ray (1994,1999), Hirschleifer (1995), and Skarpedas (1992).

environment with a coarse information structure. The existing literature on dynamic games which builds on Green and Porter (1984)'s theory of oligopolistic competition assumes a rich information structure, and this leads to a *Bang-Bang* characterization of efficient equilibria (see Abreu, Pearce, and Stacchetti, 1986, 1990, and Sannikov, 2007a, 2007b). In the context of war and diplomacy, this information structure and its equilibrium implications may not be appropriate. First, countries often have very little information about their enemy's behavior and intentions, particularly when their enemy is not cooperating. Second, even though limited wars occur in actuality, in many environments in which war represents the worst possible outcome, the *Bang-Bang* characterization of efficient equilibria implies that limited wars do not occur.⁴ In this paper, we show that under a coarse information structure, the *Bang-Bang* property need not hold since the prospect for error is large, and consequently, limited war can occur along the equilibrium path.⁵

The paper is organized as follows. Section 2 describes the model. Section 3 defines and provides necessary and sufficient conditions for sequential equilibria. Section 4 describes how concessions can forestall war in a sequential equilibrium. Section 5 analyzes efficient sequential equilibria. Section 6 presents empirical evidence which supports the model. Section 7 concludes. The Appendix contains additional proofs not included in the text.

2 Model

Consider two countries $i = \{1, 2\}$ and time periods $t = \{0, \dots, \infty\}$. In every date t , country 1 publicly chooses $W_t = \{0, 1\}$. If $W_t = 1$, war takes place, country i receives w_i , and the period ends. Alternatively, if $W_t = 0$, peace occurs, and country 2 publicly makes a concession to country 1 of size $x_t \in [0, \bar{x}]$. Country 1 receives x_t and country 2 receives $-x_t - c(x_t, s_t)$ for $c(x_t, s_t)$ which represents country 2's *private* additional cost of making a concession x_t which is a function of the state $s_t = \{0, 1\}$. s_t is observed by country 2 but not by country 1. Let $c(x, 0) = \bar{c} > 0$ for $x > 0$ and let $c(0, 0) = c(\cdot, 1) = 0$. s_t is stochastic and determined as follows. If $W_t = 0$, then prior to the choice of x_t , nature chooses s_t with $\Pr\{s_t = 1\} = \pi \in (0, 1)$.

⁴That is if total war is the min-max. This characterization applies only to efficient equilibria. Efficient equilibria as opposed to other often-examined equilibria such as Markovian equilibria or trigger strategy equilibria are a useful selection device in our setting since rival countries have long memories of their past interaction.

⁵For applied work on dynamic contracts with imperfect information, see Atkeson (1991), Atkeson and Lucas (1992), Golosov, Kocherlakota, and Tsyvinski (2003), Hauser and Hopenhayn (2004), Phelan and Townsend (1991), Spear and Srivastava (1987), and Thomas and Worrall (1990).

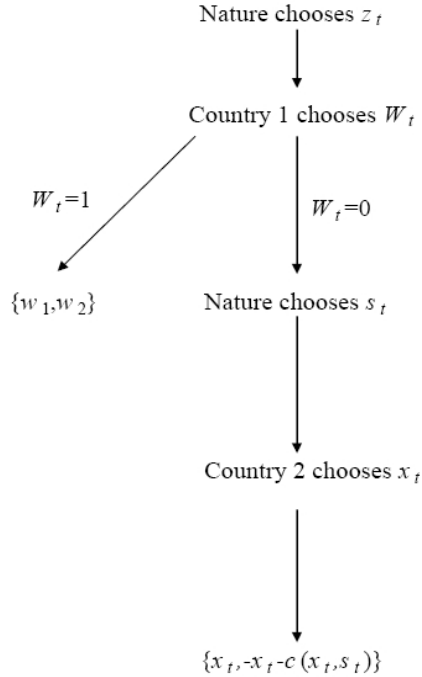
See Fudenberg and Levine (2007) and Sannikov and Skrzypacz (2007) for an additional discussion of the characteristics of equilibria under different information structures.

Concessions by country 2 are more costly if $s_t = 0$, but this cannot be verified by country 1. For example, imagine if $\bar{c}$ is very high and imagine if this implies that concessions cannot be positive if $s_t = 0$. Then, if $W_t = 0$ and if country 1 receives no concessions (i.e., $x_t = 0$), country 1 cannot tell if country 2 could not make concessions since the cost was too high (i.e., $s_t = 0$) or if country 2 could make concessions but chose to not cooperate (i.e., $s_t = 1$).

We do not allow country 2 to choose to go to war or to receive concessions from country 1 only as a matter of parsimony. Under this additional refinement, the characterization of the equilibrium is identical to the one presented here, and all of our results are left unchanged.⁶

Let $z_t \in [0, 1]$ represent an i.i.d. random variable independent of s_t and all actions drawn from a continuous c.d.f. $G(\cdot)$ at the beginning of every period t . z_t is observed by both countries and can be used as a randomization device which can improve efficiency by allowing country 1 to probabilistically go to war. The game is displayed in Figure 1.

Figure 1: Game Form



⁶Such a model is isomorphic to the one here since country 1 always makes zero concessions. Details available upon request.

Let $u_i(W_t, x_t, s_t)$ represent the payoff to i at t .⁷ Each country i has a period zero welfare

$$\mathbf{E}_0 \sum_{t=0}^{\infty} \beta^t u_i(W_t, x_t, s_t), \beta \in (0, 1).$$

Assumption 1 (inefficiency of war) $\exists x \in [0, \bar{x}]$ s.t. $\pi x > w_1$ and $-\pi x > w_2$.

Assumption 2 (military power of country 1) $w_1 > 0$.

Assumption 3 (high cost of concessions) $\bar{c} > -\beta w_2 / (1 - \beta)$.

The payoffs, the structure of concessions, and the order of actions in this game are aligned with the historical facts in the motivating example discussed in the introduction. Assumption 1 implies that war is inefficient since both countries can be better off if country 2 makes a concession to country 1 in state 1. By Assumption 2, country 1's military power w_1 exceeds the economic resources under its control of size 0. The fact that w_1 exceeds w_2 (which is negative) is without loss of generality, and it is due to the normalization of both countries' resources to 0 which is purely for notational simplicity. More generally, country 2 can be more powerful and control more resources than country 1 and vice versa as long as country 1's military power exceeds its economic power.

Remark 1 $W = 1$ is the unique static Nash equilibrium. Conditional on $W = 0$, country 2 chooses $x = 0$. By Assumption 2, country 1 chooses $W = 1$.

Both countries suffer from limited commitment. Once they decide to peacefully negotiate, the two countries temporarily forestall the possibility of war. Nonetheless, country 1 can only imperfectly commit to restraining its military power, and country 2 can only imperfectly commit to making concessions. Country 2 would like to commit to making a concession to country 1 in state 1 in order to deter country 1 from war. However, if country 1 is committed to not fighting, country 2 prefers to make no concession in a static environment since concessions require country 2 to harm itself.

In the next section, we examine whether and how reputational considerations in a dynamic environment can improve upon the static Nash equilibrium. In a dynamic environment, Assumption 3 implies that if $s_t = 0$, then country 2's concessions are so prohibitively costly that even the highest reward for a positive concession and the highest punishment for zero concessions together cannot induce a positive concession by country

⁷Specifically, $u_1(W_t, x_t, s_t) = W_t w_1 + (1 - W_t)x_t$ and $u_2(W_t, x_t, s_t) = W_t w_2 - (1 - W_t)(x_t + c(x_t, s_t))$.

2.⁸ Therefore, concessions must be zero if $s_t = 0$. Consequently, if concessions fail (i.e., $x_t = 0$), country 1 cannot determine if this is unintentional because their cost is too high (i.e., $s_t = 0$) or if this is intentional because their cost is low (i.e., $s_t = 1$).

3 Sequential Equilibria

We consider equilibria in which each country conditions its strategy on past public information in order to determine whether and how concessions can prevent war. Specifically, let $h_t = \{z^{t-1}, W^{t-1}, x^{t-1}\}$, the history of public information at t prior to the realization of z_t .⁹ Define a strategy $\sigma = \{\sigma_1, \sigma_2\} = \left\{ \{W_t(h_t, z_t)\}_{t=0}^\infty, \left\{ \{x_t(h_t, z_t, s_t)\}_{s_t=0,1} \right\}_{t=0}^\infty \right\}$. σ is feasible if $\forall t \geq 0$ and $\forall (h_t, z_t)$,

$$\left\{ W_t(h_t, z_t), \{x_t(h_t, z_t, s_t)\}_{s_t=0,1} \right\} \in \left\{ \{0, 1\}, [0, \bar{x}]^2 \right\}.$$

Given σ , define the equilibrium continuation value for country i at (h_t, z_t) as

$$U_i(\sigma|_{h_t, z_t}) = \mathbf{E} \left\{ \begin{array}{l} u_i(W_t(h_t, z_t), x_t(h_t, z_t, s_t), s_t) + \\ \beta \mathbf{E} \{ U_i(\sigma|_{h_{t+1}, z_{t+1}}) | h_t, z_t, W_t(h_t, z_t), x_t(h_t, z_t, s_t) \} \end{array} \middle| h_t, z_t \right\} \quad (1)$$

for $\sigma|_{h_t, z_t}$ which is the continuation of a strategy after (h_t, z_t) has been realized. Define $U_i(\sigma|_{h_t, z_t})|_{s_t}$ as the term inside the first expectation operator on the right hand side of (1). Let $\Sigma_i|_{h_t, z_t}$ denote the entire set of feasible continuation strategies for i after (h_t, z_t) has been realized.

Definition 1 σ is a sequential equilibrium if it is feasible and if $\forall (h_t, z_t)$

$$\begin{aligned} U_1(\sigma|_{h_t, z_t}) &\geq U_1(\sigma'_1|_{h_t, z_t}, \sigma_2|_{h_t, z_t}) \quad \forall \sigma'_1|_{h_t, z_t} \in \Sigma_1|_{h_t, z_t} \text{ and} \\ U_2(\sigma|_{h_t, z_t})|_{s_t} &\geq U_1(\sigma_1|_{h_t, z_t}, \sigma'_2|_{h_t, z_t})|_{s_t} \quad \forall \sigma'_2|_{h_t, z_t} \in \Sigma_2|_{h_t, z_t} \text{ for } s_t = \{0, 1\}. \end{aligned}$$

In a sequential equilibrium, each country dynamically chooses its best response given the strategy of its rival. Because country 1's strategy is public by definition, any deviation by country 2 to a non-public strategy is irrelevant (see Fudenberg, Levine, and Maskin, 1994).

⁸This is because the discounted difference between the largest possible reward (permanent peace with continuation value of 0) and the largest possible punishment (permanent war with continuation value of $w_2(1 - \beta)$) is not sufficiently large relative to $\bar{c}$.

⁹Without loss of generality, we let $x_t = 0$ if $W_t = 1$.

In order to build a sequential equilibrium allocation, let $q_t = \{z^{t-1}, s^{t-1}\}$, the *exogenous* equilibrium history of public signals and states prior to the realization of z_t .¹⁰ Define an equilibrium allocation as a function of the exogenous history:

$$\alpha = \{W_t(q_t, z_t), \{x_t(q_t, z_t, s_t)\}\}_{t=0}^{\infty}.$$

Note that along the equilibrium path of a sequential equilibrium, even though country 1 does not necessarily know q_t , both countries have common knowledge the continuations of α since actions are a function of past public information. Let $\mathcal{F}$ denote the set of feasible allocations α with continuations which are measurable with respect to past public information. Let $U_i(\alpha|_{q_t, z_t})$ represents the equilibrium continuation value at a history (q_t, z_t) under this sequence of actions. Define

$$\underline{U}_i = \frac{w_i}{1 - \beta},$$

the payoff to i from permanent war. By our discussion in Remark 1, there exists an equilibrium with permanent war which generates $\{\underline{U}_1, \underline{U}_2\}$.

The next proposition explains that a sequential equilibrium allocation can be enforced by both countries reverting to the repeated static Nash equilibrium after any observable deviation. We show that by Assumption 3, it is necessary that $x_t(q_t, z_t, s_t = 0) = 0$, since concessions are so prohibitively costly to country 2 if $s_t = 0$ that they cannot be made. Naturally, country 2 will always be able to unobservably deviate if $s_t = 1$ by making zero concessions and behaving as if $s_t = 0$. Let $\mathbf{E}\{U_2(\alpha|_{q_{t+1}, z_{t+1}}) | q_t, z_t, s_t = 1\}$ represent the expected continuation value to country 2 at $t + 1$ conditional on q_t, z_t , and $s_t = 1$, and let $\mathbf{E}\{U_2(\alpha|_{q_{t+1}, z_{t+1}}) | q_t, z_t, s_t = 0\}$ be analogously defined for $s_t = 0$. Unless specified otherwise, all proofs are in the Appendix.

Proposition 1 (*sequential equilibria*) $\alpha \in \mathcal{F}$ is a sequential equilibrium if and only if $\forall (q_t, z_t), x_t(q_t, z_t, s_t = 0) = 0$,

$$U_i(\alpha|_{q_t, z_t}) \geq \underline{U}_i \text{ for } i = 1, 2 \text{ and} \quad (2)$$

$$-x_t(q_t, z_t, s_t = 1) + \beta \mathbf{E}\{U_2(\alpha|_{q_{t+1}, z_{t+1}}) | q_t, z_t, s_t = 1\} \geq \beta \mathbf{E}\{U_2(\alpha|_{q_{t+1}, z_{t+1}}) | q_t, z_t, s_t = 0\} \quad (3)$$

$$\text{if } W_t(q_t, z_t) = 0.$$

Proposition 1 establishes conditions which are necessary and sufficient for a sequential

¹⁰Without loss of generality, let s_t be revealed even if $W_t = 1$.

equilibrium allocation. Country 1 can always fight forever, and this creates a lower bound for its continuation values, which establishes (2) for $i = 1$. Furthermore, whenever country 2 is prescribed positive concession, it can choose zero concessions and proceed with the same strategy after the deviation. Because concessions can be zero if $s_t = 0$, such a deviation goes unobserved, and (3) ensures that it is weakly dominated. Condition (2) for $i = 2$ follows from condition (3) which implies that if country 2 makes zero concessions forever, it cannot possibly be punished by anything worse than permanent war. Note that under perfect information, (3) is an unnecessary constraint, and all deviations are observed and punished by reversion to permanent war.

Let $U_i(\alpha)$ represents the period 0 continuation value to i implied by α prior to the realization of z_0 . Define Λ as the set of sequential equilibrium allocations. Define $V = \{\{U_1(\alpha), U_2(\alpha)\} | \alpha \in \Lambda\}$ as the set of period zero continuation values for both countries. By the stationarity of the game, $\{U_1(\alpha|_{q_t, z_t}), U_2(\alpha|_{q_t, z_t})\} \in V \forall (q_t, z_t)$, and this observation is useful for the recursive representation of the equilibrium which is discussed in Section 5. The presence of the public signal z to be used as a randomization device implies that V is convex.

Lemma 1 *V is convex and compact.*

4 Can Concessions Forestall War?

Remark 1 establishes that in the static version of our game, the possibility of concessions does not forestall war. We now show that, in contrast, this is always achievable in the repeated game if war is sufficiently costly to country 2. This is because country 1 can provide dynamic incentives for country 2 to make concessions today by threatening to fight in the future if concessions fail today. More formally, we say that concessions forestall war if $\Pr\{W_0 = W_1 = \dots W_\infty = 1\} = 0$ under an allocation α .

Proposition 2 (concessions forestall war) $\exists \hat{w}_2$ s.t. if $w_2 \leq \hat{w}_2$, $\exists \alpha \in \Lambda$ in which concessions forestall war.

Proof. Construct the following equilibrium. If $s_{t-1} = 1$, then $W_t = 0$ and $x_t = x$ if $s_t = 1$ and $x_t = 0$ otherwise for $x \in [0, \bar{x}]$ which satisfies the inequalities of Assumption 1. If $s_{t-1} = 0$, both countries revert to the repeated static Nash equilibrium forever. Let $W_0 = 0$. By Assumption 1, both countries i weakly prefer equilibrium continuation values to $\underline{U}_i$, so that (2) is satisfied. To check (3), let $U_2|_{s=1}$ represent the continuation value to country 2 conditional on successful concessions yesterday. The stationarity of the equilibrium

implies

$$U_2|_{s=1} = -\pi x + \beta (\pi U_2|_{s=1} + (1 - \pi) \underline{U}_2),$$

so that (3) which requires $-x + \beta U_2|_{s=1} \geq \beta \underline{U}_2$ becomes $-x \geq \beta w_2$ which is possible for sufficiently low w_2 . ■

Proposition 3 (necessity of war) $\nexists \alpha \in \Lambda$ s.t. $W_t = 0 \forall t$.

War must be expected in the future in all periods since it is required to provide incentives for country 2 to make concessions. Without war, country 2 makes zero concessions, and by Assumption 2, country 1 cannot be satisfied by zero concessions. Together with Proposition 2, Proposition 3 means that any periods of peace are necessarily followed by periods of war. Moreover, in going to war, countries realize that cooperation occurred in the past, so that war is by no means ex-post necessary, though it is ex-ante required for the enforcement of peace.

5 Efficient Sequential Equilibria

In the introduction, we asked whether concessions can forestall war. Propositions 2 and 3 show that periods of peaceful concessions are necessarily followed by periods of war required to sustain them. Therefore, concessions can forestall war, but not forever.

In this section, we refine this question by distinguishing between limited war and total war and by determining whether concessions can forestall either. Total war starting from some date t is defined as an equilibrium in which $W_k = 1 \forall k \geq t$. Concessions forestall total war if $\Pr \{W_t = W_{t+1} = \dots = W_\infty = 1\} = 0 \forall t$ under an allocation α . Limited war starting from some date t is defined as an equilibrium in which $W_t = 1$ but $\Pr \{W_{t+k} = 0 | W_t = 1\} > 0$ for $k > 0$. Concessions forestall limited war if $\Pr \{W_t = 1, W_{t+k} = 0\} = 0 \forall t$ and $\forall k > 0$ under an allocation α . By Proposition 3, concessions could potentially forestall total war or limited war, but not both.

5.1 Equilibrium Definition

Given the multiplicity of sequential equilibria, we focus on efficient sequential equilibria. In contrast to Markovian equilibria or trigger strategy equilibria, these equilibria are allowed to feature rich history dependent dynamics, and this is a more accurate description of warring countries which are often motivated by long memories of their past interactions.¹¹

¹¹This is also the approach pursued in the related work mentioned in Footnote 5.

Definition 2 $\alpha \in \Lambda$ is an efficient sequential equilibrium if $\nexists \alpha' \neq \alpha$ s.t. $\alpha' \in \Lambda$, $U_i(\alpha') > U_i(\alpha)$, and $U_{-i}(\alpha') \geq U_{-i}(\alpha)$ for $i = 1, 2$.

We can write our program as maximizing the welfare of country 1 subject to providing country 2 with a minimum welfare of v_0 :

$$\max_{\alpha} U_1(\alpha) \text{ s.t. } U_2(\alpha) \geq v_0 \text{ and } \alpha \in \Lambda. \quad (4)$$

We solve this program in the following sections. We first characterize the set of continuation values V (Section 5.2). Second, we argue that continuation values travel along the contours of V in the solution to (4). As a consequence, it is possible to write (4) as a recursive program in which the entire history of the game is subsumed in a single state variable (Section 5.3). This is useful for two reasons. First, it allows us to easily characterize and understand the transitional dynamics of the equilibrium (Sections 5.4 and 5.5). Second, using this characterization, we can determine whether limited war or total war can be avoided (Section 5.6).

Assumption 4 $\exists \alpha \in \Lambda$ s.t. $U_1(\alpha) > \underline{U}_1$.

This assumption is necessary if the set of values generated by peace in period 0 is not a singleton. We make this assumption in order to ensure interesting transitional dynamics, and without this assumption, the efficient equilibrium takes the stationary form as the example in the proof of Proposition 2.¹²

5.2 Shape of V

In order to characterize the shape of V , let $J(v)$ represent the highest welfare achievable by country 1 conditional on providing country 2 a welfare of v , that is, the solution to (4) subject to $U_2(\alpha) = v \geq v_0$. $J(v)$ is defined for $v \in [\underline{U}_2, \overline{U}_2]$ for some $\overline{U}_2 > \underline{U}_2$ which represents country 2's highest period zero sequential equilibrium continuation value.

Lemma 2 $J(\underline{U}_2) = J(\overline{U}_2) = \underline{U}_1$.

The reason that $J(\underline{U}_2) = \underline{U}_1$ is a consequence of the fact that the unique method of providing country 2 with a continuation value of $\underline{U}_2$ is through total war which provides country 1 with a continuation value of $\underline{U}_1$. $\underline{U}_2$ cannot be delivered to country 2 via peace,

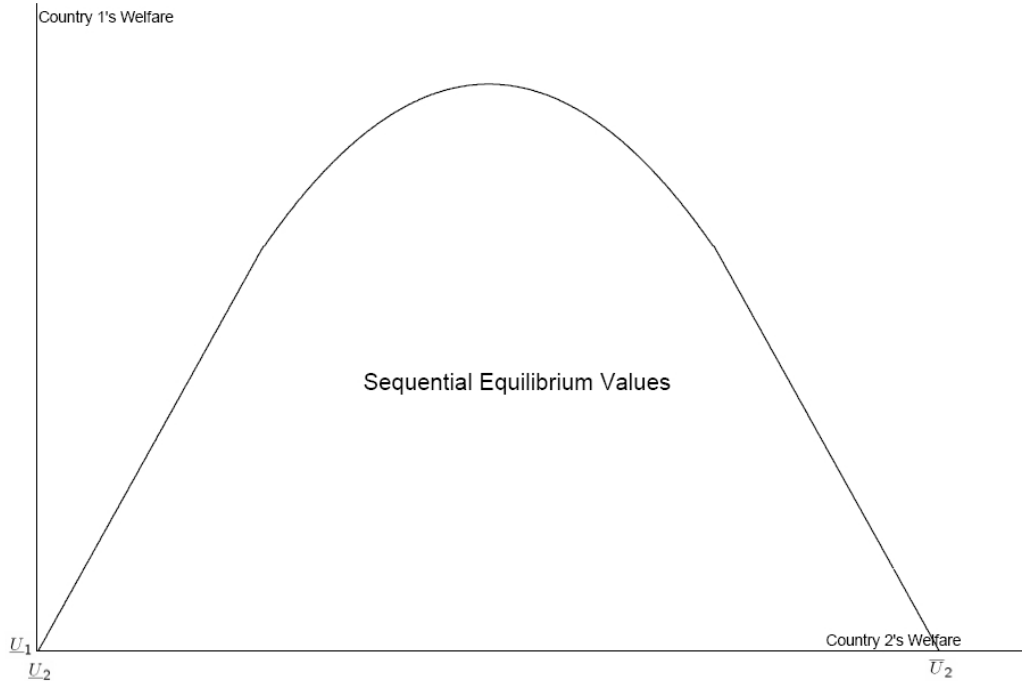
¹²This assumption is guaranteed by $-w_1/\pi > \beta w_2$ so that an equilibrium with the structure of the example in the proof of Proposition 2 satisfies Assumption 4.

because it is not possible to provide incentives for country 2 to make concessions in this circumstance. More specifically, the continuation value to country 2 cannot decline in the event of a failed concession since the continuation value is already at a minimum.

The fact that $J(\bar{U}_2) = \underline{U}_1$ is a consequence of Assumption 2. It is always possible to increase country 2's welfare at the expense of country 1's welfare by reducing country 2's concessions to country 1. This transfer of welfare is without efficiency loss, since war in the future is never utilized to provide incentives to country 1. Note that, in contrast, it is not always possible to reduce country 2's welfare without an efficiency loss, since reducing country 2's welfare will entail a higher probability of war in the future.

The implications of Lemmas 1 and 2 are displayed in Figure 2. The y -axis represents $J(v)$ and the x -axis represents v . All of the points underneath $J(v)$ and above the x -axis represent the space of sequential equilibrium continuation values V . $J(v)$ is increasing for low values of v and decreasing for high values of v . The increasing portion of $J(v)$ is the consequence of the "value burning" associated with the use of war to discipline concession-making by country 2. Any efficient equilibrium must begin on the downward sloping portion of $J(v)$ where it is not possible to make one country strictly better off without making the other country strictly worse off.

Figure 2: $J(v)$



5.3 Recursive Program

Given the shape of V , equilibrium continuation values in the solution to (4) should always be on the contours of V after every history (q_t, z_t) . If this were not the case, then it would be possible to strictly increase country 1's welfare while holding country 2's welfare constant at the history where this feature is violated. Intuitively, variations in welfare are only useful in the provision of incentives to country 2 which has private information and may wish to misrepresent itself as not being able to make a concession. An important implication of this fact is that the equilibrium continuation value to country 2, which can be denoted as v , subsumes the entire history of the game. We can therefore write $J(v)$ as the solution to a recursive program:

$$J(v) = \max_{\{W_z, v_z^W, x_z, v_z^H, v_z^L\}_{z \in [0,1]}} \int_0^1 \left(\begin{array}{c} W_z [w_1 + \beta J(v_z^W)] + \\ (1 - W_z) [\pi (x_z + \beta J(v_z^H)) + (1 - \pi) \beta J(v_z^L)] \end{array} \right) dG_z \quad (5)$$

s.t.

$$v = \int_0^1 (W_z [w_2 + \beta v_z^W] + (1 - W_z) [\pi (-x_z + \beta v_z^H) + (1 - \pi) \beta v_z^L]) dG_z, \quad (6)$$

$$J(v_z^W), J(v_z^H), J(v_z^L) \geq \underline{U}_1 \quad \forall z \in [0, 1], \quad (7)$$

$$v_z^W, v_z^H, v_z^L \geq \underline{U}_2 \quad \forall z \in [0, 1], \quad (8)$$

$$-x_z + \beta v_z^H \geq \beta v_z^L \quad \forall z \in [0, 1], \quad (9)$$

$$v_z^H = v_z^L \text{ if } x_z = 0 \quad \forall z \in [0, 1], \quad (10)$$

$$W_z \in \{0, 1\} \quad \forall z \in [0, 1], \text{ and } x_z \in [0, \bar{x}] \quad \forall z \in [0, 1]. \quad (11)$$

(5) takes into account the presence of the signal z which can be used as a public randomization device. W_z represents the decision to go to war for a given z . v_z^W represents the value promised to country 2 for tomorrow conditional on war taking place today for a given z . By Assumption 3, concessions are zero if $s = 0$, and with some abuse of notation, x_z represents the size of a concession by country 2 under $s = 1$ for a given z conditional on peace today. We refer to x_z as the *requested* concession. Conditional on peace today, the continuation value v_z^H follows if $s = 1$, and the continuation value v_z^L follows if $s = 0$, each for a given z . Equation (6) represents the promise keeping constraint which ensures that country 2 is achieving a continuation value of v . Equations (7) and (8) represent the recursive version of (2) for $i = 1$ and $i = 2$, respectively. Equation (9) represent the recursive version of (3). Constraint (10) ensures that the continuation equilibrium is a

function of public information. Constraints (11) ensure that the allocation is feasible.¹³

Let $\alpha^*(v)$ represent the argument which solves (5) – (11), which consists of

$$\{W_z^*(v), v_z^{W^*}(v), x_z^*(v), v_z^{H^*}(v), v_z^{L^*}(v)\}_{z \in [0,1]}.$$

Since $\alpha^*(v)$ may not be unique, we define the set of solutions for a particular v .

Definition 3 $\Psi(v) = \{\alpha^*(v) \mid \alpha^*(v) \text{ solves (5) – (11)}\}.$

Abreu, Pearce, and Stacchetti (1986,1990) and Sannikov (2007a,2007b) argue that in a broad class of dynamic games, the efficient sequential equilibrium necessarily features the *Bang-Bang* property. In order to relate our results to theirs, we define the *Bang-Bang* property in our framework.

Definition 4 $\alpha^*(v)$ has the *Bang-Bang* property if $v_z^{j*}(v) \in \text{ext}V \ \forall j = W, H, L$ and $\forall z$.

$\alpha^*(v)$ has the *Bang-Bang* property if continuation values in the future are all chosen from the extreme points of V . For example, by Lemma 2, $\alpha^*(\underline{U}_2)$ features the *Bang-Bang* property since $\underline{U}_2$ is an extreme point and since $W_z^*(\underline{U}_2) = 1$ and $v_z^{W^*}(\underline{U}_2) = \underline{U}_2 \ \forall z$ given that $\underline{U}_2$ is uniquely generated by total war. In the next sections, we characterize the entire set $\Psi(v)$, we show that the *Bang-Bang* property need not be satisfied, and we determine the implications for equilibrium dynamics and for the long run. This will allow us to determine whether concessions can forestall total war or limited war.

5.4 Equilibrium Path

Before characterizing the solution to (5) – (11), note that concessions do not destroy any surplus whereas war does. Ideally, war should never take place, and country 1 should punish country 2 for failed concessions in the past by requesting larger concessions today. However there are two constraints in our setting which limit the extent to which this is incentive compatible and efficient. First, concession-making requires incentives. Therefore if v is too low, it is not actually possible for country 1 to request concessions since a large enough threat in the event of failed concessions does not exist. This is expressed in the following remark.

Remark 2 If $W_z^*(v) = 0 \ \forall z$, then $v \geq \beta \int_0^1 v_z^{L^*}(v) dG_z \geq \beta \underline{U}_2 > \underline{U}_2$ by (6), (8), and (9).

¹³For the purposes of our discussion, we say that if $W_z = 1$, then the values of x_z , v_z^H , and v_z^L are not defined, and analogously, if $W_z = 0$, then the value of v_z^W is not defined.

Second, concession-making is limited by the size of the maximal concession $\bar{x}$ relative to the discounted pain of war βw_2 . Therefore, if v is too low, even the maximal concession today cannot make country 2 feel as much pain as war tomorrow, so that even if war is avoided today, it cannot be avoided in the future, and there is no efficiency gain from postponing it (i.e., $\int_0^1 v_z^{H*}(\beta \underline{U}_2) dG_z < \beta \underline{U}_2$ by Remark 2). The following useful definition and remark illustrate this idea.

Definition 5 $\tilde{w}_2 = -\bar{x}/\beta$.

Remark 3 If $W_z^*(\beta \underline{U}_2) = 0 \forall z$, then $\int_0^1 v_z^{H*}(\beta \underline{U}_2) dG_z \leq \underline{U}_2 + \bar{x}/\beta < \beta \underline{U}_2$ if $w_2 < \tilde{w}_2$ by (6), (8), and (9).

Our logic suggests that there exists a continuation value $\tilde{U}$ above which assured peace strictly dominates war. For v below $\tilde{U}$, assured peace will not strictly dominate war for one of two possible reasons. Either peace is not incentive compatible (i.e., $v < \beta \underline{U}_2$), or, alternatively, peace is incentive compatible but it is no more efficient than war, since even the maximal concession cannot make country 2 feel as much pain as war, and if war is avoided today, it cannot be avoided in the future (i.e., $\int_0^1 v_z^{H*}(v) dG_z < \tilde{U}$). The second reason is likely to be relevant if cost of the largest concession $\bar{x}$ is low relative to the discounted pain of war βw_2 , since country 2 cannot feel as much pain as war tomorrow via concessions today. These ideas are expressed formally in the below lemma.

Lemma 3 *The following properties hold:*

1. $\exists \tilde{U} \in (\underline{U}_2, \bar{U}_2)$ s.t. $\forall v \geq \tilde{U}$ and $\forall \alpha^*(v) \in \Psi(v)$, $W_z^*(v) = 0 \forall z$,
2. If $w_2 \geq \tilde{w}_2$, then $\tilde{U} = \beta \underline{U}_2$, and
3. If $w_2 < \tilde{w}_2$ then $\tilde{U} > \beta \underline{U}_2$.

Having determined when it is efficient for peace to occur, we can characterize the solution to the program. To simplify our discussion, we let the absence of a z subscript denotes the average (i.e., $W^*(v) = \int_0^1 W_z^*(v) dG_z$, $v^{W*}(v) = \int_0^1 W_z^*(v) v_z^{W*}(v) dG_z / W^*(v)$, etc.). Given the concavity of (5) and the convexity of (6) – (11), a solution always exists which only features randomization between war and peace:

$$\{v_z^{W*}(v), x_z^*(v), v_z^{H*}(v), v_z^{L*}(v)\} = \{v^{W*}(v), x^*(v), v^{H*}(v), v^{L*}(v)\} \quad \forall z. \quad (12)$$

Proposition 4 (equilibrium path) *If $\alpha^*(v) \in \Psi(v)$ then it satisfies (6) – (11),*

1. $W_z^*(v) = 0 \forall z$ if $v \geq \tilde{U}$,
2. $v_z^{W^*}(v) \leq \tilde{U} \forall z$,
3. $\pi(-x_z^*(v) + \beta v_z^{H^*}(v)) + (1 - \pi)\beta v_z^{L^*}(v) \leq \tilde{U} \forall z$ if $v \leq \tilde{U}$,
4. $x_z^*(v) = \bar{x}$ or $v_z^{H^*}(v) = \bar{U}_2 \forall z$, and
5. (9) binds $\forall z$ if $v \geq \tilde{U}$.

Corollary 1 *If $\alpha^*(v)$ satisfies Proposition 4's conditions and (12), then $\alpha^*(v) \in \Psi(v)$.*

Proposition 4 characterizes necessary conditions for the entire set of solutions to (5) – (11). Its corollary states that these conditions are also sufficient for efficiency if the allocation satisfies (12).¹⁴ A loose intuition for these conditions is as follows.

The first condition follows from Lemma 3. Imagine if the second condition is violated so that $v_z^{W^*}(v) > \tilde{U}$ for some z . Then a perturbation which reduces $v_z^{W^*}(v)$ and which reduces the the probability of war today strictly improves country 1's welfare. There is no sense in which postponing assured peace is efficient if it is possible to instead reduce the probability of war today. If the third condition is violated, then a perturbation which reduces the probability of war today and also reduces country 2's welfare conditional on the realization of peace also strictly improves country 1's welfare. This is because country 1 should extract as much surplus as possible from country 2 under peace if it is promising country 2 a low continuation value. If the fourth condition is violated, then a perturbation which increases $x_z^*(v)$ and $v_z^{H^*}(v)$ strictly increases country 1's welfare. To see why, imagine if $v_z^{H^*}(v)$ is below but arbitrarily close to $\tilde{U}$. Then the perturbation leaves $v_z^{H^*}(v)$ above $\tilde{U}$ which strictly improves country 1's welfare by increasing the concessions it receives today and by reducing the surplus destroyed by war tomorrow. More generally, the fourth condition guarantees that the duration of peace is prolonged as much as possible, and this benefits both countries. Finally, if the fifth condition is violated, then a perturbation which increases $v_z^{L^*}(v)$ and reduces $v_z^{H^*}(v)$ strictly improves country 1's welfare. For instance, imagine if $v_z^{L^*}(v)$ is below but arbitrarily close to $\tilde{U}$. Then the perturbation leaves $v_z^{L^*}(v)$ above $\tilde{U}$ which strictly improves country 1's welfare by reducing the surplus destroyed by war tomorrow.

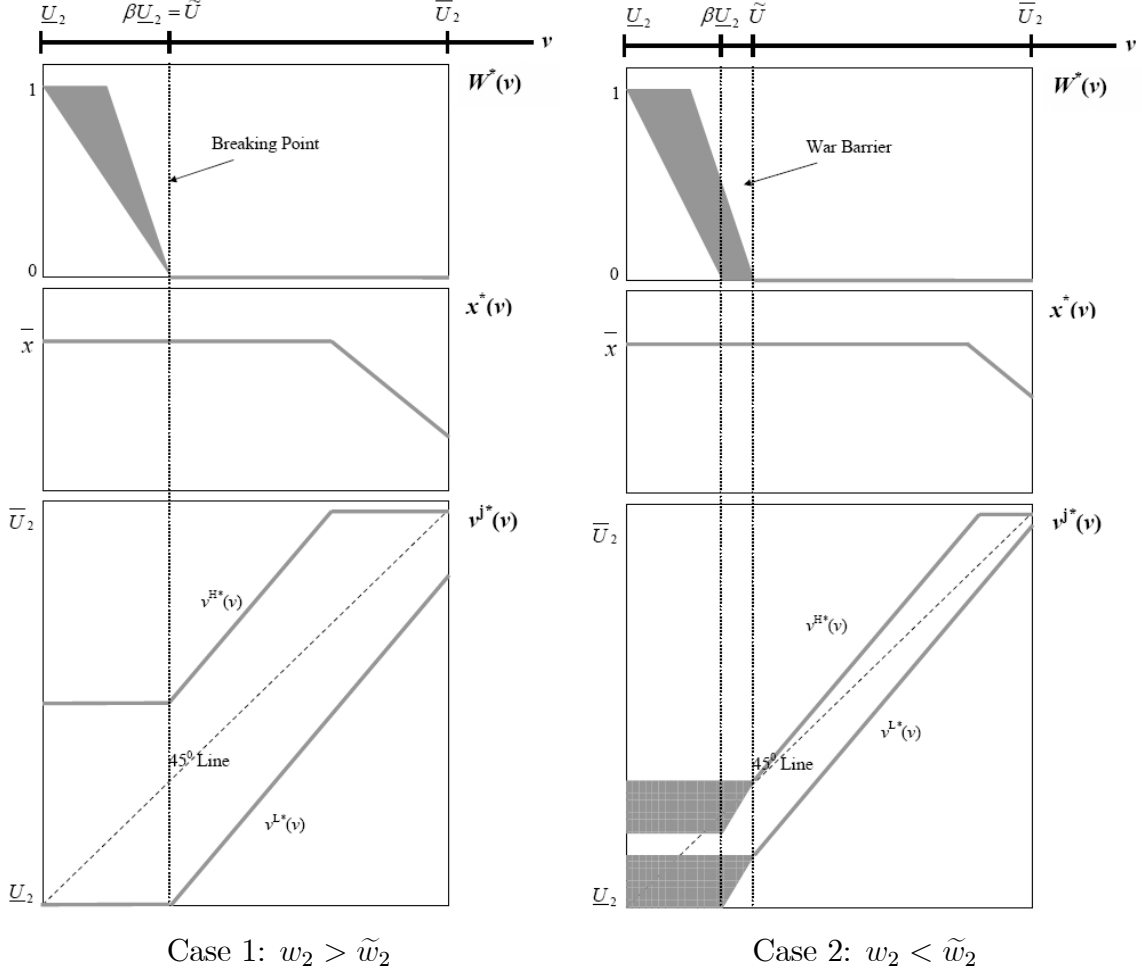
Figure 3 depicts the set of solutions for two cases: $w_2 > \tilde{w}_2$ and $w_2 < \tilde{w}_2$.¹⁵ The top, middle, and bottom panels depict the probability of war $W^*(v)$, the size of requested

¹⁴The additional sufficient condition for the entire set of solutions entails a lower bound on $v_z^{L^*}(v)$ and does not provide additional insights to what is presented here.

¹⁵The knife-edge case $w_2 = \tilde{w}_2$ combines features of these two cases and we ignore it due to space restrictions.

concessions $x^*(v)$, and continuation values $v^{j*}(v)$ on the y -axis, respectively, for different values of v on the x -axis. We do not depict $v^{W*}(v)$ due to space constraints. $v^{W*}(v)$ can take on any value in the range $\left[\underline{U}_2, \min \left\{ (v - w_2) / \beta, \tilde{U} \right\} \right]$ for all $v \in \left[\underline{U}_2, \tilde{U} \right]$.¹⁶

Figure 3: Recursive Solution



For $v > \tilde{U}$, the solutions under the two cases share the same characteristics. $W^*(v) = 0$, $x^*(v)$ is weakly decreasing in v , $v^{L*}(v)$ is strictly increasing in v , and $v^{H*}(v)$ is weakly increasing in v . Moreover, $v^{L*}(v)$ is strictly below v and $v^{H*}(v)$ is strictly above v . Therefore, a long sequence of zero concessions leads continuation values to decrease, and analogously, a long sequence of positive concessions, leads continuation values to increase. For example, in an equilibrium which begins at $v_0 = \bar{U}_2$, country 1 punishes the first failed concessions of country 2 by requesting larger concessions (i.e., $x^*(v)$ is decreasing in v).

¹⁶In Figure 3, we implicitly assume that $\beta \bar{U}_2 - \beta \underline{U}_2 > \bar{x}$ in case 1 and that $\beta \underline{U}_2 > w_2 + \beta \tilde{U}$ in case 2. Relaxing either inequality does not affect our discussion or the intuition behind our result.

for high v). After a long enough sequence of failed concessions, country 1 cannot request any more concessions from country 2 given the upper bound $\bar{x}$, and country 1 therefore punishes country 2 by reducing the reward for a successful concession (i.e., $v^{H*}(v)$ is increasing in v for intermediate v). If concessions continue to fail—as they must with some positive probability—then continuation values transition below $\tilde{U}$.

For $v < \tilde{U}$, the solutions under the two cases share some similarities and some difference. In both cases, $W^*(v)$ and $v^{W*}(v)$ can be represented by a graph. More specifically, war can occur with low probability today but with high probability in the future, or alternatively it can occur with high probability today but with low probability in the future. As v declines, the maximal and minimal probability of war today both weakly increase in order to generate sufficient punishment for country 2.

There are however four important differences. First, $v^{H*}(v) > \tilde{U}$ under case 1 but $v^{H*}(v) \leq \tilde{U}$ under case 2. Second, $v^{L*}(v) = \underline{U}_2$ under case 1, but $v^{L*}(v) > \underline{U}_2$ for some solutions for all v under case 2. Third, $W^*(v) > 0$ under case 1, but $W^*(v) = 0$ for some solutions for $v \geq \beta \underline{U}_2$ under case 2. Finally, $v^{H*}(v)$ and $v^{L*}(v)$ are uniquely defined under case 1, but they are a graph under case 2. These four facts have important implications. In case 1, once continuation values have declined below $\tilde{U}$, they can increase above $\tilde{U}$ only if country 1 makes a positive concession under peace. If country 1's concession fails, it is punished with total war. For this reason, we refer to $\tilde{U}$ as the *breaking point*. In contrast, in case 2, once continuation values have declined below $\tilde{U}$, they can never return above $\tilde{U}$, even if country 2 makes a positive concession. For this reason, we refer to the range $[\beta \underline{U}_2, \tilde{U}]$ as the *war barrier*. Though continuation values are trapped below $\tilde{U}$, country 2's failed concessions need not be punished with total war in this region as in case 1. Moreover, in contrast to case 1, under case 2, war tomorrow can follow positive concessions today (i.e., $W^*(v^{H*}(v)) > 0$), and peace tomorrow can follow zero concessions today (i.e., $W^*(v^{L*}(v)) < 1$).

An important implication of Proposition 4 is that the solution to (5) – (11) for $v > \underline{U}_2$ need not have the *Bang-Bang* property. An easy way to see this is to note that the solutions in $\Psi(v)$ with the *Bang-Bang* property do not feature limited war.

Remark 4 *If $\alpha^*(v) \in \Psi(v)$ has the Bang-Bang property, then $v^{j*}(v) = \{\underline{U}_2, \tilde{U}\}$ if $v^{j*}(v) \in [\underline{U}_2, \tilde{U}]$ since the portion of $J(\cdot)$ in the range $(\underline{U}_2, \tilde{U})$ is linear given that probabilistic war is efficient. Since $W_z^*(v) = 0 \forall z$ if $v \geq \tilde{U}$, then a solution to (4) which satisfies the Bang-Bang property at every history entails $\Pr\{W_{t+k} = 0 | W_t = 1\} = 0 \forall k$.*

Nonetheless, in our framework, a path with limited war is always efficient, since solutions which admit $v^{W*}(v) > \underline{U}_2$ exist for $v > \underline{U}_2$. This is stated formally below.

Proposition 5 (*limited war*) $\exists$ a solution to (4) for some v_0 s.t.

$$\Pr \{W_{t+k} = 0 | W_t = 1\} > 0 \text{ for } k > 0.$$

Proposition 5 implies that there are initial conditions v_0 with solutions which features limited war.¹⁷ Intuitively, it is just as efficient to punish the first few failed concessions with limited war with high probability as it is to punish these failed concessions with total war with low probability, which is the solution which satisfies the *Bang-Bang* property. Total war does not strictly dominate limited war since no information is released if concessions are not made, and the coarseness of information leaves the possibility for error. This result is in contrast to that of Abreu, Pearce, and Stacchetti (1986,1990) and Sannikov (2007a,2007b) since in their environment, the richness of information is used to optimally minimize the probability of error associated with the use of extreme rewards and extreme punishments. We discuss these differences in greater detail in Section 5.6.

5.5 Examples

5.5.1 Case 1: $w_2 > \tilde{w}_2$

Using Figure 3, we generate sample paths for concessions and for war under case 1. This is depicted in Figures 4 and 5 in which requested concessions are on the y -axis, time is on the x -axis, and the vertical dotted lines coincide with shocks to the cost of concessions s_t and decisions to go to war W_t .¹⁸ In both figures, we let $v_0 = \bar{U}_2$ so that the initial requested concession is minimal.

Figure 4 represents a solution to (4) which does not feature limited war, such as a solution which satisfies the *Bang-Bang* property. Country 1 begins by requesting some minimal concession x^p . As long as this concession succeeds, country 1 continues to request x^p . After the first failed concession, country 1 again requests x^p plus yesterday's concession plus interest x^p/β . If this concession succeeds, the equilibrium restarts and country 1 requests x^p . If instead this concession fails—as it does in the figure—country 1 requests x^p plus past concessions plus interest $x^p/\beta + x^p/\beta^2$. If this concession fails again, country 1 requests an even bigger concession, and so on. After a long sequence of failed concessions, country 1 cannot request more than $\bar{x}$ since this is infeasible. If at any point, country 2's concessions succeed, then country 1 forgives country 2, and it requests fewer concessions

¹⁷In practice, limited war may not occur under case 1 if $U_2(\alpha) = v_0$ and $v_0 \in \{\beta U_2, \beta^2 U_2, \dots\}$. However it can always occur for $v_0 + \epsilon$ for some $\epsilon \geq 0$ which is arbitrarily small in absolute value.

¹⁸Formally, "Successful Concessions" refers to $s_t = 1$ and "Unsuccessful Concessions" refers to $s_t = 0$. "Requested Concessions" corresponds to x_t conditional on $s_t = 1$.

in the future. However, with positive probability, country 2's concessions continue to fail, and it becomes necessary for country 1 to punish country 2 with total war.

Figure 4: No Limited War Under $w_2 > \tilde{w}_2$

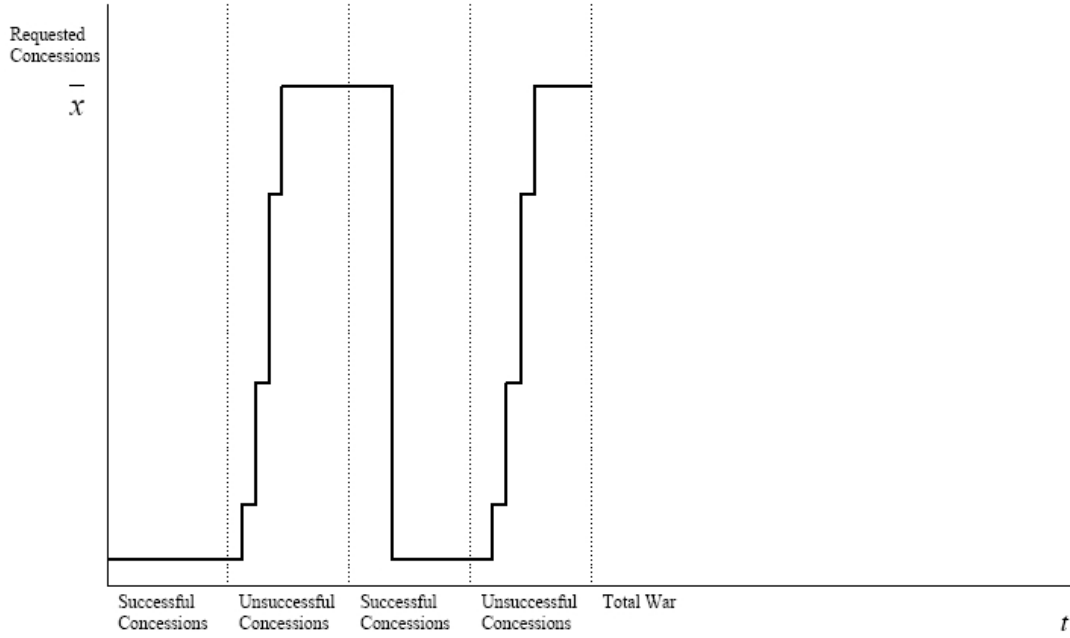


Figure 5: Limited War Under $w_2 > \tilde{w}_2$

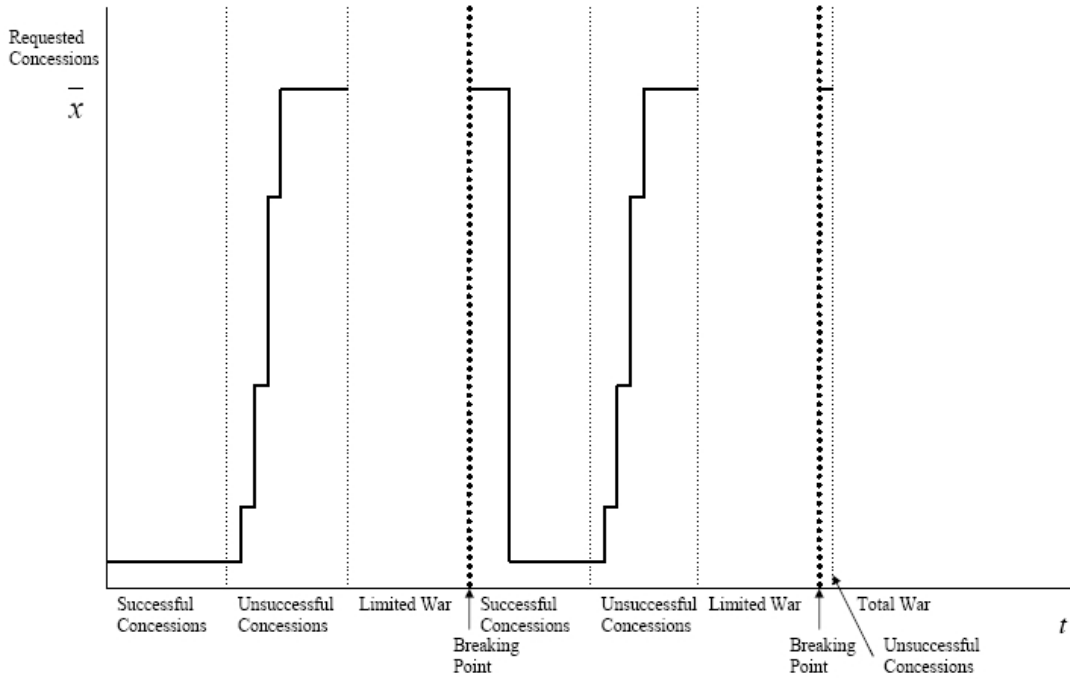


Figure 5 represents a solution to (4) which features limited war along the equilibrium path and which therefore violates the *Bang-Bang* property. Country 1 can punish the first few failed concessions with limited war, and this limited war culminates at the breaking point. At the breaking point, country 1 requests a concession and it promises peace in the future if this concession succeeds, but it threatens total war if the concession fails. In the example, country 2 is able to make a positive concession at the breaking point to avoid total war after the first phase of limited war. However, the second phase of limited war leads to the breaking point again, and country 2 is unable to make a positive concession, and this forces the two countries into total war. Note that while the two sample paths in Figures 4 and 5 are different, they both culminate in total war.

5.5.2 Case 2: $w_2 < \tilde{w}_2$

Figures 6 and 7 under case 2 are analogous to Figure 4 and 5, respectively, under case 1. The primary difference between Figures 4 and 6 is that in Figure 6, once a sufficiently large number of concessions have failed, country 1 cannot forgive country 2 by requesting lower concessions in the future. The reason is as follows. Under case 1, country 2 can make a very large concession today so as to be forgiven with lower requested concessions in the future since $-\bar{x}$ is sufficiently low relative to the discounted pain of war βw_2 . In contrast, under case 2, such a large concession cannot be made and forgiveness in the form of lower requested concessions in the future is not possible.

The most important difference between the two cases is in the difference between Figures 5 and 7. Prior to the first phase of limited war, equilibrium dynamics in Figure 7 resemble those of Figure 5. However, in Figure 7, the two countries move beyond the war barrier during the first phase of limited war, and a breaking point which can lead the two countries back above the war barrier does not exist. Consequently, country 2 cannot make a concession today which is so large that it be rewarded with assured peace and fewer requested concessions in the future. Moreover, in contrast to Figure 5, unsuccessful concessions after phases of limited war do not necessarily lead to total war. In fact, unsuccessful concessions do not need to be followed by limited war with certainty, and successful concessions do not need to be followed by peace with certainty. The realization of war is less predictable than in case 1, and there is no force which pushes the two countries towards total war.¹⁹ In sum, while the two countries can converge to total war,

¹⁹It is still true that if concessions fail at t , then the "discounted average probability of war" starting from $t + 1$, $(1 - \beta) \mathbf{E}_t \sum_{k=t+1}^{\infty} \beta^{k-t-1} W_k$, is higher if a concession fails at t .

they can also fluctuate between periods of peace and periods of war forever.

Figure 6: No Limited War Under $w_2 < \tilde{w}_2$

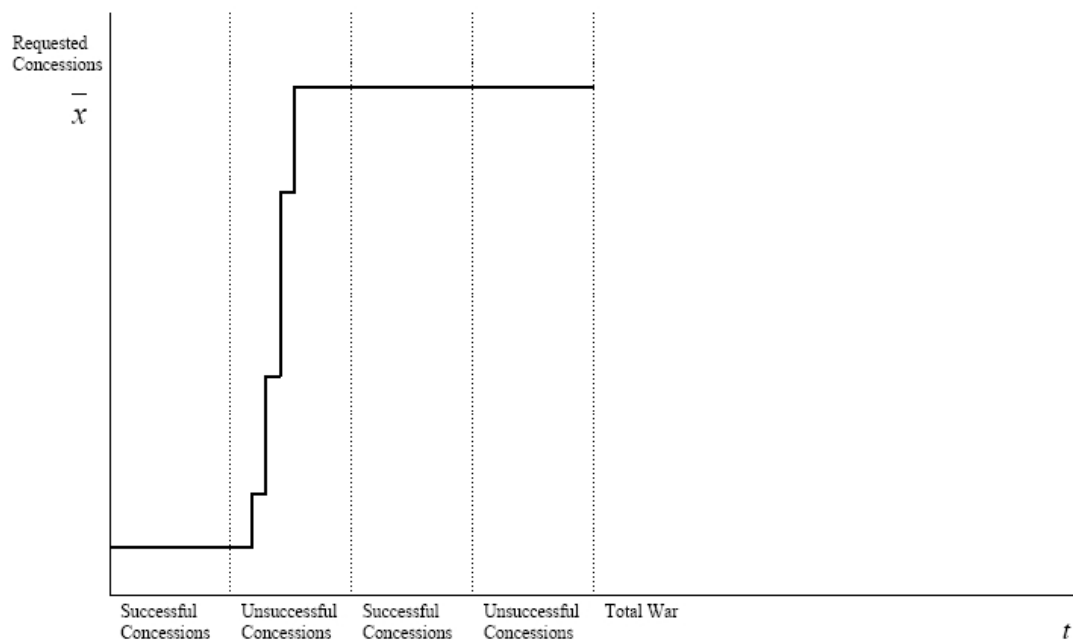
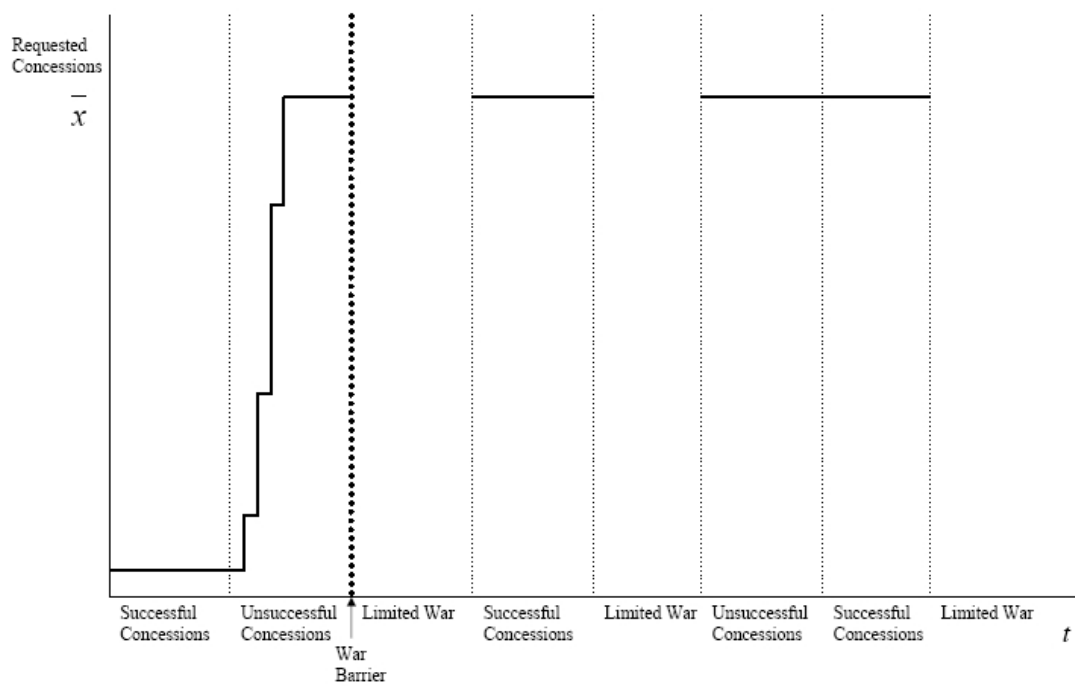


Figure 7: Limited War Under $w_2 < \tilde{w}_2$



5.6 Long Run Dynamics

Our sample paths illustrates a general result that the two countries must converge to total war in case 1, but they need not converge to total war in case 2. The argument follows from Proposition 4. Consider case 1. In order to sustain peace, the two countries must expect to go to war in the future. Any phase of limited war culminates at the breaking point, and from the breaking point, a transition to total war occurs with positive probability whenever concessions are too costly and are not made. In the long run, the probability with which the two countries are able to transition out of the breaking point to a phase of peace converges to zero. In contrast, consider case 2. One can construct an example in which the two countries converge to total war as in Figure 6. However, once continuation values have transitioned beyond the war barrier, total war is not useful for providing incentives for country 2, and equilibria such as in Figure 7 in which failed concessions are punished with limited war are efficient. This is stated formally in the below theorem.

Theorem 1 (*long run*) $\forall v_0$,

1. If $w_2 \geq \tilde{w}_2$, $\nexists$ a solution to (4) s.t. $\lim_{t \rightarrow \infty} \Pr \{W_t = 0\} > 0$,
2. If $w_2 < \tilde{w}_2$, $\exists$ a solution to (4) s.t. $\lim_{t \rightarrow \infty} \Pr \{W_t = 0\} = 0$, and
3. If $w_2 < \tilde{w}_2$, $\exists$ a solution to (4) s.t. $\lim_{t \rightarrow \infty} \Pr \{W_t = 0\} > 0$.

The intuition for the first part of the theorem is as follows. Country 1 utilizes the extreme threat of total war in the long run so as to extract as much surplus from country 2 along the equilibrium path. Therefore, the two countries make sacrifices in the long run so as to increase their welfare in the short run. Eventually, it becomes necessary for country 1 to punish country 2 for a long sequence of failed concessions. Moreover, βw_2 , the magnitude of country 2's suffering under war tomorrow, is not very high relative to $\bar{x}$, to the maximal amount that can be extracted by country 1 today. Consequently, the potential welfare cost of making an error and going to total war relative to the added benefit of using extreme threats to induce concessions is small. Therefore, even though information in our environment is coarse, and even though the *Bang-Bang* property can be violated along the equilibrium path, the efficient equilibrium satisfies the *Bang-Bang* property in the long run since continuation values converge to the repeated static Nash equilibrium. Therefore, concessions cannot forestall total war in this case.²⁰

²⁰Note that though transitions to permanent war are required in the long run, the probability with which this occurs goes to zero as the discount factor goes to 1. This is because asymptotically $v^{L*}(v) \rightarrow v$, so that continuation values never decline.

The second part of the theorem is a direct application of the *Bang-Bang* property since equilibria in which limited war is avoided are always efficient. The intuition for the third part of the theorem is as follows. The threat of total war in the long run is not necessary to enforce efficient cooperation in the short run since the possibility of temporary war is sufficiently distasteful so as to induce country 2 to make a concession. Eventually it becomes necessary for country 1 to punish country 2 for a long sequence of failed concessions, yet total war does not dominate limited war in the long run. More specifically, βw_2 , the magnitude of country 2's suffering under war tomorrow, is high relative to $\bar{x}$, the maximal amount that can be extracted by country 1 today. Consequently, the potential welfare cost of making an error and going to total war relative to the added benefit of using extreme threats to induce concessions is large. Therefore, the coarseness of information in this case implies that the *Bang-Bang* property can be violated both along the equilibrium path and in the long run. The two countries can fluctuate between periods of war and periods of peace forever, and concessions can forestall total war.

One way to illustrate the difference between these two cases is to consider a simple two period version of our model. To facilitate the discussion, imagine if country 1 can commit to its actions and if $w_1 < 0$, so that the welfare of both countries is increased by reducing the realization of war in the second period.²¹ Consider the efficient equilibrium which maximizes country 1's welfare. If $w_2 \geq \tilde{w}_2$, it can be shown that the solution to this problem admits a version of the *Bang-Bang* property: assured peace in the second period rewards positive concessions in the first period and assured war in the second period punishes zero concessions in the first period. In contrast, if $w_2 < \tilde{w}_2$, then zero concessions in the first period are punished with probabilistic war in the second period. In the first case, extreme punishments are useful since the maximal concessions $\bar{x}$ is large relative to the worst punishment βw_2 , and in the second case, extreme punishments are not useful since $\bar{x}$ is small relative to βw_2 .²²

To see why the separation between these two cases depends on the coarseness of information, imagine the following modification of this two periods example which brings the environment more in line with the setting studied by Abreu, Pearce, and Stacchetti (1986,1990) and Sannikov (2007a,2007b). Introduce a continuous public signal y which is revealed if concessions are zero. The signal y is informative about the state s with higher

²¹This captures the fact that country 1 benefits from a reduction in the probability of war and an increase in country 2's welfare for low levels of v in an infinitely repeated setting.

²²We have implicitly assumed that that $\pi > (1 - \pi)w_1/w_2$ so that the efficient equilibrium admits positive concessions in the first period. The constraint that $x \leq \bar{x}$ binds in the second case but not in the first case. In the second case, the probability of war after zero concessions is $-\bar{x}/(\beta w_2)$.

values of y being more likely if $s = 1$ (i.e., the cost of concessions is low).²³ Now the *Bang-Bang* property holds in both cases: assured peace rewards successful concessions as well as failed concessions if $y \leq y^*$, and assured war punishes failed concessions if $y > y^*$ for some threshold y^* . Since information is no longer coarse, country 1 never fights probabilistically after any public history. Instead, it provides country 2 with extreme rewards and punishments while simultaneously utilizing the information in y to optimally reduce the probability of error in going to war.

6 Evidence

We have characterized a model which describes conditions under which total war can be avoided and countries can engage in limited war forever. In our model, total war corresponds to an equilibrium in which war occurs in every period, whereas limited war corresponds to an equilibrium in which the realization of war is always followed by a positive probability realization of peace in the future. In practice, countries engage in wars of differing intensities in the resolution of their disputes. While countries clearly do not "fight forever" as they do in our stylized model, they do engage in wars of differing lengths and of differing levels of destructiveness. Since total war is longer and destroys more surplus than limited war in our model, one can interpret total war in the world as corresponding to a war with a longer duration and with more casualties than limited war.

With this interpretation in mind, our model predicts that short wars with few casualties are more likely to occur if the aggressive country can inflict significant damage on its rival. This is because limited wars are very costly to the rival country, and therefore a long war with many casualties is not necessary to induce concessions. In contrast, long wars with many casualties are more likely to occur if the aggressive country cannot inflict significant damage on its rival. In this case, diplomatic concessions can temporarily forestall the realization of war, however, these concessions are sustained by the extreme threat of total war which must eventually be realized. In this section, we investigate whether these predictions are in line with the available data on international conflicts.

We collect observations from the Correlates of War Inter-State War Data which provides information on the duration and casualty cost of wars from 1816 onward.²⁴ For all of these wars, the database provides us with information regarding the identity of the ag-

²³Specifically, y has full support conditional on s and it satisfies the monotone likelihood ratio property. I thank Andrew Atkeson for pointing out this example.

²⁴Data available at <http://cow2.la.psu.edu> and corresponds to the COW Inter-State War Data, 1816-1997, version 3.0.

gressor in the conflict and the identity of the aggressor’s allies. In aligning this data with our model, we are met with two complications. First, countries sometimes form alliances in wars, and wars are fought by groups of countries. Moreover, countries do not always fight about the same issue. Consequently, constructing the time-series of the interaction between two sides of a conflict is not a straightforward task. In order to deal with this issue, we code every war as constituting a single observation corresponding to the resolution of a single conflict between two sides, where each side in the data (which may consist of several countries) is analogous to a single country in our model. As a second complication, the ex-ante cost of war to the non-aggressive country in our model, w_2 , cannot be directly observed, so that we must consider observable variables which are related to the size of w_2 . To do this, we calculate the ratio of the size of the aggressive side’s army at the start of the conflict to the size of the non-aggressive side’s army at the start of the conflict. This variable which is related to the relative strength of the aggressive side can serve as a proxy for the *expected* damage to be suffered by the non-aggressive side. Clearly, the size of the army is not a perfect measure of relative strength, and moreover, relative strength is not a perfect measure of the cost of war to the rival, since this also depends on other factors, such as the location of the war or differences in military technology, for instance. However, to the extent that a larger infantry by the aggressor creates an additional cost of war to the non-aggressive country, our constructed variable is useful proxy for w_2 .²⁵

In Tables 1a and 1b we list all of the 73 wars which were fought between 1816 and 1990. The third column reports the ratio of the military personnel on the aggressive side of the conflict to the military personnel on the non-aggressive side of the conflict. In the fourth column, we report the length of the war in days. In the fifth column, we report the number of battle deaths in thousands. These tables show that there is significant variation in the relative strength of the aggressor, the duration, and the number of casualties in different wars. As an illustration, World War I which began in 1914 lasted over 1,500 days and entailed over 8 million battle deaths, whereas the Turco-Cypriot war of 1974 lasted only 13 days and entailed about 2,000 battle deaths. In World War I, the aggressor’s relative military strength (i.e., the Central Powers’ military personnel relative to the Entente Powers’ military personnel) was 0.27. In contrast, in the Turco-Cypriot war, the aggressor’s relative military strength (i.e., Turkey’s military personnel relative to Cyprus’s military personnel) was 40.91. A similar comparison can be drawn between the Anglo-Persian war of 1856 and the Iran-Iraq war of 1980, both which involved aggression against present day Iran. In the Anglo-Persian war, Great Britain was very strong relative

²⁵See McNeill (1982) for a discussion of the rise of the importance of infantry in the nineteenth century.

to Persia, and the war was short and involved few casualties. In contrast, in the Iran-Iraq war, Iraq was not much stronger than Iran, and the war was long and involved many casualties.

Our model predicts that conflicts in which the aggressor is very strong should be associated with a shorter duration of war and with fewer casualties. This is supported by the evidence in Tables 1a and 1b, which show that our comparison of World War I and the Turco-Cypriot war and the Anglo-Persian war and the Iran-Iraq war are indicative of a more general pattern. For wars in which the relative strength of the aggressor is above the sample median, the average duration of war is 340 days and the average number of casualties is 50,335. In comparison, for wars in which the relative strength of the aggressor is below the sample median, the average duration of war is 566 days and the average number of casualties is 828,638. These differences are significant. For example, the correlation between the log relative strength of the aggressor and the log duration of war is -0.20 with a p-value of 0.08. Furthermore, the correlation between the log relative strength of the aggressor and log battle deaths is -0.24 with a p-value of 0.04. Therefore, disputes involving a stronger aggressor are likely to involve shorter wars with fewer battle deaths. These results suggest that the mechanism described in our model may be operational in practice.

7 Conclusion

We have analyzed a dynamic model to determine whether and how concessions to an aggressive country can be used to forestall war. We have argued that the realization of war in the future is necessary to sustain peace today, so that concessions can only temporarily forestall war. In the efficient sequential equilibrium, concessions cannot forestall total war if the cost of war to the non-aggressive country is low. However, concessions can forestall total war if the cost of war to the non-aggressive country is high.

In our analysis, we have explicitly characterized the transitional dynamics of war and the size of requested concessions. Our model predicts that situations in which the aggressive country can only inflict small military damage must be characterized by escalating demands by the aggressor, potential limited war, and culmination in total war. In contrast, in situations in which the aggressive country can inflict large military damage, total war can be avoided, and countries can engage in limited war forever. In the last section, we have provided evidence that our model may explain some of the empirical evidence on war. Our analysis is linked to the large literature on dynamic games with imperfect

public monitoring, and our model provides a special example of a game with a coarse information structure in which convergence to the repeated static Nash equilibrium is only necessary if the threat of the repeated Nash equilibrium is not large enough.

We highlight some important caveats in interpreting our results. First, in choosing to focus on the role of diplomatic concessions, we have ignored the fact that military concessions such as disarmament could also serve to avert conflict by altering the payoff from war.²⁶ Second, we have implicitly assumed that there is a single good over which the two countries bargain. One can imagine a natural extension of this framework in which each country controls different goods, so that bilateral concessions are necessary to sustain peace. Finally, in the interest of parsimony, we have ignored issues regarding military strategy during war, and we have abstracted from the mechanism by which total war is resolved by defining it as equivalent to permanent war.²⁷ A thorough investigation of the implications of these issues for our results would be interesting for future research.

²⁶See for example Baliga and Sjoström (2007).

²⁷See for example Leventoglu and Slantchev (2006).

Table 1a: Wars 1816-1990

War	Year	Relative Strength of Aggressor	War Length	Battle Deaths
Franco-Spanish	1823	2.25	221	1
Russo-Turkish	1828	2.48	507	415
Mexican-American	1846	1.95	632	19
Austro-Sardinian	1848	0.19	143	8
First Schleswig-Holstein	1848	5.00	247	6
Roman Republic	1849	70.69	55	3
La Plata	1851	4.00	200	1
Crimean	1853	1.17	861	264
Anglo-Persian	1856	21.22	141	2
Italian Unification	1859	0.69	75	23
Spanish-Moroccan	1859	14.69	260	20
Italo-Roman	1860	9.63	19	1
Italo-Sicilian	1860	1.99	97	1
Franco-Mexican	1862	19.26	1757	20
Ecuadorian-Columbian	1863	0.67	15	1
Lopez	1864	0.88	1936	310
Second Schleswig-Holstein	1864	15.94	111	5
Spanish-Chilean	1865	5.64	197	1
Seven Weeks	1866	1.02	42	44
Franco-Prussian	1870	1.30	223	204
First Central American	1876	3.00	30	4
Pacific	1879	0.12	1762	14
Anglo-Egyptian	1882	16.47	67	2
Sino-French	1884	0.53	291	12
Second Central American	1885	3.00	19	1
Franco-Thai	1893	0.12	53	2
Sino-Japanese	1894	0.16	1615	1015
Greco-Turkish	1897	0.15	1256	52
Spanish-American	1898	1.54	114	4
Boxer Rebellion	1900	2.56	59	3
Sino-Russian	1900	1.14	55	4
Russo-Japanese	1904	0.19	586	152
Third Central American	1906	1.75	55	1
Fourth Central American	1907	1.00	64	1
Italo-Turkish	1911	0.87	386	20
First Balkan	1912	0.49	185	82
Second Balkan	1913	1.02	31	61

Table 1b: Wars 1816-1990, Continued

War	Year	Relative Strength of Aggressor	War Length	Battle Deaths
World War I	1914	0.27	1567	8578
Franco-Turkish	1919	18.33	720	40
Hungarian-Allies	1919	16.37	111	11
Russo-Polish	1919	5.17	613	100
Lithuanian-Polish	1920	6.00	140	1
Sino-Soviet	1929	0.33	109	3
Manchurian	1931	0.17	505	60
Chaco	1932	0.43	1093	93
Saudi-Yemeni	1934	0.22	55	2
Italo-Ethiopian	1935	13.80	220	20
Changkufeng	1938	4.23	14	2
Nomonhan	1939	0.53	129	28
Russo-Finnish	1939	48.35	104	75
World War II	1939	0.72	2173	16569
First Kashmir	1948	1.21	169	2
Palestine	1948	1.34	143	8
Korean	1950	1.00	1130	910
Russo-Hungarian	1956	23.72	23	4
Sinai	1956	18.19	9	3
Assam	1962	6.49	34	2
Second Kashmir	1965	3.95	50	7
Vietnamese	1965	15.74	3735	1021
Six Day	1967	0.18	6	20
Football	1969	1.20	5	2
Israeli-Egyptian	1969	3.53	520	5
Bangladesh	1971	2.51	15	11
Yom Kippur	1973	5.34	19	16
Turco-Cypriot	1974	40.91	13	2
Vietnamese-Cambodian	1975	8.75	1348	8
Ethiopian-Somalian	1977	0.13	226	6
Ugandan-Tanzanian	1978	2.15	165	3
Sino-Vietnamese	1979	4.27	33	25
Iran-Iraq	1980	1.01	2890	1250
Falklands	1982	0.47	88	1
Israel-Syria (Lebanon)	1982	0.78	138	1
Gulf War	1990	0.09	253	26

8 Appendix

8.1 Definitions

The following definitions simplify notation.

$$\begin{aligned}\Upsilon^+(v, \epsilon) &= \frac{J(v + \epsilon) - J(v)}{\epsilon} \\ \Upsilon^-(v, \epsilon) &= \frac{J(v) - J(v - \epsilon)}{\epsilon}\end{aligned}$$

$$v^{\max} = \min_v \left\{ v \in [\underline{U}_2, \overline{U}_2] \text{ s.t. } v = \arg \max_v J(v) \right\}$$

$$\begin{aligned}v^{F*}(v) &= w_2 + \beta v^{W*}(v) \\ v^{P*}(v) &= \pi(-x^*(v) + \beta v^{H*}(v)) + (1 - \pi)\beta v^{L*}(v)\end{aligned}$$

8.2 Proofs of Section 3

8.2.1 Proof of Proposition 1

Step 1. If α is a sequential equilibrium, then $x_t(q_t, z_t, s_t = 0) = 0 \forall (q_t, z_t)$. If instead $x_t(q_t, z_t, s_t = 0) > 0$, consider a deviation by country 2 at $(q_t, z_t, s_t = 0)$ to $x'_k(q_k, z_k, s_k) = 0 \forall k \geq t$ and $\forall (q_k, z_k, s_k)$ which yields a minimum continuation value of $\beta \underline{U}_2$. Since $x_t(q_t, z_t, s_t = 0)$ is bounded from below by 0 so that $\mathbf{E}\{U_2(\alpha|_{q_{t+1}, z_{t+1}}) | q_t, z_t, s_t = 1\}$ is bounded from above by 0, if this deviation is weakly dominated, then it must be that $-c(x, 0) \geq \beta \underline{U}_2$ for $x > 0$, but this violates Assumption 3.

Step 2. The necessity of (2) for $i = 1$ follows from the fact that country 1 can choose $W'_k(q_k, z_k) = 1 \forall k \geq t$ and $\forall (q_k, z_k)$ and this delivers continuation value $\underline{U}_1$. The necessity of (2) for $i = 2$ follows from the fact that country 2 can choose $x'_k(q_k, z_k, s_k) = 0 \forall k \geq t$ and $\forall (q_k, z_k, s_k)$, and this delivers a minimum continuation value $\underline{U}_2$. The necessity of (3) follows from the fact that conditional on $W_t(q_t, z_t) = 0$, country 2 can unobservably deviate to $x'_t(q_t, z_t, s_t = 1) = x_t(q_t, z_t, s_t = 0) = 0$ and follow the equilibrium strategy associated with $(q_t, z_t, s_t = 0)$ thereafter.

Step 3. For sufficiency, consider an allocation in which $x_t(q_t, z_t, s_t = 0) = 0 \forall (q_t, z_t)$ which also satisfies (2) and (3), and construct the following off-equilibrium strategy. Any observable deviation results in a reversion to the repeated static Nash equilibrium. We only consider single period deviations since $\beta < 1$ and since $U_i(\alpha)$ is bounded for $i = 1, 2$. If $W_t(q_t, z_t) = 1$, a deviation to $W'_t(q_t, z_t) = 0$ is strictly dominated by (2) since

$\beta \underline{U}_1 < \underline{U}_1$. If $W_t(q_t, z_t) = 0$, a deviation to $W'_t(q_t, z_t) = 1$ is weakly dominated by (2). If $W_t(q_t, z_t) = 0$, any deviation to $x'_t(q_t, z_t, s_t = 1) > 0$ is weakly dominated by a deviation to $x'_t(q_t, z_t, s_t = 1) = x_t(q_t, z_t, s_t = 0) = 0$ since $\mathbf{E}\{U_2(\alpha|_{q_{t+1}, z_{t+1}}) | q_t, z_t, s_t = 0\} \geq \underline{U}_2$. A deviation to $x'_t(q_t, z_t, s_t = 1) = 0$ is weakly dominated by (3). Any deviation to $x'_t(q_t, z_t, s_t = 0) \neq x_t(q_t, z_t, s_t = 1)$ is strictly dominated since $c(x, 0) > 0$. Since $\mathbf{E}\{U_2(\alpha|_{q_{t+1}, z_{t+1}}) | q_t, z_t, s_t = 1\} \leq 0$, by Assumption 3 and (2), a deviation to $x'_t(q_t, z_t, s_t = 0) = x_t(q_t, z_t, s_t = 1)$ is strictly dominated. **Q.E.D.**

8.2.2 Proof of Lemma 1

Step 1. Consider two continuation value pair $\{U'_1, U'_2\} \in V$ and $\{U''_1, U''_2\} \in V$ with corresponding allocations α' and α'' . It must be that

$$\{U_1^\kappa, U_2^\kappa\} = \{\kappa U'_1 + (1 - \kappa) U''_1, \kappa U'_2 + (1 - \kappa) U''_2\} \in V \quad \forall \kappa \in (0, 1).$$

Define $\alpha^\kappa = \{\alpha^\kappa|_{q_0, z_0}\}_{z_0 \in [0, 1]}$ as follows:

$$\alpha^\kappa|_{q_0, z_0} = \begin{cases} \alpha'|_{q_0, z_0} & \text{if } z_0 = 0 \\ \alpha'|_{q_0, \frac{z_0}{\kappa}} & \text{if } z_0 \in (0, \kappa) \\ \alpha''|_{q_0, \frac{z_0 - \kappa}{1 - \kappa}} & \text{if } z_0 \in [\kappa, 1] \end{cases},$$

where $\alpha'|_{q_0, \frac{z_0}{\kappa}}$ for $z_0 \in (0, \kappa)$ is identical to $\alpha'|_{q_0, z_0}$ with the exception that $\frac{z_0}{\kappa}$ replaces z_0 in all information sets q_t , and $\alpha''|_{q_0, \frac{z_0 - \kappa}{1 - \kappa}}$ for $z_0 \in [\kappa, 1]$ is analogously defined. α^κ achieves $\{U_1^\kappa, U_2^\kappa\}$, and since $\alpha', \alpha'' \in \Lambda$, then $\alpha^\kappa \in \Lambda$.

Step 2. V is bounded since $U_i(\alpha)$ is bounded for $i = 1, 2$.

Step 3. To show that V is closed, consider a sequence $V'_j \in V$ such that $\lim_{j \rightarrow \infty} V'_j = V'$. There exists a corresponding sequence of allocations α'_j which converges to α'_∞ since $U_i(\alpha'_j)$ is continuous in α'_j . Since every element of α'_j at (q_t, z_t) is contained in $\{0, 1\} \times [0, \bar{x}]^2$, and since (2) and (3) are weak inequalities, then Λ is closed and $\alpha'_\infty \in \Lambda$. Since $\beta \in (0, 1)$, then by the Dominated Convergence Theorem, $U_i(\alpha'_\infty) = U'_i$ for $i = 1, 2$. Therefore $V' \in V$. **Q.E.D.**

8.3 Proofs of Section 4

8.3.1 Proof of Proposition 3

Imagine if $W_t(q_t, z_t) = 0 \quad \forall (q_t, z_t)$. Country 2 can deviate to $x'_k(q_k, z_k, s_k) = 0 \quad \forall k \geq t$ and $\forall (q_k, z_k, s_k)$, which delivers a continuation value to country 2 equal to 0. There-

fore, $U_2(\alpha|_{q_t, z_t}) \geq 0 \ \forall (q_t, z_t)$. Since $u_1(\cdot) + u_2(\cdot) \leq 0$ by Assumption 1, $U_1(\alpha|_{q_t, z_t}) + U_2(\alpha|_{q_t, z_t}) \leq 0$, which means that $U_1(\alpha|_{q_t, z_t}) \leq 0 \ \forall (q_t, z_t)$, but this violates (2) by Assumption 2. **Q.E.D.**

8.4 Proofs of Section 5

8.4.1 Proof of Lemma 2

Step 1. It is not possible that $J(\cdot) < \underline{U}_1$ since this violates (2) for $i = 1$.

Step 2. Imagine if $J(\underline{U}_2) > \underline{U}_1$ and consider the associated α . By Assumptions 1 and 2 and equation (2) for $i = 2$, equation (3) implies that $U_2(\alpha|_{q_0, z_0}) \geq \beta \underline{U}_2 > \underline{U}_2$ if $W_0(q_0, z_0) = 0$. Since $U_2(\alpha|_{q_0, z_0}) \geq \underline{U}_2$, then $U_2(\alpha|_{q_0, z_0}) = \underline{U}_2$ and $W_0(q_0, z_0) = 1 \ \forall (q_0, z_0)$. This requires $\mathbf{E}\{U_2(\alpha|_{q_1, z_1}) | q_0, z_0\} = \underline{U}_2 \ \forall (q_0, z_0)$ and therefore $U_2(\alpha|_{q_1, z_1}) = \underline{U}_2 \ \forall (q_1, z_1)$. Forward induction on this argument implies that $W_t(q_t, z_t) = 1 \ \forall (q_t, z_t)$ so that $J(\underline{U}_2) = \underline{U}_1$.

Step 3. Imagine if $J(\overline{U}_2) > \underline{U}_1$ and consider the associated α . Since $U_2(\alpha|_{q_0, z_0}) \leq \overline{U}_2$, then $U_2(\alpha|_{q_0, z_0}) = \overline{U}_2 \ \forall (q_0, z_0)$ in order that $U_2(\alpha) = \overline{U}_2$. If $W_0(q_0, z_0) = 1$, then $\overline{U}_2 = w_2 + \beta \mathbf{E}\{U_2(\alpha|_{q_1, z_1}) | q_0, z_0\} \leq w_2 + \beta \overline{U}_2$, which contradicts Assumption 4. Therefore, $W_0(q_0, z_0) = 0$. Furthermore, $x_0(q_0, z_0) = 0$ and $\mathbf{E}\{U_2(\alpha|_{q_1, z_1}) | q_0, z_0\} = \overline{U}_2$, otherwise it is possible to reduce $x_0(q_0, z_0)$ or increase $\mathbf{E}\{U_2(\alpha|_{q_1, z_1}) | q_0, z_0\}$ while maintaining (2) and (3) and strictly increasing $U_2(\alpha)$. This means that $\overline{U}_2 = \beta \overline{U}_2$, but this violates the fact that $\overline{U}_2 < 0$, since $\overline{U}_2 = 0$ is not incentive compatible by the proof of Proposition 3. **Q.E.D.**

8.4.2 Proof of Lemma 3

Step 1. Consider $\alpha^*(v) \in \Psi(v)$ for $v > \underline{U}_2$ for which $W^*(v) \in (0, 1)$. Since a perturbation to which satisfies (12) satisfies (6) – (11), optimality and the concavity of $J(\cdot)$ require

$$J(v) = W^*(v) J(v^{F^*}(v)) + (1 - W^*(v)) J(v^{P^*}(v)). \quad (13)$$

By (13) and the concavity of $J(\cdot)$, if $v^{F^*}(v) < v^{P^*}(v)$, then $\Upsilon^+(v^{F^*}(v), \epsilon) = \Upsilon^-(v^{P^*}(v), \epsilon)$ for $\epsilon > 0$ arbitrarily small.

Step 2. By (6), if $W^*(v) = 1$ for $v > \underline{U}_2$, then $v^{W^*}(v) = \frac{v - w_2}{\beta} > v$, so that $\frac{J(v) - J(\underline{U}_2)}{v - \underline{U}_2} = \frac{J(\frac{v - w_2}{\beta}) - J(\underline{U}_2)}{\frac{v - \underline{U}_2}{\beta}}$. By the concavity of $J(\cdot)$ and Assumption 4,

$$J\left(\frac{v - w_2}{\beta}\right) = \underline{U}_1 + m \left(\frac{v - w_2}{\beta} - \underline{U}_2\right) \text{ for } m > 0. \quad (14)$$

Define $\tilde{U}$ as

$$\Upsilon^+(\tilde{U}, \epsilon) < m \text{ and } \Upsilon^-(\tilde{U}, \epsilon) = m. \quad (15)$$

By (14), $\tilde{U} > \max_v v^{F*}(v) \geq \underline{U}_2$. By the concavity of $J(\cdot)$ and Assumption 4, $\tilde{U} \leq v^{\max} < \bar{U}_2$. If instead $\nexists v > \underline{U}_2$ s.t. $W^*(v) = 1$, then $\max_v v^{F*}(v) = \underline{U}_2$. By (9), $v^{P*}(v) \geq \beta v^{L*}(v) \geq \beta \underline{U}_2 > \underline{U}_2$. Therefore, $W^*(v) \in (0, 1)$ and (13) applies $\forall v \in (\underline{U}_2, \beta \underline{U}_2)$, so that (14) holds $\forall v \in (\underline{U}_2, w_2 + \beta^2 \underline{U}_2)$, and $\tilde{U} \in (\underline{U}_2, \bar{U}_2)$ by Assumption 4 and Lemma 2.

Step 3. $\forall \alpha^*(v) \in \Psi(v)$ s.t. $\forall v \geq \tilde{U}$, $W^*(v) = 0$. The possibility that $W^*(v) = 1$ is ruled out since $\max_v v^{F*}(v) < \tilde{U}$. If $W^*(v) \in (0, 1)$, then since $v^{F*}(v) < v < v^{P*}(v)$, $\Upsilon^+(v^{F*}(v), \epsilon) = \Upsilon^-(v^{P*}(v), \epsilon)$ from step 1. However, this contradicts (15) and the concavity of $J(\cdot)$. This establishes the first part of the lemma.

Step 4. Imagine if $\tilde{U} < \beta \underline{U}_2$. This violates (8) and (9) which require $\tilde{U} \geq \beta v^{L*}(\tilde{U}) \geq \beta \underline{U}_2$.

Step 5. $J(v + \epsilon) > J(v) - \epsilon$ for $\epsilon > 0$. If instead $\exists v$ s.t. $\Upsilon^-(v, \epsilon) \leq -1$, then by the concavity of $J(\cdot)$ and Lemma 2, $\exists \hat{U} \in [v^{\max}, \bar{U}_2)$ s.t. $\Upsilon^+(\hat{U}, \epsilon) \leq -1$ and $\Upsilon^-(\hat{U}, \epsilon) > -1$, where $\hat{U} \geq \tilde{U}$ so that $W^*(\hat{U}) = 0$ by step 3. Consider $\alpha^*(\hat{U}) \in \Psi(\hat{U})$ which satisfies step 1 and under which (9) binds, which is always weakly optimal by the weak concavity of the program and convexity of the constraint set. If $x^*(\hat{U}) > 0$, then optimality requires that

$$J(\hat{U} + \epsilon) \geq \pi \left(x^*(\hat{U}) - \epsilon + \beta J(v^{H*}(\hat{U})) \right) + (1 - \pi) \beta J \left(v^{L*}(\hat{U}) + \frac{\epsilon}{\beta} \right) \quad (16)$$

since a perturbation to $x'(\hat{U} + \epsilon) = x^*(\hat{U}) - \epsilon$, $v^{H'}(\hat{U} + \epsilon) = v^{H*}(\hat{U})$, and $v^{L'}(\hat{U} + \epsilon) = v^{L*}(\hat{U}) + \frac{\epsilon}{\beta}$ satisfies (6) – (11) for $v = \hat{U} + \epsilon$. Subtraction of $J(\hat{U})$ from both sides of (16) yields $\Upsilon^+(\hat{U}, \epsilon) \geq -\pi + (1 - \pi) \Upsilon^+(\frac{\hat{U}}{\beta}, \frac{\epsilon}{\beta}) > -1$, which is a contradiction, where by Proposition 3, $v < 0$, so that $\frac{v}{\beta} < v$. If instead $x^*(\hat{U}) = 0$ so that $v^{H*}(\hat{U}) = v^{L*}(\hat{U}) = \frac{\hat{U}}{\beta}$, then analogous arguments can be made with a perturbation to $x'(\hat{U} + \epsilon) = x^*(\hat{U})$ and $v^{H'}(\hat{U} + \epsilon) = v^{L'}(\hat{U} + \epsilon) = \frac{v}{\beta} + \frac{\epsilon}{\beta}$, so that $\Upsilon^+(\hat{U}, \epsilon) \geq \Upsilon^+(\frac{\hat{U}}{\beta}, \frac{\epsilon}{\beta}) > -1$ which is also a contradiction.

Step 6. $\forall v \in [\underline{U}_2, \bar{U}_2]$ and $\forall \alpha^*(v) \in \Psi(v)$, if $W_z^*(v) = 0$ then $x_z^*(v) = \bar{x}$ or $v_z^{H*}(v) = \bar{U}_2 \forall z$. If $x^*(v) < \bar{x}$ and $v^{H*}(v) < \bar{U}_2$, then consider a perturbation to $x'_z(v) = x^*(v) + \epsilon$, $v_z^{H'}(v) = v^{H*}(v) + \epsilon/\beta$, and $v_z^{L'}(v) = v^{L*}(v) \forall z$. Such a perturbation satisfies (6) – (11) and strictly improves welfare by step 5.

Step 7. Consider if $w_2 \geq \tilde{w}_2$ and imagine if $\tilde{U} > \beta \underline{U}_2$. Let $J^P(v)$ denote the value of the constrained program (5) – (11) s.t. $W(v) = 0$. By steps 1-3, $J^P(v) \leq J(v)$ for $v < \tilde{U}$ and $J^P(v) = J(v)$ for $v \geq \tilde{U}$. Therefore,

$$m = \Upsilon^-\left(\tilde{U}, \epsilon\right) \leq \frac{J^P(\tilde{U}) - J^P(\tilde{U} - \epsilon)}{\epsilon} \quad (17)$$

for $\epsilon > 0$ arbitrarily small. By step 6, $x^*(v) = \bar{x}$ or $v^{H^*}(v) = \bar{U}_2$, and by the same reasoning as step 5, (9) can bind. Therefore, if $\frac{\tilde{U} + \bar{x}}{\beta} > \bar{U}_2$, then $\frac{J^P(\tilde{U}) - J^P(\tilde{U} - \epsilon)}{\epsilon} = -\pi + (1 - \pi) \Upsilon^-\left(\frac{\tilde{U}}{\beta}, \frac{\epsilon}{\beta}\right) < m$, but this contradicts (17). If $\frac{\tilde{U} + \bar{x}}{\beta} \leq \bar{U}_2$, then $\frac{J^P(\tilde{U}) - J^P(\tilde{U} - \epsilon)}{\epsilon} = \pi \Upsilon^-\left(\frac{\tilde{U} + \bar{x}}{\beta}, \frac{\epsilon}{\beta}\right) + (1 - \pi) \Upsilon^-\left(\frac{\tilde{U}}{\beta}, \frac{\epsilon}{\beta}\right) < m$, but this also contradicts (17). This establishes the second part of the lemma.

Step 8. Consider if $w_2 < \tilde{w}_2$ and imagine if $\tilde{U} = \beta \underline{U}_2$. This implies that (15) holds for $\tilde{U} = \beta \underline{U}_2$. By steps 1-3, $W^*(\beta \underline{U}_2) = W^*(\beta \underline{U}_2 + \epsilon) = 0$ for $\epsilon > 0$ arbitrarily small. By step 6, $x^*(v) = \bar{x}$ or $v^{H^*}(v) = \bar{U}_2$ and (9) may bind for $v = \beta \underline{U}_2$ and $v = \beta \underline{U}_2 + \epsilon$ by the same reasoning as step 5. Since $\frac{\tilde{U} + \bar{x}}{\beta} < \beta \underline{U}_2 \leq \bar{U}_2$, then $\Upsilon^+\left(\tilde{U}, \epsilon\right) = \pi \Upsilon^+\left(\frac{\tilde{U} + \bar{x}}{\beta}, \frac{\epsilon}{\beta}\right) + (1 - \pi) \Upsilon^+\left(\frac{\tilde{U}}{\beta}, \frac{\epsilon}{\beta}\right) = m$, but this contradicts (15). **Q.E.D.**

8.4.3 Proof of Proposition 4

Step 1. The necessity of (6) – (11) follows by definition. The necessity of $W_z^*(v) = 0 \forall z$ if $v \geq \tilde{U}$ follows from Lemma 3. The necessity of $x_z^*(v) = \bar{x}$ or $v_z^{H^*}(v) = \bar{U}_2 \forall z$ follows from step 6 of the proof of Lemma 3.

Step 2. Imagine if $v_z^{W^*}(v) > \tilde{U}$. Perturb the allocation as in step 1 of the proof of Lemma 3. By (14), $\Upsilon^-(v^{W^*}(v), \epsilon) = m$, which by the concavity of $J(\cdot)$ implies $v^{W^*}(v) \leq \tilde{U}$. In order that this perturbation not strictly improve welfare, it is necessary that $\Upsilon^+(v_z^{W^*}(v), \epsilon) = m \forall z$ which by (15) implies a contradiction.

Step 3. Imagine if $\pi(-x_z^*(v) + \beta v_z^{H^*}(v)) + (1 - \pi) \beta v_z^{L^*}(v) > \tilde{U}$. Perturb the allocation as in step 1 of the proof of Lemma 3. By step 1 of the proof of Lemma 3, $\Upsilon^-(v^{P^*}(v), \epsilon) = m$ and which by the concavity of $J(\cdot)$ implies $v^{P^*}(v) \leq \tilde{U}$. In order that this perturbation not strictly improve welfare, it is necessary that

$$\Upsilon^+(\pi(-x_z^*(v) + \beta v_z^{H^*}(v)) + (1 - \pi) \beta v_z^{L^*}(v), \epsilon) = m \forall z$$

which by (15) implies a contradiction.

Step 4. Imagine if (9) does not bind for some z if $v \geq \tilde{U}$. If $x^*(v) < \bar{x}$, consider a

perturbation to $x'_z(v) = x^*(v)$, $v_z^{H'}(v) = \frac{v+x^*(v)}{\beta} < v^{H^*}(v)$, and $v_z^{L'}(v) = \frac{v}{\beta} > v^{L^*}(v) \forall z$. Such a perturbation satisfies (6) – (11) and weakly increases welfare by the concavity of $J(\cdot)$. However, the perturbed allocation is suboptimal by step 6 of the proof of Lemma 3 since $x'_z(v) < \bar{x}$ and $v_z^{H'}(v) < \bar{U}_2$.

Step 5. If instead $x^*(v) = \bar{x}$, denote $v = U'$. If the perturbation of step 4 does not strictly improve welfare, then

$$\Upsilon^+(v, \epsilon) = \Upsilon^+\left(\frac{v + \bar{x}}{\beta}, \epsilon\right) = \Upsilon^+\left(\frac{v}{\beta}, \epsilon\right) \quad (18)$$

for $\epsilon > 0$ arbitrarily low and $v = U'$, and with some abuse of notation, define $\Upsilon^+\left(\frac{v+\bar{x}}{\beta}, \epsilon\right) = \Upsilon^-\left(\frac{v+\bar{x}}{\beta}, \epsilon\right)$ if $\frac{v+\bar{x}}{\beta} = \bar{U}_2$. By steps 1, 5, and 6 of the proof of Lemma 3, $\forall v \in [\tilde{U}, U']$, there exists a solution to (5) – (11) s.t. $x^*(v) = \bar{x}$ for which (9) binds so that $v^{H^*}(v) = \frac{v+\bar{x}}{\beta}$ and $v^{L^*}(v) = \frac{v}{\beta}$. Therefore, $\forall v \in [\tilde{U}, U']$,

$$\Upsilon^+(v, \epsilon) = \pi \Upsilon^+\left(\frac{v + \bar{x}}{\beta}, \frac{\epsilon}{\beta}\right) + (1 - \pi) \Upsilon^+\left(\frac{v}{\beta}, \frac{\epsilon}{\beta}\right). \quad (19)$$

However, if (18) is satisfied for $v = U'$, then it must be satisfied for $v = \frac{U'}{\beta}$ if $\frac{U'}{\beta} \geq \tilde{U}$. This follows from the concavity of $J(\cdot)$ which implies $\Upsilon^+\left(\frac{U'}{\beta}, \epsilon\right) \geq \Upsilon^+\left(\frac{\frac{U'}{\beta} + \bar{x}}{\beta}, \frac{\epsilon}{\beta}\right) \geq \Upsilon^+\left(\frac{U' + \bar{x}}{\beta}, \frac{\epsilon}{\beta}\right) = \Upsilon^+\left(\frac{U'}{\beta}, \epsilon\right)$ and from (19) which implies $\Upsilon^+\left(\frac{\frac{U'}{\beta} + \bar{x}}{\beta}, \frac{\epsilon}{\beta}\right) = \Upsilon^+\left(\frac{U'}{\beta}, \frac{\epsilon}{\beta}\right)$. By forward induction, $\exists N = \{0, 1, 2, \dots\}$ s.t. $\frac{U'}{\beta^N} \in [\tilde{U}, \beta\tilde{U})$ and (18) holds for $v = \frac{U'}{\beta^N}$, and by step 2 of the proof of Lemma 3, $\Upsilon^+\left(\frac{U'}{\beta^N}, \epsilon\right) < m$. However, since $\frac{U'}{\beta^N} < \tilde{U}$, $\Upsilon^+\left(\frac{\frac{U'}{\beta^N}}{\beta}, \frac{\epsilon}{\beta}\right) = m$, and (18) cannot hold for $v = \frac{U'}{\beta^N}$. **Q.E.D.**

8.4.4 Proof of Corollary 1

Step 1. If $v \geq \tilde{U}$, then consider any solution which satisfies the conditions of Proposition 4. Since a perturbation as in step 1 of Lemma 3 satisfies (12), yields a unique solution, and weakly improves welfare, then $\alpha^*(v) \in \Psi(v)$.

Step 2. If $v < \tilde{U}$, consider $w_2 \geq \tilde{w}_2$. Satisfaction of the conditions entails $v^{F^*}(v) = w_2 + \beta v^{W^*}(v) \in [\underline{U}_2, \min\{v, w_2 + \beta\tilde{U}\}]$ and $v^{P^*}(v) = \beta\underline{U}_2 = \tilde{U}$. Suboptimality implies that

$$J(v) > W^*(v) (w_1 + \beta J(v^{W^*}(v))) + (1 - W^*(v)) J(\tilde{U}), \quad (20)$$

for $W^*(v) = \frac{\tilde{U}-v}{\tilde{U}-v^{F^*}(v)}$, but (20) contradicts (14).

Step 3. If $v < \tilde{U}$, consider $w_2 < \tilde{w}_2$. Satisfaction of the conditions entails $v^{F^*}(v) = w_2 + \beta v^{W^*}(v) \in [\underline{U}_2, w_2 + \beta \tilde{U}]$ and $v^{P^*}(v) \in [\beta \underline{U}_2, \tilde{U}]$. Suboptimality and (14) imply that

$$\begin{aligned} J(v) &> W^*(v) (w_1 + \beta J(v^{W^*}(v))) \\ &\quad + (1 - W^*(v)) \left(\pi \bar{x} + \beta \left(\underline{U}_1 + m \left(\frac{\tilde{U} + \pi \bar{x}}{\beta} - \underline{U}_2 \right) \right) \right), \end{aligned} \quad (21)$$

for $W^*(v) = \frac{v^{P^*}(v)-v}{v^{P^*}(v)-v^{F^*}(v)}$ if $v^{F^*}(v) \leq v \leq v^{P^*}(v)$ and $W^*(v) = \frac{v^{F^*}(v)-v}{v^{F^*}(v)-v^{P^*}(v)}$ if $v^{F^*}(v) \geq v \geq v^{P^*}(v)$. However, step 1 and (14) imply that,

$$J(\tilde{U}) = \pi \bar{x} + \beta \left(\underline{U}_1 + m \left(\frac{\tilde{U} + \pi \bar{x}}{\beta} - \underline{U}_2 \right) \right),$$

and since $m = \frac{J(\tilde{U})-\underline{U}_1}{\tilde{U}-\underline{U}_2}$, then by some algebra, $m = -\frac{\pi \bar{x}-w_1}{\pi \bar{x}+w_2}$, and (21) contradicts (14).

Q.E.D.

8.4.5 Proof of Proposition 5

Step 1. By Proposition 4 and Corollary 1, $\exists \alpha^*(v) \in \Psi(v)$ s.t. $W^*(v) > 0$ and $v^{W^*}(v) > \underline{U}_2 \forall v \in (\underline{U}_2, \tilde{U})$. Therefore, if $\Pr \{W_{t+k} = 0 | W_t = 1\} = 0 \forall t$ and $\forall k > 0$, then $\Pr \{v_t \in (\underline{U}_2, \tilde{U})\} = 0$.

Step 2. Consider a solution which satisfies (12). Since (9) binds for $v \geq \tilde{U}$, then $\Pr \{v_{t+1} = v/\beta | v_t = v\} = 1 - \pi > 0$. If $\Pr \{v_t \in (\underline{U}_2, \tilde{U})\} = 0$, then $\Pr \{v_{t-1} \in (\beta \underline{U}_2, \beta \tilde{U})\} = 0$ and $\Pr \{v_{t-2} \in (\beta^2 \underline{U}_2, \beta^2 \tilde{U})\} = 0$ and so on. Therefore, $\Pr \{v_t \notin \{\underline{U}_2, \beta \underline{U}_2, \dots\}\} = 0$. Choose $v_0 \geq v^{\max}$ s.t. $v_0 \notin \{\underline{U}_2, \beta \underline{U}_2, \dots\}$. This yields $\Pr \{W_{t+k} = 0 | W_t = 1\} > 0$. **Q.E.D.**

8.4.6 Proof of Theorem 1

Step 1. If $w_2 \geq \tilde{w}_2$, imagine if $\exists$ a solution to (4) s.t. $\lim_{t \rightarrow \infty} \Pr \{W_t = 0\} > 0$, and consider a potential long run distribution of v . Since $\Pr \{v_{t+1} = \underline{U}_2 | v_t = \underline{U}_2\} = 1$, by step 2 of the proof of Lemma 2, then $\Pr \{v_t = \underline{U}_2\} = 0$ under this long run distribution.

Step 2. If $v \in (\underline{U}_2, \beta \underline{U}_2)$, then from Proposition 4, $v_z^{W^*}(v) \leq \beta \underline{U}_2 \forall z$ and $v_z^{L^*}(v) = \underline{U}_2$

$\forall z$. Therefore,

$$\Pr \{v_{t+1} = \underline{U}_2 | W_t = 0, v_t \in (\underline{U}_2, \beta \underline{U}_2)\} = 1 - \pi, \text{ and} \quad (22)$$

$$\Pr \{v_{t+1} \leq \beta \underline{U}_2 | W_t = 1, v_t \in (\underline{U}_2, \beta \underline{U}_2)\} = 1 \quad (23)$$

under the long run distribution of v .

Step 3. From (22),

$$\Pr \{v_{t+1} = \underline{U}_2\} \geq \Pr \{v_t \in (\underline{U}_2, \beta \underline{U}_2)\} \times \Pr \{W_t = 0 | v_t \in (\underline{U}_2, \beta \underline{U}_2)\} \times (1 - \pi)$$

under the long run distribution. In order that $\Pr \{v_{t+1} = \underline{U}_2\} = 0$, it is necessary that $\Pr \{W_t = 0 | v_t \in (\underline{U}_2, \beta \underline{U}_2)\} = 0$. This is because $\Pr \{v_t \in (\underline{U}_2, \beta \underline{U}_2)\} > 0$ since $\Pr \{v_t = \underline{U}_2\} = 0$ and since Proposition 3 and Lemma 3 imply that $\Pr \{v_t \in [\underline{U}_2, \beta \underline{U}_2)\} = \Pr \{v_t = \underline{U}_2\} + \Pr \{v_t \in (\underline{U}_2, \beta \underline{U}_2)\} > 0$.

Step 4. The fact that $\Pr \{W_t = 1 | v_t \in (\underline{U}_2, \beta \underline{U}_2)\} = 1$ combined with (23) implies $\Pr \{v_{t+1} \in (\underline{U}_2, \beta \underline{U}_2) | v_t \in (\underline{U}_2, \beta \underline{U}_2)\} = 1$, and by forward induction

$$\Pr \{W_k = 1 \ \forall k \geq t + 1 | v_t \in (\underline{U}_2, \beta \underline{U}_2)\} = 1.$$

Since $\Pr \{v_t \in (\underline{U}_2, \beta \underline{U}_2)\} > 0$, then $\Pr \{W_k = 1 \ \forall k \geq t + 1\} = \Pr \{v_{t+1} = \underline{U}_2\} > 0$ which is a contradiction. This establishes the first part of the theorem.

Step 5. If $w_2 < \tilde{w}_2$, by Corollary 1, $\forall v \in (\underline{U}_2, \tilde{U})$, $\exists \alpha^*(v) \in \Psi(v)$ s.t. $W^*(v) = \frac{\tilde{U}-v}{\tilde{U}-\underline{U}_2}$, $v_z^{W^*}(v) = \underline{U}_2$, $x_z^*(v) = \bar{x}$, $v_z^{H^*}(v) = \frac{\tilde{U}+\bar{x}}{\beta}$, and $v_z^{L^*}(v) = \frac{\tilde{U}}{\beta} \ \forall z$.

Step 6. Construct an equilibrium with the property of step 5, and imagine if $\lim_{t \rightarrow \infty} \Pr \{W_t = 0\} > 0$. Then it must be that $\Pr \{v_t = \underline{U}_2\} = 0$ under the long run distribution. However, in such an equilibrium, $\Pr \{v_{t+1} = \underline{U}_2 | v_t \in (\underline{U}_2, \tilde{U})\} = \frac{\tilde{U}-v_t}{\tilde{U}-\underline{U}_2} > 0$ under the long run distribution. By Proposition 3 and Lemma 3, $\Pr \{v_t \in [\underline{U}_2, \tilde{U})\} > 0$. Consequently, $\Pr \{v_t = \underline{U}_2\} > 0$ under the long run distribution. This establishes the second part of the theorem.

Step 7. If $w_2 < \tilde{w}_2$, by Corollary 1, $\forall v \in (\underline{U}_2, \tilde{U})$, $\exists \alpha^*(v) \in \Psi(v)$ s.t. $W^*(v) = \frac{\tilde{U}-v}{\tilde{U}-(\underline{U}_2+\epsilon)}$, $v_z^{W^*}(v) = v^{W^*}(\underline{U}_2 + \epsilon) > \underline{U}_2 + \epsilon$, $x_z^*(v) = \bar{x}$, $v_z^{H^*}(v) = \frac{\tilde{U}+\bar{x}}{\beta}$, and $v_z^{L^*}(v) = \frac{\tilde{U}}{\beta} \ \forall z$ for $\epsilon > 0$ which is arbitrarily small.

Step 8. Construct an equilibrium with the property of step 7, and imagine if $\lim_{t \rightarrow \infty} \Pr \{W_t = 0\} = 0$. Then it must be that $\Pr \{v_t = \underline{U}_2\} > 0$ under the long run distribution. However, in such an equilibrium, $\Pr \{v_{t+1} = \underline{U}_2 | v_t \in (\underline{U}_2, \tilde{U})\} = 0$. More-

over, by Corollary 1, $\Pr \left\{ v_{t+1} = \underline{U}_2 | v_t \in \left[\tilde{U}, \bar{U}_2 \right] \right\} = 0$. Since $U_2(\alpha) \geq v^{\max} > \underline{U}_2$, then $\Pr \{v_t = \underline{U}_2\} = 0$ under the long run distribution. **Q.E.D.**

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