Dividing and Discarding:
A Procedure for Taking Decisions with Non-transferable Utility*

(Preliminary Draft)

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Abstract

We consider a setting in which two players must take a single action. The analysis is done within a private values model in which (i) the players' preferences over actions are private information, (ii) utility is quadratic (non-transferable), (iii) implementation is bayesian and (iv) the welfare criterion is utilitarian. We characterize an optimal monotonic allocation rule. Instead of asking the agents to directly report their types, this allocation can be implemented dynamically. The agents are asked if they are to the left or to the right of the midpoint of the interval of possible types ($eg.1/2$ for the initial interval $[0, 1]$). If both reports agree, the section of the interval which none preferred is discarded and the remaining interval is divided in two parts and the process continued until one agent chooses left and the other right. In that case, the midpoint of this remaining interval is implemented. This implementation can be carried out by a Principal who lacks commitment, implying this process is an optimal communication protocol.

When players interact repeatedly, the first best can be arbitrarily approximated (but never attained) as the agents become patient. We also show that the optimal provision of intertemporal incentives eventually leads to a dictatorial mechanism.

1 Introduction

Many situations of interest require two agents to take a joint action. Examples abound– managers of two different divisions within a firm, parents deciding on some aspects of their children's education, Monetary Union members deciding on monetary policy, parties in a political coalition, and others.

This paper analyzes such types of situations for the case in which the players' preferences over actions are private information and uniformly distributed, utility is non-transferable and the common action to be

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chosen belongs to an interval.[1] The objective is to find an allocation that maximizes the sum of the players ex-ante utility within a Bayesian Implementation setting.

The environment we consider makes the task of finding an optimal joint action hard because the scope for the players to misreport their preferences is very large and the instruments available to induce truthfulness are very limited. We further restrict attention to the case in which the Agent's preferences only depend on their own private information, so that, unlike other papers that analyze decisions in committees, there is no advantage in sharing information to uncover some underlying truth.[2] Note that, again, this assumption lowers the incentives for agents to be truthful about their types.

We first analyze the problem as a mechanism design problem and characterize the optimal mechanism. We show that, in spite of there being a continuum of states (i.e., preferences over actions by the players), the optimal allocation can be implemented by simultaneously asking the players if they are to the left or to the right of the midpoint over the remaining choice set. If they both agree on the side of the coarse partition they prefer, we discard the section of the interval which none preferred and continue dividing the remaining interval in this way until one chooses left and the other right. In that case, the midpoint of the remaining interval is implemented. We therefore name this mechanism the Divide and Discard mechanism (DD for short).

We find the DD appealing for a number of reasons. In spite of the complexity brought up by the lack of side payments, the DD is an extremely simple mechanism. Most importantly, its dynamic implementation resembles many real world situations (such as bilateral trade agreements, firm and union bargaining, and others) in which there are many rounds of negotiations and alternatives are successively discarded until an agreement is reached. The principal or mediator provides a way to mitigate the conflict that arises between the agents' incentives to share their information in order to achieve a better allocation and their fear that if they reveal too much information about their preferred actions the other player may manipulate the allocation to his advantage. This is resolved by having information being released very coarsely at first and only as agents learn that their interests are aligned they gradually refine their reports. Indeed, the DD hints on why contracting parties that negotiate sequentially could commit not to consider choices that were eliminated in previous rounds, i.e., agree to rule out – Hart and Moore (2007) –: as players move along further rounds of negotiation, they can be confident about the alignment of their interests. Therefore, whenever an agreement is reached, it will necessarily deliver an outcome that cannot be Pareto dominated by those that were ruled out.

An additional advantage is that the amount of communication required by the DD is very low. In fact, with the dynamic implementation, the players just need to communicate one byte of information per round. Furthermore, as soon as there is disagreement there is no need to keep conveying more refined information about the players' preferences. In settings for which, in addition to efficiency, a designer is concerned with informational requirements of the candidate mechanisms, such a feature is desirable.[3]

The analysis of the problem in terms of mechanism design presumes that the players, or alternatively a third party – i.e., a Principal – can commit to a mechanism. For many circumstances, this may not be

[1] If the decision is contractible and choices binary, a simple voting mechanism is able to attain the efficient outcome. See, for example, the analysis of enforceable voting by Maggi and Morelli (2006). Alternatively, if transfers are possible and players have quasilinear utility, the problem could be easily solved using the expected externality mechanisms proposed by Arrow (1979), and d'Aspremont and Gerard-Varet (1979).

[2] See, for example, Persico (2004).

[3] For communication requirements of arbitrary social choice rules, see Segal (2006).

a reasonable assumption. Within a firm, for instance, it is not clear that a principal with authority will commit to not overrule the divisions' managers. Therefore, it is important to figure out what is the optimal allocation for the case in which the Principal cannot commit to a mechanism.

We are able to show that the allocation implied by the DD mechanism can be implemented without commitment. Communication now takes the form of cheap talk. Each round the Agent's publicly and simultaneously report to the principal if they are above or below the relevant midpoint. As long as they report to be in the same interval they continue to divide the remaining interval and report which segment they prefer. If they ever report different segments they stop reporting. Once the principal stops receiving reports, she implements the best allocation given her beliefs about the agents' types. It turns out that this leads to the Principal implementing the last midpoint, just as in the DD allocation.

Remarkably, the same expected value can be attained with just one round of cheap talk. The allocation that attains this value was derived by Alonso, Dessein and Matouschek (2007) (ADM from now on).[4] Therefore, in this environment we have as a result not only that the same value can be attained with or without commitment but furthermore, that one round of cheap talk is actually sufficient to attain this value. An advantage of the long cheap talk is that the resulting allocation is renegotiation proof. Instead, with just one round of cheap talk both players could be reporting that their preferred allocation is in the same partition but would not be allowed to divide this partition further into smaller subdivisions in search of a better allocation. Also, as mentioned before, it seems natural that as the Agents learn that their interests are more aligned they are willing to reveal more and more details of their true preferences. For this, long conversations are needed.

Repeated interaction is natural for many environments in which agents must agree on joint actions. For example two division heads in a company or members of a Monetary Union. With this in mind, in Section 5 we study the infinitely repeated version of the stage game. Now, even though players cannot transfer money between them, they can transfer continuation utilities. This allows for better per period allocations and actually as players become very patient, the best per period expected values approach first best. Intuitively, as players become more patient the inefficiencies that arise because it is utility and not money that is being transferred start disappearing. The transfer of continuation utilities becomes closer and closer to a monetary transfer with quasilinear preferences.

Higher continuation values for an agent are delivered by having his preferences have more weight on the determination of future allocations. Continuation values are used to provide incentives and vary from period to period. They tend to increase for the agent that reports a less extreme type, and this asymptotically leads to one of the agents becoming a dictator. That is, eventually only the preferences of one agent are taken into account to determine allocations. Dictatorship is incentive compatible hence there is no need to vary continuation values to implement it. This implies it is an absorbing state. Therefore, once an agent gets all the decision rights he will have them for ever.

In the next section we introduce the model. The optimal mechanism is characterized in Section 3. Section 4 describes the dynamic implementation of the optimal mechanism. We show it is implementable even if the principal lacks commitment and discuss the use of long versus one round of cheap talk. In Section 5, we present the infinitely repeated version of the model, show that one can approach (but never attain) the efficient allocation when the players become arbitrarily patient, and that the provision of intertemporal

[4] Their very interesting paper focuses mainly on the issue of centralized vs. decentralized decision making. In both cases, they consider only decision making with no commitment and one round of communication.

incentives leads to a dictatorial allocation. All proofs are relegated to the Appendix.

2 The Model

We consider a setting in which two ex-ante symmetric players, $i = 1, 2$, have to take a joint action a. Player $i's$ type is determined by his favorite action, θ_i, which is uniformly distributed over $U[0,1]$. Types are independent and privately known by the players. Their (Bernoulli) utility function is[5]

$$u_i(a, \theta_i) = -(a - \theta_i)^2.$$

The objective is to maximize the ex-ante sum of their utilities

$$\sum_i E[u_i(a, \theta_i)]$$

by choosing an action schedule (an enforceable contract) that maps the players' reported types $\tilde{\theta}$, and an independent randomization device $x \sim U[0,1]$ into an action:[6]

$$a\left(\tilde{\theta}_i, \tilde{\theta}_{-i}; x\right) : [0,1]^2 \times [0,1] \to \Re.$$

As mentioned in the introduction we do not allow for transfers. This greatly increases the difficulty of the problem. In particular, since the set of IC schedules is not convex, we cannot rely on Lagrangian Theorems to find the optimal contract.

3 The Optimal Mechanism:

Given the difficulties outlined above instead of trying to characterize the whole allocation directly in one step we proceed by deriving properties of an optimal allocation. In turn, this then allows us to completely describe the optimal mechanism. Before proceeding in this direction, it is convenient to recast the problem in term of the agents' virtual utilities.[7]

Preliminaries

The objective is to find the incentive compatible allocation rule $a(\theta; x)$ that solves the following problem:

$$\begin{aligned} & \max_{a(\theta;x)} E_{\theta,x}\left(u(\theta_i, a) + u(\theta_{-i}, a)\right) \\ s.t.\theta_i \quad \in \quad & \arg\max_{\tilde{\theta}_i} E_{\theta_{-i},x}\left[-\left(a(\tilde{\theta}_i, \theta_{-i}; x) - \theta_i\right)^2\right] \ \forall i \end{aligned}$$

In order to get the virtual valuation representation, we start by making use of the following standard result:

[5] The results can be extended to the case where utilities are single peaked and the loss from not taking the preferred action is an increasing function of the distance to the peak.

[6] The randomization device is simply used to preserve the symmetry in the allocation. There is no need for randomization device from an efficiency standpoint.

[7] See Myerson (1981) and Myerson's notes on virtual utility at http://home.uchicago.edu/~rmyerson/research/virtual.pdf

Lemma 1 (IC Representation) *Letting*

$$U(\theta_i, a(\theta; x)) = \max_{\tilde{\theta}} E_{\theta_{-i}} \left[-\left(a(\tilde{\theta}, \theta_{-i}; x) - \theta_i \right)^2 \right] = E_{\theta_{-i}} \left[-\left(a(\theta_i, \theta_{-i}; x) - \theta_i \right)^2 \right],$$

Incentive Compatibility is equivalent to:

$$U(\theta_i, a(\theta; x)) = \begin{cases} U(\theta', a(\theta; x)) + 2 \int_{\theta'}^{\theta_i} E_{\theta_{-i}, x} \left[a(\tau, \theta_{-i}; x) - \tau \right] d\tau, & \text{if } \theta_i > \theta' \\ U(\theta', a(\theta; x)) - 2 \int_{\theta_i}^{\theta'} E_{\theta_{-i}, x} \left[a(\tau, \theta_{-i}; x) - \tau \right] d\tau, & \text{if } \theta_i < \theta' \end{cases},$$

where $\theta' \in [0, 1]$, *and* $E_{\theta_{-i}, x} \left[a(\theta_i, \theta_{-i}; x) \right]$ *non-decreasing in* θ_i.

Proof. The proof follows from Milgrom and Segal (2002) and is detailed in the Appendix. ■

Incentive compatibility only requires that allocations be non-decreasing in expectation. We restrict a bit further the set feasible allocations by considering only schedules that are non-decreasing, that is, schedules that satisfy:

$$a(\theta_i, \theta_{-i}; x) \text{ non-decreasing in } \theta_i \text{ for all } \theta_{-i}. \qquad \text{(Monotonicity)}$$

Beyond the fact that this condition facilitates our analysis in this section it also allows us to directly connect our results with those of optimal allocations with no commitment (see Section 4.2). When the planner cannot commit he will always choose an allocation that is monotonic on his beliefs of the player's types.[8]

After integrating by parts the representation of $U(\theta_i, a(\theta; x))$ induced by Lemma (1), one can recast our program of interest as:

$$\max_{a(\theta; x)} \sum_{i=1}^{2} \left\{ \begin{array}{c} -E_{\theta_{-i}} \left[a\left(\theta', \theta_{-i}; x\right) - \theta' \right]^2 \\ +2 \Pr(\theta_i > \theta') E_{\theta} \left[\left(a(\theta; x) - \theta_i \right) (1 - \theta_i) | \theta_i > \theta' \right] \\ -2 \Pr(\theta_i < \theta') E_{\theta} \left[\left(a(\theta; x) - \theta_i \right) \theta_i | \theta_i < \theta' \right] \end{array} \right\}, \qquad (1)$$

$$\begin{aligned} &s.t.\ a(\theta; x) \text{ being Incentive Compatible} \\ &\text{and satisfying Monotonicity} \qquad (2) \end{aligned}$$

where $\theta' \in [0, 1]$ can be chosen arbitrarily.

We pause here to make a subtle, but important, point. In the program above, in spite of using the incentive compatible representation of the players' payoffs we still impose the constraint that the schedule must be Incentive Compatible. Such constraint is not redundant. In fact, one can evaluate the incentive compatible representation of the player's utility at schedules which are *not* incentive compatible. Even though it is possible to evaluate the above representation at a non-IC schedule, it carries no relevant meaning in such a case. Only at incentive compatible schedules the above representation is equivalent to the player's expected utility. In settings with quasilinear preferences and side payments any allocation schedule can be made IC by a proper choice of a transfer function. Therefore, in such cases, it is correct to proceed by simply maximizing pointwise the Incentive Compatible representation of the players' utility, as it can always be made equivalent to the players' expected utility by choosing the right side payments. The fact that players cannot make side

[8] Note that although we restrict our attention to monotonic allocations, we are quite general in the sense of not requiring the allocation to be differentiable nor continuous.

payments forces us to consider the Incentive Compatibility Constraint explicitly, which, in turn, makes the problem fairly hard.[9]

The Optimal Monotonic Mechanism

The symmetry of the problem makes it natural to pick $\theta' = \frac{1}{2}$ as the reference type in the optimization problem. Using this together with the Law of Iterated Expectations, we can, ignoring constant terms, write the objective functional in (1) as:

$$\left(\begin{array}{c} \underbrace{\sum_i \left[-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x<\frac{1}{2}\right]-\frac{1}{2}E_{\theta,x}\left[a\left(\theta;x\right)\left(\theta_i\right)|\theta_i,\theta_{-i}<\frac{1}{2}\right]\right]}_{A} \\ \underbrace{\sum_i \left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x<\frac{1}{2}\right]+\frac{1}{2}E_{\theta,x}\left[a\left(\theta;x\right)\left(1-\theta_i\right)|\theta_i,\theta_{-i}>\frac{1}{2}\right]\right]}_{B} \\ \underbrace{\sum_i \frac{1}{2}\left[E_{\theta,x}\left[\left[a\left(\theta_i,\theta_{-i};x\right)\right]\left(1-\theta_i\right)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]-E_{\theta,x}\left[\left[a\left(\theta_i,\theta_{-i};x\right)\right]\theta_i|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right]\right]}_{C} \\ \underbrace{\sum_i \left[-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x>\frac{1}{2}\right]-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x>\frac{1}{2}\right]\right]}_{D} \end{array}\right), \tag{3}$$

Now, consider the terms C and D.

For any non-decreasing schedule $a(\theta_i, \theta_{-i})$, one has that

$$\begin{aligned} & E_{\theta,x}\left[a\left(\theta_i,\theta_{-i};x\right)\left(1-\theta_i\right)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right] \\ \leq\ & E_{\theta,x}\left[a\left(\theta_i,\theta_{-i};x\right)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]E_{\theta}\left[\left(1-\theta_i\right)|\theta_i>\frac{1}{2}\right], \end{aligned}$$

for the expected value of the product of a non-decreasing function and a decreasing function is no larger than the product of the expected values.

A similar reasoning can be applied to the other term in C, so it follows that

$$\begin{aligned} & -E_{\theta,x}\left[\left[a\left(\theta_i,\theta_{-i};x\right)-\theta_i\right]\theta_i|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right] \\ \leq\ & -E_{\theta,x}\left[a\left(\theta_i,\theta_{-i};x\right)|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right]E_{\theta}\left[\theta_i|\theta_i\leq\frac{1}{2}\right]. \end{aligned}$$

This discussion suggests that the term C – that is associated with the region in which $\theta_i < 1/2 < \theta_{-i}$ – would be maximized by the choice of a constant action. Note that a constant action of $\frac{1}{2}$ in this region also maximizes the term D in the objective. From the work in social choice theory by Moulin (1980) we know that if we required instead ex-post incentive compatibility the optimal allocation would also have 1/2 of diagonals.[10]

[9] The set of IC schedules is not convex so we cannot directly rely on Lagrangian methods to find the optimal scheme.

[10] See also Barberà and Jackson (1994), and Barberà (2001) for detailed discussions of strategy-proof social choice functions.

Nonetheless, once we consider Bayesian implementation, it is not obvious that it is optimal to set 1/2 as the allocation in these regions. We could expect that by perturbing the allocation slightly in these off-diagonal regions one could improve the attainable values on the on-diagonals $\left(\left(\frac{1}{2},1\right)^2 \text{ and } \left(0,\frac{1}{2}\right)^2\right)$ once incentive constraints are taken explicitly into account. Suppose we were to carry out such a perturbation in the region where $\theta_i < \frac{1}{2} < \theta_{-i}$. Note that we start from $a(\theta) = \frac{1}{2} < \theta_{-i}$ hence, if we were to make the allocation strictly increasing in θ_{-i} in this region player $-i$ would have more incentives to claim his type is higher than it actually is. This would not help us bring the allocation in $\left(\frac{1}{2},1\right)^2$ any closer to first best since the problem with the first best allocation is exactly that types in this region would want to pretend they are higher than they actually are. Therefore, within the class of weakly increasing allocation rules it is best to set a constant $\left(a(\theta) = \frac{1}{2}\right)$ on the off-diagonals. The following lemma establishes this formally.

Lemma 2 (1/2 off-diagonals) *Given any symmetric incentive compatible allocation $a(\theta;x)$ that satisfies Monotonicity we can find an alternative incentive compatible allocation $\tilde{a}(\theta;x)$ which is weakly better and satisfies $\tilde{a}(\theta;x) = \frac{1}{2}$ for all x, when $\theta_i > \frac{1}{2} > \theta_{-i}$, and $\tilde{a}\left(\frac{1}{2},\theta_{-i};x\right) = \frac{1}{2}$ when $x > \frac{1}{2}$.*

This is a very powerful result towards the full characterization of the optimal allocation. Once one knows that setting $\frac{1}{2}$ off-diagonals is optimal – so that this region plays no role in terms of providing incentives over the main diagonal – the problem is separable. The following result states this in a precise way.

Lemma 3 (Separability) *Let $a^*(\theta,x)$ be an allocation that solves the program of interest. If $a^*(\theta;x) = \frac{1}{2}$ for all $\theta \in \left[0,\frac{1}{2}\right) \times \left(\frac{1}{2},1\right]$, and $a^*\left(\frac{1}{2},\theta_{-i},x\right) = \frac{1}{2}$ for all θ_{-i} whenever $x > \frac{1}{2}$ then:*

$$\begin{aligned} \textit{for } \theta &\in (0,1/2)^2,\ a^*(\theta,x) \in \arg\max_{a()} \sum_i E\left[u_i(a,\theta_i)\,|\theta \in (0,1/2)^2\right] \\ \textit{s.t. IC for } i &= 1,2 \textit{ given } \theta_{-i} \in (0,1/2) \textit{ and monotonicity} \end{aligned}$$

and

$$\begin{aligned} \textit{for } \theta &\in (1/2,1)^2,\ a^*(\theta,x) \in \arg\max_{a()} \sum_i E\left[u_i(a,\theta_i)\,|\theta \in (1/2,1)^2\right] \\ \textit{s.t. IC for } i &= 1,2 \textit{ given } \theta_{-i} \in (1/2,1) \textit{ and monotonicity} \end{aligned}$$

Furthermore, the problem over $\left[0,\frac{1}{2}\right]^2$,and respectively $\left[\frac{1}{2},1\right]^2$ is, subject to rescaling, exactly the same as the original problem (the problem over $[0,1]^2$). So we can sequentially apply appropriately rescaled versions of Lemmas (2) and (3) to those regions.

The DD Allocation

Consider the self-similar allocation which results from the following algorithm:

(1) Divide the set of types into four quadrants using the midpoint for each type to define the quadrants.

(2) Set the allocation in the top-left and bottom-right quadrants to the midpoint used to define the quadrants.

(3) Restart the process from step (1) for the top-right and bottom left quadrants.

The limiting allocation can alternatively be described as requiring the Agents to report their type, doing a binary decomposition of their reports and then choosing the point that corresponds to the first digit for which they differ. More precisely, for almost all θ_i we can redefine the type to be k_i as follows:

$$\theta_i = k_i b$$

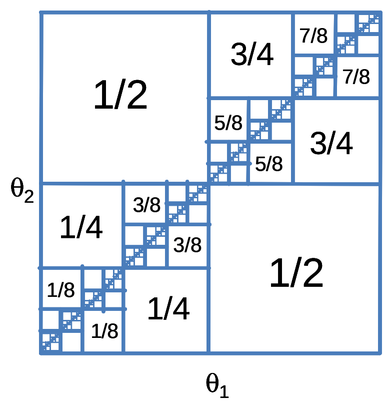
where k_i is an infinite row vector of zeros and ones

$$k_i = \begin{bmatrix} k_i^1 \\ k_i^2 \\ \dots \\ k_i^n \\ \dots \end{bmatrix} \text{and } b = \begin{bmatrix} 1/2 \\ 1/4 \\ \dots \\ (1/2)^n \\ \dots \end{bmatrix}$$

Now $a^{DD}\left(\tilde{\theta}_i, \tilde{\theta}_{-i}\right) = \bar{k}b$. Where $\bar{k}$ is a row vector for which:

$$\bar{k}^n = \begin{cases} k_i^n & if\ \forall\ m \leq n\ \ \tilde{k}_i^m = \tilde{k}_{-i}^m \\ 1 & if\ \forall\ m < n\ \ \tilde{k}_i^m = \tilde{k}_{-i}^m\ and\ \tilde{k}_i^n \neq \tilde{k}_{-i}^n \\ 0 & otherwise \end{cases}$$

The resulting allocation from this mechanism is graphed below.



The DD Allocation

Note that, given the other player's report, by misreporting the agent can only move the allocation further away from his preferred action.[11]

Given the properties of the DD allocation and the implications of Lemmas (2) and (3) for the characterization of an optimal, we have our main result:

Theorem 1 *The DD allocation is optimal in the class of non-decreasing Incentive Compatible allocations.*

[11] See Proposition (1) for a more formal treatment.

4 Dynamic Implementation, Lack of Commitment and Long Cheap Talk

4.1 The DD Mechanism: Dynamic Implementation

As a principal, consider proceeding in the following way. Instead of asking the agents to report their type, start by simply asking agents to simultaneously report if they prefer an allocation below or above the midpoint of the interval of possible types ($\frac{1}{2}$ for the initial interval $[0, 1]$). If their reports fall on different sides of the midpoint, then implement the midpoint. If they both report to be on the same side of the midpoint then restart the process considering only the interval they both preferred. For example the $[0, \frac{1}{2}]$ interval if they both reported "below" or the $[\frac{1}{2}, 1]$ interval if they reported "above". Iterate this process until agents eventually will report to be on different sides of the relevant midpoint.[12] Also, if at any point a player is indifferent between reporting either "below" or "above" we assume he flips a fair coin to decide. In terms of the direct mechanism, this was essentially the role played by the random device x. *The use of x to determine the allocation is purely to preserve symmetry since we could use the other player's report to generate any randomization needed.*

Proposition 1 (Dynamic IC) *The dynamic implementation of the DD allocation is Incentive Compatible.*

The proof is in the Appendix but the result is straightforward. At each stage, reporting the truth on average brings the allocation closer to the Agent's preferred allocation. The only types who are indifferent are the cutoff types.

A natural interpretation of this mechanism is that decisions are taken in rounds, and at each round a mediator asks the players whether the action should be above or below a certain threshold. At a stage in which the players disagree, the action is set to be equal to that round's threshold. Otherwise, negotiations move on and another $\{above, below\}$ question is made by the mediator. Given the property that the interval of possible allocations is divided into two at each stage and the segment not chosen is discarded we refer to it as the Divide and Discard Mechanism –DD for short. The DD captures in a simple way the property that negotiations often take place in rounds, and choices are successfully eliminated over time until an agreement is reached.

The principal or mediator provides a way to resolve the conflict that arises between the agents' incentive to share their information in order to achieve a better allocation and their fear that if they reveal too much information about their preferred actions the other player may misreport in order to manipulate the allocation in his advantage. This is resolved by having information being released very coarsely at first and as agents learn that their interests are aligned they gradually refine their reports.

An additional advantage of this feature is that the amount of communication required by the DD is very low. In the dynamic implementation, at each point in time the players only need to communicate one byte of information, so that in each round two bytes of information are communicated.[13]

[12] With zero probability the types are the same. In that case, the procedure described above would never stop, then simply take the limiting point as the allocation.

[13] For a discussion of communication requirements in mechanisms, see Segal (2005).

4.2 Optimal Allocations with no Commitment

In many instances allowing the Principal to commit to a given mechanism strictly improves the efficiency of the resulting allocations. This is the case because, without commitment, once the principal knows the agent's type she might wish to implement something different than originally promised. In this environment for example if both players report their types truthfully the principal would then want to implement the first best allocation $a = \frac{\theta_i + \theta_{-i}}{2}$. Of course, if this were expected, the agents would not report truthfully in the first place. Hence, with no commitment, we would not be to implement the DD allocation simply by having both agents report their types to the principal. She would deviate and implement the average instead. Nonetheless, we can still implement the DD allocation. The key for this is that, instead of revealing their types fully in one round of communication, the agents must reveal it step by step, as characterized in the previous section. Revealing information very coarsely at each stage eliminates the principal's temptation to deviate.

Every round, both players publicly and simultaneously reveal if they are above or below the midpoint of the remaining interval. As long as they reported the same in the last round they continue reporting. The new midpoint now being the one corresponding to the smaller segment to which both reported they belong. If at some point the players announce they belong to different segments of the remaining interval they stop communicating. Once the agents know that they are on opposite sides of the midpoint, there is no way for the principle to extract any more valuable information from them, even with commitment.[14] Given then, that all the Principal knows is that the players are in different sides of a given midpoint, his optimal choice for an allocation is the midpoint itself. This establishes the following result:

Proposition 2 *The DD allocation can be implemented even when the Principal cannot commit to an allocation.*

4.2.1 Short and Long Cheap Talk

Even more remarkable is the fact that actually the same expected value that is attained with the DD allocation with many rounds of communication can be attained with just one round of cheap talk. In recent work ADM studied the optimal allocation without commitment restricting their analysis to only one round of cheap talk by the Agents. As they acknowledge:

> "It is well known in the literature on cheap talk games that repeated rounds of communication may expand the set of equilibrium outcomes even if only one player is informed. However, even for a simple cheap talk game such as the leading example in Crawford and Sobel (1982), it is still an open question as to what is the optimal communication protocol."

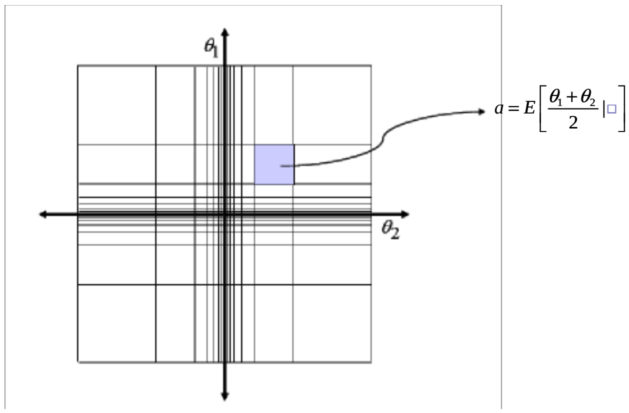
Surprisingly, the value attained with the allocation they characterize (for the extreme case that corresponds to our model) is exactly the same as the one we attain with the DD allocation. This implies that, in this environment, one round of communication is actually sufficient.[15] This implies that the results presented

[14] Even if the principal had commitment the best he can do when the agents are on different sides is to implement the midpoint. This is actually neccessary for Lemma (2) to be true.

[15] See for example Aumann and Hart (2003) or Krishna and Morgan (2004) for more on the potential benefits of long cheap talk.

in ADM are actually much stronger since the limitation of their analysis to one round of communication is actually of no consequence in terms of ex-ante payoffs.

Nonetheless, we believe that the dynamic implementation of the DD allocation with several rounds of communication has some additional appealing features. Before making our case it is useful to recall what the allocation characterized by ADM looks like. Essentially, *the interval of types is partitioned, each agent reports the element of the partition to which his favorite action belongs,* and then the Principal implements as an allocation the average type given the reported rectangle. Partitions are very fine close to the middle of the interval since incentive constraints are not very binding for those types and progressively become coarser towards the extremes. Below we replicate Figure 2 from their paper.



ADM Allocation

The first important thing to note is that the DD allocation is renegotiation proof, while the ADM allocation is not. For example, if both players report to be in the shaded area in the figure above they would have a strong incentive to communicate further. Essentially, they are facing the same situation they were facing originally albeit within a smaller range. Instead, with the DD allocation as long as agents report to be in the same quadrant they would keep on refining their reports until it is clear their interests are in conflict. This happens when there is a value (the midpoint) that objectively separates both types.

Second, the amount of communication that needed to be transmitted to implement the DD allocation is much lower than the one need for the ADM allocation. Lastly, although ADM provide a very simple difference equation to characterize their allocation we find the simplicity of the DD mechanism very appealing.

5 The Infinitely Repeated Game

Repetition is key in many of the settings in which agents have to agree on a joint action. It is natural then to ask if repetition allows for better per period allocations, and what are the properties of the best allocation rule. Similar questions have been risen in the social choice literature. For example, Jackson and Sonnenschein (2006) have shown in a more general setting that the inefficiencies arising from incentive compatibility considerations become arbitrary small when there are a large number of independent decisions to be taken simultaneously. This is established by defining a mechanism in which agents must budget their representations of preferences so that the frequency of preferences across problems mirrors the underlying distribution of preferences, and then arguing that agents' incentives are to satisfy their budget by being as truthful as possible. In a related paper Casella (2005) consider a repeated setting in which a binary choice must be taken each period and agents can different intensity of preferences then given them a number of votes to use across decisions increases efficiency since the agents in equilibrium use their votes mainly when they feel strongly about an issue.

Our approach to the analysis of the optimal allocation rule in the repeated game relies on the factorization results of Abreu, Pearce and Stacchetti (1990) which split the agent's payoff in a current value and a continuation value. The continuation values of an agent play the same role as the number of remaining votes in Casella (2005) since we show that agents with larger continuation values get more future decision power. Furthermore, we show that in the limit one of the agents will eventually "run out of votes" which implies that the other agent essentially dictates all future allocations. This result is similar to the immiseration results found in many dynamic insurance problems, see for example Thomas and Worral (1990). An agent that reports to have an extreme type in a given period is like an agent that reports to have a low income realization, the optimal mechanism will respond by giving that agent more weight in the current allocation decision (similarly a higher transfer today) and in order to preserve incentives he "pays" for this by forgoing future weight in the allocation decision (future consumption). Our environment has a few differences with that of Thomas and Worral (1990). First, the problem is symmetric, we have two agents rather than a principal and an agent. Second, dictatorship is an absorbing state in the sense that once an agent is granted all the decision rights incentive constraints will not bind any longer and the continuation values will be constant rather than constantly drifting towards $-\infty$ (*immiseration*).

The other main result we establish in this section is that as the discount factor δ goes towards 1 we can attain values arbitrary close to those achievable with the first best allocation. What happens is that as $\delta \to 1$ the value of the current period which is weighted by $(1-\delta)$ becomes insignificant relative to the continuation values. Hence, in order to guarantee truth-telling in the current period continuation values have to vary only minimally. Since even without transfers the utility possibility frontier is locally linear, this implies that the associated losses from the variation in continuation values become negligible. Hence, we have a similar result to Jackson and Sonnenschein (2006) Corollary 2 in which they prove that for any $\varepsilon > 0$ there is a δ large enough and finite but large number of periods such that there is less than ε inefficiency relative to the first best. The sources of the inefficiency are quite different though. In their setting, what happens when you get close to the last periods is that agents are not allowed to report truthfully since they might have run out of their budgeted reports for a particular type. In particular both agents could have the same type yet the allocation be different. Instead, in our setting the inefficiency arrises from two sources. On one hand we slowly drift towards one of the agents becoming a dictator and on the other relative to the first best the allocation is biased towards the center in order to provide incentives.

5.0.2 The Setup:

Consider repeating the game described in Section 2 an infinite number of periods. At each period $t \in \{0, 1, ...\}$ the two agents receive new i.i.d. preference shocks θ_i drawn from a uniform distribution over $[0,1]$. After they observe their preference shocks, they make reports $\tilde{\theta}_1$ and $\tilde{\theta}_2$. Given the reports and the history of the game denoted by h^t and defined below, a history dependent allocation is chosen according to a sequence of functions

$$\left\{a_t\left(\tilde{\theta}_i, \tilde{\theta}_{-i}, x^t, h^{t-1}\right) : [0,1]^3 \times [0,1]^{3(t-1)} \to [0,1]\right\}_{t=1}^{\infty}.$$

This allocation rule is chosen a priori before the agents learn their preference shocks.

Public Histories, Public Strategies and Equilibrium Payoffs

The public history at time t, h^t, is a sequence of (i) past announcements of the two players, (ii) past realized actions, and (iii) the realizations of a public random device x that serves convexification purposes:

$$h^t = \{\varnothing, \left(\tilde{\theta}_1^1, \tilde{\theta}_2^1, a^1, x^1\right), ..., \left(\tilde{\theta}_1^{t-1}, \tilde{\theta}_2^{t-1}, a^{t-1}, x^{t-1}\right)\}.$$

Let H^t be the set of all public histories of length t. A public strategy for player i is a sequence of functions $\{\tilde{\theta}_i^t(.,.)\}_{t,}$ where

$$\tilde{\theta}_i^t : H^t \times [0,1] \to [0,1] .$$

Each pair of strategies $\tilde{\theta} = \left(\left\{\tilde{\theta}_1^t\right\}, \left\{\tilde{\theta}_2^t\right\}\right)$ defines a probability distribution over public histories and, as consequence, player i's expected payoff is given by:

$$E\left[(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i^t(a(\tilde{\theta}^t);\theta_i^t) \mid \widehat{\theta}\right].$$

We analyze this game using the recursive methods developed by Abreu, Pearce and Stacchetti (1990). More specifically, letting $W \subset \Re^2$ be the set of Public Pure Strategy Equilibria Payoffs for the Players[16], we can decompose the payoff into a current utility $u_i(a,\theta)$ and a continuation value $v_i(\tilde{\theta}) \in W$,

$$E_\theta[(1-\delta)u_i(a,\theta_i) + \delta v_i(\tilde{\theta}_i, \tilde{\theta}_{-i})],$$

In other words, any PPS equilibrium can be summarized by the actions to be taken in the current period and equilibrium continuation values as function of the announcements.

We can use this decomposition to write the Bellman equation that characterizes the frontier of equilibrium values that can be attained in this environment. Let $V(v)$ be the highest value to player 1 given that player 2 expected value is v. Clearly, the expected value for a player that always picks his favorite allocation is zero. Now, define v^s as the expected value for a player when the other one always chooses the allocation. Simple calculations show that $v^s = -\frac{1}{6}$.

We can write $V(v)$ as:

$$V(v) = \max_{\{a(\theta,x),w(\theta,x)\}_\theta} -E_\theta\left[(1-\delta)\left(a(\theta,x)-\theta_1\right)^2 + \delta V\left(w(\theta,x)\right)\right]$$

s.t.

$$-E_\theta\left[(1-\delta)\left(a(\theta,x)-\theta_2\right)^2 + \delta w(\theta,x)\right] = v \qquad \text{(Promise Keeping)}$$

$$-(1-\delta)2E\left(\left[\left(a(\theta,x)-\theta_2\right)\frac{da(\theta,x)}{d\theta_2}|\theta_2\right] + \delta\frac{d}{d\theta_2}\left(E\left[w(\theta,x)\right]|\theta_2\right)\right) = 0. \qquad \text{(IC Local)}$$

$$-(1-\delta)2E\left(\left[\left(a(\theta,x)-\theta_1\right)\frac{da(\theta,x)}{d\theta_1}|\theta_1\right] + \delta\frac{d}{d\theta_1}\left(E\left[V\left(w(\theta,x)\right)|\theta_1\right]\right)\right) = 0 \qquad \text{(IC Local)}$$

$$\left(V\left(w(\theta)\right), w(\theta)\right) \in W \ for\ all\ \theta, \qquad \text{(Feasibility)}$$

$$E_{\theta_{-i}}\left[a\left(\theta_i,\theta_{-i}\right)\right]\ is\ non-decreasing \qquad \text{(Expected Monotonicity)}$$

[16] This set is not empty as the infinite repetition of the actions taken in the static game constitutes a PPS equilibrium, and there are plenty of equilibria in the stage game.

Incentive compatibility is substituted by the first order conditions that capture local incentive compatibility together with expected monotonicity which, together, are sufficient to guarantee global incentive compatibility.

Properties of the Optimal Allocation Rules

When Agents interact repeatedly, continuation values can play a similar role to the one side payments play in standard incentive problems. The difference between side payments and continuation values is that the latter can only imperfectly transfer utility across players. In particular, to transfer continuation utility from player 1 to player 2 in any period t, *allocations* for periods $\tau > t$ must be altered. When $\delta \to 1$ what happens is that the value of the current period which is weighted by $(1-\delta)$ becomes insignificant relative to the continuation values. Hence, in order to guarantee truth-telling in the current period continuation values have to vary only minimally. Since locally $V(v)$ is linear, this implies that the associated losses from the variation in continuation values becomes negligible.

Proposition 3 *As $\delta \to 1$, the first best payoffs can be arbitrarily approximated by PPE payoffs.*

In practice, higher continuation values for an Agent are delivered by having his preferences have more weight on the determination of the future allocations. In general, the continuation values are used to provide incentives and vary from period to period. Continuation values tend to increase for the Agent that reports a less extreme type. This leads to the following result: asymptotically, one of the Agents becomes a dictator, that is, eventually only his preferences are taken into account to determine the current and future allocations.

Proposition 4 (Dictatorship in the limit) *The provision of intertemporal incentives necessarily leads to a dictatorial mechanism: In the limit as $t \to \infty$ either v or $V(v)$ converge to 0 almost surely.*

The proof is detailed in the appendix but the main ingredients are as follows. In solving for the optimal allocation rule and continuation values one of the conditions for optimality is:

$$E_\theta \left[V'(w(\theta))\right] = V'(v),$$

so that the marginal value for player 1 follows a non-positive martingale.

The optimal provision of incentives also requires that, for any given $w \in \left(-\frac{1}{6}, 0\right)$, there is a positive mass of types for which $w(\theta) > v$, and another for which $w(\theta) < v$.[17]. This implies that, with probability 1, eventually either

$$w(\theta) \to 0, \text{ or } V(w(\theta)) \to 0.$$

Whenever an Agent is promised continuation values of 0, it must be the case that his most favorite action is taken from then on.

This result is similar to the immiseration results found in many dynamic insurance problems, see for example Thomas and Worral (1990). The difference with the hidden income models is that instead of having incentive to claim to be poorer than one actually is, here the incentives are to claim one is more extreme than

[17] As a side remark, we point here that it is optimal to provide continuation values that not only depend on the realization of types but also depend on the identity of the players. Indeed, it can be easily shown that the best equilibrium for the case in which continuation values are symmetric across players' identity replicates the DD mechanism at every period irrespective of the discount factor.

one actually is. To restore incentive compatibility, extreme types lose future decision rights and eventually the point is reached where they have no say at all on the future allocations. Endogenous participation constraints as those considered in the Monetary Union model of Fuchs and Lippi (2005), which we abstract from in the present setting, would provide a countervailing force against one of the agents becoming too powerful.

6 Concluding Remarks

This paper considered a setting in which two agents have to take a commonly agreed action for the case in which the players' preferences over actions are private information, there are no transfers, the action to be chosen rather than being binary belongs to an interval, and the welfare criterion is utilitarian.

The main results are as follows. When Agents interact once, we have shown that the optimal allocation (which we label the DD allocation) can be implemented by simultaneously asking the players if they are to the left or to the right of the midpoint over the remaining choice set. If they both agree on the side of the coarse partition they prefer, we discard the section of the interval which none preferred and continue dividing the remaining interval in this way until one chooses left and the other right. In that case, the midpoint of the remaining interval is implemented. The DD allocation can also be implemented by a Principal without commitment, and, surprisingly, yields the same expected value as the one attained with just one round of cheap talk between the Agents, as in the mechanism in ADM.0

When players interact repeatedly, first best can be approximated (but not attained) when players get sufficiently patient. The variation in continuation values that allows to transfer utilities across Agents, and that yields improvements over the one-shot interaction has a surprising implication: the optimal scheme necessarily converges to a dictatorial mechanism. In other words, in the long-run, only the preferences of one of the players are taken into account to determine allocations.

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7 Appendix

Appendix A: Preliminary Results

In this appendix, we prove some of the preliminary results we will need to prove the optimality of the DD.

Proof Lemma (IC Representation)

Proof. The proof is standard. For necessity, just notice that the integral formula is implied by Milgrom and Segal's (2002) Envelope Theorem. The monotonicity condition follows because, if $a(\theta)$ is Incentive Compatible, for any $\theta' > \theta''$, one must have

$$E_{\theta_{-i}}\left[-\left(a(\theta',\theta_{-i})-\theta'\right)^2\right] \geq E_{\theta_{-i}}\left[-\left(a(\theta'',\theta_{-i})-\theta'\right)^2\right] \tag{IC$_{\theta'\theta''}$}$$

and

$$E_{\theta_{-i}}\left[-\left(a(\theta'',\theta_{-i})-\theta''\right)^2\right] \geq E_{\theta_{-i}}\left[-\left(a(\theta',\theta_{-i})-\theta''\right)^2\right] \tag{IC$_{\theta''\theta'}$}$$

Combining both expressions one has, after a few algebraic manipulations, that

$$2\left[\theta'-\theta''\right]\left\{E_{\theta_{-i}}\left[a(\theta',\theta_{-i})\right]-E_{\theta_{-i}}\left[a(\theta'',\theta_{-i})\right]\right\} \geq 0$$

which implies that

$$E_{\theta_{-i}}\left[a(\theta',\theta_{-i})\right] \geq E_{\theta_{-i}}\left[a(\theta'',\theta_{-i})\right],$$

For sufficiency, let $\theta_i > 0.5$, *and consider* $0.5 < \widehat{\theta} < \theta_i$.

$$\begin{aligned}
U_i(\theta_i) - U_i(\widehat{\theta}) &= 2\int_{\widehat{\theta}}^{\theta_i} E_{\theta_{-i}}\left[a\left(\tau,\theta_{-i}\right)-\tau\right]d\tau \geq \\
2\int_{\widehat{\theta}}^{\theta_i} E_{\theta_{-i}}\left[a\left(\widehat{\theta},\theta_{-i}\right)-\tau\right]d\tau &= 2\left[E_{\theta_{-i}}\left(a\left(\widehat{\theta},\theta_{-i}\right)\right)\left[\theta_i-\widehat{\theta}\right]-\left[\frac{\theta_i^2}{2}-\frac{\widehat{\theta}^2}{2}\right]\right] = \\
& E_{\theta_{-i}}\left[-\left(a(\widehat{\theta},\theta_{-i})-\theta_i\right)^2\right]+E_{\theta_{-i}}\left[\left(a(\widehat{\theta},\theta_{-i})-\widehat{\theta}\right)^2\right],
\end{aligned}$$

where the first inequality follows from the expected monotonicity of the allocation.

It then follows that

$$U_i(\theta_i) \geq E_{\theta_{-i}}\left[-\left(a(\widehat{\theta},\theta_{-i})-\theta_i\right)^2\right].$$

The analysis for all other cases is analogous. ■

To prove Proposition1, it is convenient to show the following

Lemma 4 (Dynamic IC) *Thruthtelling is IC given the "dynamic" implementation of the DD allocation rule.*

Proof. Consider a truncated version of the DD allocation rule in which for N rounds the players are asked if they are above or below the the midpoint. If in any of these rounds the announcements fell in different sides of the relevant midpoint then that is the allocation chosen. If for N rounds they have always picked the same side then at round $N+1$ the players are asked to announce their types. The allocation chosen is then the median announcement where there is phantom announcement in the midpoint of the remaining interval. Clearly as $N \to \infty$ this alternative mechanism converges to the DD mechanism. We will show that for any $N \geq 0$ this mechanism is IC and therefore the DD mechanism is IC.

$N = 0$: It is clearly IC to reveal the true type since your type only affects the allocation when the allocation equals your type. Note that this is regardless of where the phantom voter is chosen.

$N = 1$: Consider a player with $\theta_i < 1/2$ (the other case is symmetric). If he announces bottom then with probability 1/2 the game is ended $\theta_j \geq 1/2$ and the allocation is $\frac{1}{2}$. If he announces top then also with probability 1/2 the game is ended $\theta_j < 1/2$. Now suppose that the game is not ended after his first announcement then if he was truthful the phantom voter in the second round is set at 1/4 if he lied in the first round the phantom voter will be set at 3/4. Clearly given the allocation rule in the final round he wants to reveal truthfully and the phantom voter being at 3/4 gives him lower expected utility.

The induction step. Suppose that when there were $N > 0$ rounds the players had no incentive to lie. Consider a new game with $N + 1$ rounds. If you are truthful in the first round and the game does not end then we have already shown that the player will not lie in the remaining rounds since the game looks the same except that the type space is a smaller interval. Actually, even if he deviates by mistake in the first round he won't have an incentive to deviate in the remaining rounds. Hence, we can simply check if he would benefit from deviating only once. Deviating only once is not profitable because it will make the possible stopping points be further away in expectation from the Agent's true type. To illustrate consider a type close to 1/2 (but lower) if this type lies in the first round and the game does not end he will be truthful from then on hence with probability 1/4 the allocation will be 3/4 with probability 1/8 it will be 5/8 and so on. Suppose instead he was truthful in the first round and the game didn't end then with probability 1/4 the allocation would be 1/4 with probability 1/8 it would be 3/8 and so on. Note that the expected allocation is closer to the Agent's type if he is truthful hence he wouldn't want to lie at the first stage either. ■

Proof of Proposition 1. Lemma (Dynamic IC) shows that, at any given round, truthtelling is optimal given the player´s information set at that round (what he knows about the other player's type). By the Law of Iterated Expectations, it follows that the player´s ex-ante (i.e., before having more information about his opponent's type) payoff is maximized when he announces truthfully. ■

Appendix B: The Optimality of the DD

In this appendix, we show that the DD is optimal among the class on non-decreasing Incentive Compatible allocations. Our strategy of proof involves two main steps.

The first step shows that, for any given class of allocations, if it is optimal to set $\frac{1}{2}$ off-diagonals, then the DD must be an optimal allocation within that class. The idea of the proof is straightforward. Given an uniform distribution, the problem of finding an optimal allocation, *conditional* on both agents having their types on the same region – say, above $\frac{1}{2}$ – and given that a constant action is being chosen when they are in different regions, is exactly the same as the original problem. In other words, if a constant action is chosen off-diagonals, so that the schedule off-diagonals plays no role in the provision of incentives, the problem self-replicates. Hence, the DD must be optimal.

In the second step, we show that if a non-decreasing schedule does not have $\frac{1}{2}$ off-diagonals, it cannot be optimal. We proceed in the following way. Starting with an arbitrary Incentive Compatible, non-decreasing, and continuous schedule $a(.)$, we show, using the same arguments as in the text, and ignoring Incentive

Compatibility issues, that an improvement can be attained if one sets $\frac{1}{2}$ off-diagonals. We then move towards showing that further modifications of $a(.)$ along the main diagonals can be made so to guarantee both Incentive Compatibility, and that the new Incentive Compatible allocation still fares better than $a(.)$. We then use limiting arguments to argue that the same reasoning applies to starting schedules which are non-decreasing, but discontinuous.

Steps 1 and 2 together prove Theorem1.

Separability and the Optimality of $\frac{1}{2}$ Off-Diagonals

Proof of Lemma Separability. The program of interest is:

$$\underset{a(.,.)\text{ is IC over }[0,1]^2}{\arg\max}\left(\begin{array}{c}\sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}\le\frac{1}{2},x<\frac{1}{2}\right]-\frac{1}{2}E_\theta\left[a(\theta)(\theta_i)|\theta_i,\theta_{-i}<\frac{1}{2}\right]\right]\\ \sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x<\frac{1}{2}\right]+\frac{1}{2}E_\theta\left[a(\theta)(1-\theta_i)|\theta_i,\theta_{-i}>\frac{1}{2}\right]\right]\\ \sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}\le\frac{1}{2},x>\frac{1}{2}\right]-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x>\frac{1}{2}\right]\right]\\ \sum_i\frac{1}{2}\left[E_\theta\left[\left[a(\theta_i,\theta_{-i})\right](1-\theta_i)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]-E_\theta\left[\left[a(\theta_i,\theta_{-i})\right]\theta_i|\theta_i\le\frac{1}{2},\theta_{-i}>\frac{1}{2}\right]\right]\end{array}\right)$$

Given $a^*(\theta,x)$ sets 1/2 off-diagonals what remains to be proven is that

$$a^*(\theta,x)\in\underset{a(.,.)}{\arg\max}\left(\begin{array}{c}\sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}\le\frac{1}{2},x<\frac{1}{2}\right]-\frac{1}{2}E_\theta\left[a(\theta)(\theta_i)|\theta_i,\theta_{-i}<\frac{1}{2}\right]\right]\\ \sum_i\left[-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x<\frac{1}{2}\right]+\frac{1}{2}E_\theta\left[a(\theta)(1-\theta_i)|\theta_i,\theta_{-i}>\frac{1}{2}\right]\right]\\ s.t.\ IC\ for\ i=1,2\ given\ \theta_{-i}\in(0,1)\ and\ monotonicity\\ and\ a(\theta,x)=1/2\text{ off-diagonals}\end{array}\right)$$

Since any schedule which is incentive compatible over $[0,1]^2$ and has a constant off-diagonals must, in fact, be incentive compatible over $\left[0,\frac{1}{2}\right]^2$ and $\left[\frac{1}{2},1\right]^2$. The program above implies:

$$\begin{aligned}for\ \theta &\in (0,1/2)^2,\ a^*(\theta,x)\in\arg\max_{a()}\sum_i E\left[u_i(a,\theta_i)|\theta\in(0,1/2)^2\right]\\ s.t.\ IC\ for\ i &= 1,2\ given\ \theta_{-i}\in(0,1/2)\ and\ monotonicity\end{aligned}$$

and

$$\begin{aligned}for\ \theta &\in (1/2,1)^2,\ a^*(\theta,x)\in\arg\max_{a()}\sum_i E\left[u_i(a,\theta_i)|\theta\in(1/2,1)^2\right]\\ s.t.\ IC\ for\ i &= 1,2\ given\ \theta_{-i}\in(1/2,1)\ and\ monotonicity\end{aligned}$$

as desired. ■

Lemma (1/2 off-diagonals): *Given any incentive compatible allocation $a(\theta;x)$ that satisfies Monotonicity we can find an alternative incentive compatible allocation $\tilde{a}(\theta;x)$ which is weakly better and satisfies $\tilde{a}(\theta;x)=\frac{1}{2}$ for all x, when $\theta_i>\frac{1}{2}>\theta_{-i}$, and $\tilde{a}\left(\frac{1}{2},\theta_{-i};x\right)=\frac{1}{2}$ when $x>\frac{1}{2}$.*

Proof Lemma (1/2 off-diagonals): Without loss of generality we focus on the case in which the starting $a(\theta;x)$ is symmetric across players and around $\frac{1}{2}$ i.e.

$$\begin{aligned}a(\theta_i,\theta_{-i};x) &= a(\theta_{-i},\theta_i;x)\\ a(\theta_i,\theta_{-i};x) &= 1-a(1-\theta_i,1-\theta_{-i};x)\end{aligned}$$

We first consider the case where the starting $a(\theta;x)$ is continuous and show later (in Step 3) the result extends to non-continuous allocations. Furthermore, let us first point out that it is without loss to start with a schedule that is not constant off-diagonals. Indeed, if $a(\theta;x)=c\in\Re$ for all θ in $\left[0,\frac{1}{2}\right]\times\left[\frac{1}{2},1\right]$ and $\left[\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right]$, we could set $\frac{1}{2}$ off-diagonals (and $\frac{1}{2}$ fifty percent of time whenever a player announces $\frac{1}{2}$) without affecting incentives and such a change would generate a gain.

We now proceed to prove the result through a sequence of steps.

Step 1 (1/2 off-diagonals generates an improvement): *In this first step we show that a strict improvement can be attained if we replace the off-diagonal values in the original allocation by* 1/2. Formally,

$$a_{1/2}(\theta,x)=\begin{cases}\frac{1}{2} & \text{if } \theta_i>\frac{1}{2}>\theta_{-i} \text{ or if } \theta_i=\frac{1}{2} \text{ and } x>\frac{1}{2}\\ a(\theta;x) & \text{otherwise}\end{cases}$$

We should note however that $a_{1/2}(\theta,x)$ may not be IC (we will address this in the next step).

By picking $\theta'=\frac{1}{2}$, and ignoring constant terms we can write the objective functional as:

$$V(a)=\left(\begin{array}{c}\sum_i\left[-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x<\frac{1}{2}\right]-\frac{1}{2}E_{\theta,x}\left[a(\theta;x)(\theta_i)|\theta_i,\theta_{-i}<\frac{1}{2}\right]\right]\\ \sum_i\left[-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x<\frac{1}{2}\right]+\frac{1}{2}E_{\theta,x}\left[a(\theta;x)(1-\theta_i)|\theta_i,\theta_{-i}>\frac{1}{2}\right]\right]\\ \sum_i\left[-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x>\frac{1}{2}\right]-\frac{1}{4}E_{\theta_{-i},x}\left[\left(a\left(\frac{1}{2},\theta_{-i};x\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x>\frac{1}{2}\right]\right]\\ \sum_i\frac{1}{2}\left[E_{\theta,x}\left[\left[a(\theta_i,\theta_{-i};x)\right](1-\theta_i)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]-E_{\theta,x}\left[\left[a(\theta_i,\theta_{-i};x)\right]\theta_i|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right]\right]\end{array}\right)$$

For any non-decreasing $a(\theta,x)$, one has that

$$\begin{aligned}&E_{\theta,x}\left[a(\theta_i,\theta_{-i})(1-\theta_i)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]\\ \leq\ &E_\theta\left[a(\theta_i,\theta_{-i})|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right]E_\theta\left[(1-\theta_i)|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right],\end{aligned}$$

and

$$\begin{aligned}&-E_\theta\left[a(\theta_i,\theta_{-i})\theta_i|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right]\\ \leq\ &-E_\theta\left[a(\theta_i,\theta_{-i})|\theta_i\leq\frac{1}{2},\theta_{-i}>\frac{1}{2}\right]E_\theta\left[\theta_i|\theta_i>\frac{1}{2},\theta_{-i}<\frac{1}{2}\right].\end{aligned}$$

with strict inequality holding whenever $a(\theta_i,\theta_{-i})$ is not constant over the region under analysis.

Hence, over

$$\left[0,\frac{1}{2}\right)\times\left(\frac{1}{2},1\right]\cup\left(\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right)$$

the optimal schedule is constant. Moreover, by setting the constant to $\frac{1}{2}$ when one announces $\frac{1}{2}$, and $x>\frac{1}{2}$, the term

$$-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}\leq\frac{1}{2},x>\frac{1}{2}\right]-\frac{1}{4}E_{\theta_{-i}}\left[\left(a\left(\frac{1}{2},\theta_{-i}\right)-\frac{1}{2}\right)^2|\theta_{-i}>\frac{1}{2},x>\frac{1}{2}\right]$$

is also maximized.

Therefore the modified allocation has (a) a constant action off-diagonals and (b) prescribes, at least half of the time, type $\frac{1}{2}'s$ most preferred actions. Hence,

$$V\left(a_{1/2}(\theta, x)\right) > V\left(a(\theta; x)\right).$$

■

Step 2 (Restoring IC): In this step we show that we can modify $a_{1/2}(\theta, x)$ in a way that restores IC, preserves 1/2 off-diagonals and does strictly better than the original allocation $a(\theta, x)$. Throughout, we specify the changes only for the top quadrant $(\left[\frac{1}{2}, 1\right]^2)$ as the required changes over the bottom quadrant are similar.

We start by constructing an alternative schedule as follows.

For an integer N, consider the partition of $\left[\frac{1}{2}, 1\right]$ given by $\{A_i\}_i$, where $i \in \{1, ..., N\}$; and, for $i < N-1$, $A_i = \left[\frac{1}{2} + \frac{i-1}{2N}, \frac{1}{2} + \frac{i}{2N}\right)$ and $A_N = \left[\frac{1}{2} + \frac{N-1}{2N}, 1\right]$. Let $a_{ij} = E_{\theta,x}\left[a(\theta; x) | \theta \in A_i \times A_j\right]$ denote the expected action in each square $A_i \times A_j$ under the original allocation. Now consider, the schedule

$$\overline{a}_N(\theta, x) = \begin{cases} \frac{1}{2} & \text{if } \theta_i > \frac{1}{2} > \theta_{-i} \ \ or\ if\ \theta_i = \frac{1}{2}\ and\ x > \frac{1}{2} \\ a_{ij} & \text{if } \theta \in A_i \times A_j \subset \left(\frac{1}{2}, 1\right]^2 \\ a_{1j} & \text{if } \theta_i = \frac{1}{2},\ \theta_j \in A_j\ and\ x < \frac{1}{2} \end{cases} \tag{A1}$$

There exists a $\overline{N}$ and a $\gamma > 0$ so that for all $N > \overline{N}$,

$$V\left(\overline{a}_N(\theta, x)\right) > V\left(a(\theta, x)\right) + \gamma.$$

This follows from noting that $\overline{a}_N(\theta, x)$ converges to $a(\theta; x)$ over $\left(\frac{1}{2}, 1\right]^2$ when N goes to the infinity, so that:

$$\lim_{N \to \infty} \overline{a}_N(\theta, x) = a_{1/2}(\theta, x),$$

Since

$$V\left(a_{1/2}(\theta, x)\right) > V(a),$$

making use of the Dominated Convergence Theorem, it then follows that there exists a $\overline{N}$ and a $\gamma > 0$ so that if $N > \overline{N}$

$$V\left(\overline{a}_N(\theta, x)\right) > V\left(a(\theta, x)\right) + \gamma$$

as stated.

Although better than $a(\theta; x)$ for N large, $\overline{a}_N(\theta, x)$ might not be IC.

To re-establish incentive compatibility, we need all the "cutoff types" $\frac{1}{2} + \frac{i}{2N}$, $i \in \{1, .., N-1\}$, to be indifferent between reporting "left" or "right" together with expected monotonicity of the allocation. As a first step we show that it is sufficient to modify the allocation by adding (respectively subtracting) constants δ_i, along the diagonal squares $A_i \times A_i$, $i \geq 1$ (resp. $i \leq -1$) to satisfy the indifference conditions. As a second step, we show that such δ_i can be chosen to be positive and such that the resulting allocation fares strictly better than $a(\theta; x)$. Finally, the last step shows that expected monotonicity is indeed satisfied. In what follows, we restrict attention to the region $\left[\frac{1}{2}, 1\right]^2$. The analysis for the region $\left[0, \frac{1}{2}\right]^2$ is analogous.

Step 2A:

In order for the IC constraints to be satisfied, the $\{\delta_i\}_i$ must be chosen to guarantee

$$\begin{aligned} & -\frac{1}{N}\sum_{j=1,j\neq i}^{N}\left(a_{i,j}-\left(\frac{1}{2}+\frac{i}{N}\right)\right)^2-\frac{1}{2N}\left(a_{ii}+\delta_i-\left(\frac{1}{2}+\frac{i}{N}\right)\right)^2 \\ = & -\frac{1}{N}\sum_{j=1,j\neq i+1}^{N}\left(a_{i+1,j}-\left(\frac{1}{2}+\frac{i}{N}\right)\right)^2-\frac{1}{2N}\left(a_{i+1i+1}+\delta_{i+1}-\left(\frac{1}{2}+\frac{i}{N}\right)\right)^2, \end{aligned}$$

so that the cut-off types are indifferent between reporting left and right.

The above conditions induce a difference equation of the following form

$$\begin{aligned} & -\frac{1}{N}\sum_j a_{i,j}^2+2\left(\frac{1}{2}+\frac{i}{N}\right)\frac{1}{N}\sum_j a_{i,j}-\frac{1}{N}\delta_i^2+2\frac{1}{N}\left(\frac{1}{2}+\frac{i}{N}\right)\delta_i-2\frac{1}{N}a_{ii}\delta_i \\ = & -\frac{1}{N}\sum_j a_{i+1,j}^2+2\frac{1}{N}\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j a_{i+1,j}-\frac{1}{N}\delta_{i+1}^2+2\frac{1}{N}\left(\frac{1}{2}+\frac{i}{N}\right)\delta_{i+1}-2\frac{1}{N}a_{i+1i+1}\delta_{i+1} \end{aligned} \tag{A2}$$

or

$$\begin{aligned} \delta_{i+1} &= \frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\pm\frac{N}{2}\left(\begin{array}{c} \frac{4}{N^2}\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)^2 \\ -4\frac{1}{N}\left[\begin{array}{c} \frac{2\delta_i}{N}\left[\left(\frac{1}{2}+\frac{i}{N}\right)-a_{ii}\right]-\frac{1}{N}\delta_i^2+ \\ \frac{1}{N}\sum_j\left(a_{i+1,j}^2-a_{i,j}^2\right) \\ -\frac{2}{N}\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j\left(a_{i+1,j}-a_{i,j}\right) \end{array}\right] \end{array}\right)^{1/2} \\ \delta_{i+1} &= \frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\pm\frac{1}{2}\left(\begin{array}{c} 4\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)^2 \\ -4\left[\begin{array}{c} 2\delta_i\left[\left(\frac{1}{2}+\frac{i}{N}\right)-a_{ii}\right]-\delta_i^2+ \\ \sum_j\left(a_{i+1,j}^2-a_{i,j}^2\right) \\ -2\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j\left(a_{i+1,j}-a_{i,j}\right) \end{array}\right] \end{array}\right)^{1/2}, \end{aligned}$$

(where we have dropped the superscripts in the summation sign).

Hence

$$\frac{\delta_{i+1}}{\sqrt{N}}=\begin{array}{c} \frac{1}{\sqrt{N}}\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right) \\ \pm\frac{1}{2}\left(\frac{4}{N}\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)^2-4\left[\begin{array}{c} \frac{2\delta_i}{N}\left[\left(\frac{1}{2}+\frac{i}{N}\right)-a_{ii}\right]-\frac{\delta_i^2}{N} \\ +\frac{1}{N}\sum_j\left(a_{i+1,j}^2-a_{i,j}^2\right) \\ -\frac{2}{N}\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j\left(a_{i+1,j}-a_{i,j}\right) \end{array}\right]\right)^{1/2} \end{array} \tag{A3}$$

We next show that, for a properly chosen δ_1, one can find a sequence of $\{\delta_i\}_i$ with $\delta_i \geq 0$ for all i.

Step 2B: There exists a sequence of non-negative $\{\delta_i\}$ that (i) solve Equation $(A3)$, and (ii) lead to a schedule that improves strictly upon the initial $a(\theta; x)$. Moreover, the $\{\delta_i\}$ can be chosen to be $O\left(\sqrt{N}\right)$.

We establish this result in 3 Claims.

Claim 1: If δ_1 is strictly positive and $O\left(\sqrt{N}\right)$, one can find, for all $i \geq 2$, a sequence of strictly positive$\{\delta_i\}_{i\geq 2}$, where each δ_i is also $O\left(\sqrt{N}\right)$.

Proof: Since $a(\theta, x)$ is a continuous function over a compact set, it is uniformly continuous. Therefore, for any $\varepsilon > 0$, there exists a N'so that, if $N > N'$,

$$|a_{i,j}-a_{i+1,j}|<\varepsilon$$

and

$$\left|a_{i,j}^2 - a_{i+1,j}^2\right| < \varepsilon.$$

for *all* i,j.

Hence,

$$\frac{1}{N}\left|\sum_j \left(a_{i,j}^2 - a_{i+1,j}^2\right)\right| \leq \frac{1}{N}\sum_j \left|\left(a_{i,j}^2 - a_{i+1,j}^2\right)\right| < \varepsilon$$

and

$$\frac{1}{N}\left|\sum_j \left(a_{i,j} - a_{i+1,j}\right)\right| \leq \frac{1}{N}\sum_j \left|\left(a_{i,j}^2 - a_{i+1,j}^2\right)\right| < \varepsilon$$

This implies that, for all i, j,

$$\frac{1}{N}\left|\sum_j \left(a_{i,j}^2 - a_{i+1,j}^2\right)\right| = O\left(\frac{1}{N}\right)$$

and

$$\frac{1}{N}\left|\sum_j \left(a_{i,j} - a_{i+1,j}\right)\right| = O\left(\frac{1}{N}\right).$$

Now, consider A3 when $i = 1$.

If δ_1 is $O\left(\sqrt{N}\right)$, δ_1^2 is $O\left(N\right)$. It then follows that $-2\frac{1}{2N}\left(a_{11} - \left(\frac{1}{2} + \frac{1}{N}\right)\right)\delta_1 = O\left(\frac{1}{\sqrt{N}}\right)$, and $-\frac{1}{2N}\delta_1^2 = O\left(1\right)$

Moreover, from the discussion in the beginning of the proof, one has that

$$-\frac{1}{N}\sum_j a_{1,j}^2 + \frac{1}{N}\sum_j a_{2,j}^2 = O\left(\frac{1}{N}\right),$$

and

$$2\left(\frac{1}{2} + \frac{1}{N}\right)\frac{1}{N}\left[\sum_j \left(a_{1,j} - a_{2,j}\right)\right] = O\left(\frac{1}{N}\right).$$

Therefore, whenever δ_1 is $O\left(\sqrt{N}\right)$,

$$\frac{\frac{1}{\sqrt{N}}\left[\left(\frac{1}{2} + \frac{1}{N} - a_{22}\right)\right]}{\pm\sqrt[2]{\begin{array}{c}\frac{1}{N}\left(\frac{1}{2} + \frac{1}{N} - a_{22}\right)^2 \\ -2\left[\frac{\delta_1}{N}\left[\left(\frac{1}{2} + \frac{1}{N}\right) - a_{11}\right] - \frac{1}{2}\frac{\delta_1^2}{N} + \frac{1}{2}\frac{1}{N}\sum_j \left(a_{2,j}^2 - a_{1,j}^2\right) - \frac{1}{N}\left(\frac{1}{2} + \frac{1}{N}\right)\sum_j \left(a_{2,j} - a_{1,j}\right)\right]\end{array}}}$$

is $O\left(1\right)$ (and well defined for N large, as the term inside the square root will be strictly positive). Hence, $\frac{\delta_2}{\sqrt{N}}$ must be $O\left(1\right)$, which implies that δ_2 must also be $O\left(\sqrt{N}\right)$. It is easy to see that δ_2 can be chosen to be positive by picking the positive term of the square root.

Proceeding inductively, the result follows.■

Denote by $\hat{N}$ the value such that for $N > \hat{N}$ the sequence $\{\delta\}$ is well defined and positive.

Claim 2: There exists a strictly positive δ_1, which is $O\left(\sqrt{N}\right)$ so that the schedule defined by

$$\widetilde{a}_1\left(\theta, x\right) = \begin{cases} \overline{a}_N\left(\theta, x\right) + \delta_1 \text{ if } \theta \in A_1 \times A_1 \\ \overline{a}_N\left(\theta, x\right) \textit{ otherwise} \end{cases}$$

satisfies

$$V\left(\widetilde{a}_1\left(\theta;x\right)\right) \geq V\left(a\left(\theta;x\right)\right).$$

Proof: This follows immediately from Step 1. In fact, since for all $N > \max\left(\overline{N},\hat{N}\right)$,

$$V\left(\overline{a}_N\left(\theta,x\right)\right) > V\left(a\left(\left(\theta;x\right)\right)\right) + \gamma$$

for γ strictly positive.

By adding a strictly positive number over $A_1 \times A_1$, one will decrease type $\frac{1}{2}'$s payoff.

Since, from type $\frac{1}{2}'s$ perspective, the harm caused by such change will occur with probability $\frac{1}{N}$, and

$$V\left(\overline{a}_N\left(\theta,x\right)\right) > V\left(a\left(\left(\theta;x\right)\right)\right) + \gamma$$

the positive δ_1 necessary to satisfy

$$V\left(\widetilde{a}_1\left(\theta;x\right)\right) \geq V\left(a\left(\theta;x\right)\right).$$

can be made $O\left(\sqrt{N}\right)$.■

Claim 3: One can find a schedule that satisfies the indifference condition (Equation ($A3$)) and fares strictly better than $a\left(\theta;x\right)$.

Proof: For some $N > \overline{\overline{N}} = \max\left(\overline{N},\hat{N}\right)$, define a new schedule that is equal to $\overline{a}_N\left(\theta,x\right)$ except at the squares $A_i \times A_i$, where it is equal to $a_{ii}+\delta_i$ for $i \geq 1$, where the δ_i is given by the sequence defined in Claim 1 for the δ_1.in Claim 2. Denoting this schedule by $\widetilde{a}\left(.,x\right)$ one has that

$$V\left(\widetilde{a}\left(\theta;x\right)\right) > V\left(\widetilde{a}_1\left(\theta;x\right)\right) \geq V\left(a\left(\theta;x\right)\right).$$

This follows because (i) $V\left(.\right)$ is linear in $a\left(.\right)$ for $\theta \in \left[\frac{1}{2},1\right]^2$, and, finally, (ii) the adding of the $\delta_i's$ for $i \geq 2$ does *not* affect the utility of type $\frac{1}{2}$.■

We have just shown that starting from an arbitrary continuous and non-decreasing schedule a, we can construct an alternative schedule $\widetilde{a}$ that has $\frac{1}{2}$ off-diagonals, satisfies Local Incentive Compatibility and fares better than the initial a. What is left to show is that $\widetilde{a}$ satisfies expected monotonicity. We know argue that this is in fact the case.

Step 2C (Monotonicity): For all i, there is $\widetilde{N}$ so that, for $N > \widetilde{N}$,

$$\sum_{j=1}^{N}\widetilde{a}_{i+1j} \geq \sum_{j=1}^{N}\widetilde{a}_{ij}$$

Proof: First note that the indifference condition Eq.[$A2$] can be read as

$$\begin{aligned} 0 = & \sum_j\left(a_{i+1,j}^2 - a_{i,j}^2\right) - 2\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j\left(a_{i+1,j}-a_{i,j}\right) \\ & -\left(\delta_i^2-\delta_{i+1}^2\right) + 2\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)\left(\delta_i-\delta_{i+1}\right) - 2\delta_i\left(a_{ii}-a_{i+1i+1}\right)\end{aligned}$$

Now we do a Taylor series expansion of $a_{i+1,j}^2$ and δ_{i+1}^2,

$$a_{i+1,j}^2 = a_{i,j}^2 + 2a_{i,j}\left[a_{i+1,j}-a_{i,j}\right] + O\left(\left(a_{i+1,j}-a_{i,j}\right)^2\right),$$

$$\delta_{i+1}^2 = \delta_i^2 + 2\delta_i\left[\delta_{i+1}-\delta_i\right] + O\left(\left(\delta_{i+1}-\delta_i\right)^2\right)$$

Therefore, one can write the indifference condition as

$$\begin{aligned} 0 &= \sum_j \left(2a_{i,j}\left[a_{i+1,j}-a_{i,j}\right]+O\left(\left(a_{i+1,j}-a_{i,j}\right)^2\right)\right)-2\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j\left(a_{i+1,j}-a_{i,j}\right) \\ &+2\delta_i\left[\delta_{i+1}-\delta_i\right]+O\left(\left(\delta_{i+1}-\delta_i\right)^2\right)+2\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)\left(\delta_i-\delta_{i+1}\right)-2\delta_i\left(a_{ii}-a_{i+1i+1}\right) \end{aligned}$$

or

$$\begin{aligned} 0 &= \sum_j \left(2a_{i,j}\left[a_{i+1,j}-a_{i,j}\right]+O\left(\left(a_{i+1,j}-a_{i,j}\right)^2\right)\right)-2\left(\frac{1}{2}+\frac{i}{N}\right)\sum_j\left(a_{i+1,j}-a_{i,j}\right) \\ &-2\left[\delta_i-\delta_{i+1}\right]\left[\delta_i-\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)\right]+O\left(\left(\delta_{i+1}-\delta_i\right)^2\right)-2\delta_i\left(a_{ii}-a_{i+1i+1}\right) \end{aligned} \tag{4}$$

Now, assume, towards a contradiction, that expected monotonicity is violated i.e. there is an i such that for all N :

$$\sum_{j=1}^N \widetilde{a}_{ij} > \sum_{j=1}^N \widetilde{a}_{i+1j}$$

This, in turn, implies that

$$\delta_i-\delta_{i+1} > \sum_{j=1}^N \left(a_{i+1j}-a_{ij}\right) \geq 0.$$

For N large and δ of order $O\left(\sqrt{N}\right)$:

$$\delta_i-\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)>0,$$

and therefore:

$$-2\left[\delta_i-\delta_{i+1}\right]\left[\delta_i-\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)\right] < -2\left[\delta_i-\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)\right]\sum_{j=1}^N\left(a_{i+1j}-a_{ij}\right). \tag{5}$$

Also, since $\left[a_{i+1,j}-a_{i,j}\right]\geq 0\ \forall j$

$$\sum_j\left(a_{i,j}\left[a_{i+1,j}-a_{i,j}\right]\right)\leq \max_j a_{i,j}\sum_j\left(a_{i+1,j}-a_{i,j}\right). \tag{6}$$

From Eq. [4] and the last two inequalities we get:

$$\begin{aligned} 0 &< 2\sum_j\left(a_{i+1,j}-a_{i,j}\right)\left[\max_j a_{i,j}-\left(\frac{1}{2}+\frac{i}{N}\right)-\left[\delta_i-\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)\right]\right] \\ &+\sum_j O\left(\left(a_{i+1,j}-a_{i,j}\right)^2\right)+O\left(\left(\delta_{i+1}-\delta_i\right)^2\right)-2\delta_i\left(a_{ii}-a_{i+1i+1}\right) \end{aligned}$$

Finally show that for large N the right hand side of this inequality is smaller than zero, which leads to the desired contradiction. Towards this note that for large N the following are true:

1. $$\left[\max_j a_{i,j}-\left(\frac{1}{2}+\frac{i}{N}\right)-\left[\delta_i-\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)\right]\right]\sum_{j=1}^N\left(a_{i+1j}-a_{ij}\right)<0$$

   and of order $O\left(\sqrt{N}\right)$. This follows from $\left[\max_j a_{i,j}-\left(\frac{1}{2}+\frac{i}{N}\right)-\left[\delta_i-\left(\frac{1}{2}+\frac{i}{N}-a_{i+1i+1}\right)\right]\right]$ being $O\left(\sqrt{N}\right)$, and $\sum_{j=1}^N\left(a_{i+1j}-a_{ij}\right)$ being $O\left(1\right)$.

2. $-2\delta_i\left(a_{ii}-a_{i+1i+1}\right)$ is $O\left(\frac{1}{\sqrt{N}}\right)$, since δ_i is $O\left(\sqrt{N}\right)$ and $\left(a_{ii}-a_{i+1i+1}\right)$ is $O\left(\frac{1}{N}\right)$.

3. $\sum_j O\left(\left(a_{i+1,j}-a_{i,j}\right)^2\right)$ is $O\left(\frac{1}{N}\right)$.

4. $O\left(\left(\delta_{i+1}-\delta_i\right)^2\right)$ is $O(1)$.

Hence, there exists an N large such that the right hand side of the inequality is negative which implies, that for all i, there must be an $\widetilde{N}$ so that, for $N>\widetilde{N}$,

$$\sum_{j=1}^{N}\widetilde{a}_{i+1j}\geq\sum_{j=1}^{N}\widetilde{a}_{ij}.\blacksquare$$

All the results above establish the proof for the case in which the initial $a(.)$ is continuous.

In the next Step 3 we deal with the case in which $a(.)$ is non-decreasing but potentially discontinuous.

Step 3: *Given any incentive compatible allocation $a(\theta;x)$ that satisfies Monotonicity we can find an alternative incentive compatible allocation $\tilde{a}(\theta;x)$ which is weakly better and satisfies $\tilde{a}(\theta;x)=\frac{1}{2}$ for all x, when $\theta_i>\frac{1}{2}>\theta_{-i}$, and $\tilde{a}\left(\frac{1}{2},\theta_{-i};x\right)=\frac{1}{2}$ when $x>\frac{1}{2}$.*

Consider an initial allocation $a(\theta,x)$ which satisfies monotonicity and incentive compatibility but is not continuous in θ. Lemmas 5 and 6 below imply that there exists a sequence of non-decreasing continuous functions $\{f_n\}_n$ such that

$$f_n(\theta,x)\rightarrow a(\theta,x)\text{ for all }\theta.$$

Now applying the Steps 1 & 2 detailed above to each function f_n we get a sequence of functions $\left\{\tilde{f}_n\right\}_n$ with the property that:

$$V\left(\tilde{f}_n\right)>V(f_n)\quad\forall n$$

Since (by continuity and the fact that $[0,1]^2$ is compact) there exists $k<\infty$ so that

$$\max_{\theta}|f_n(\theta)|<k\text{ for all }n,$$

one has, by the Dominated Convergence Theorem,

$$V(f_n)\rightarrow V(a)\tag{A9}$$

This implies that there exists an $\bar{n}$ such that for $n>\bar{n}$

$$V\left(\tilde{f}_n\right)\geq V(a).$$

Finally note that for all $\tilde{f}_n$ the values set in the off-diagonals is $\frac{1}{2}$. Hence, setting $\frac{1}{2}$ off-diagonal is weakly better even if the starting allocation was not continuous.$\blacksquare$

We now show that, for any non-decreasing function $g:[0,1]^2\rightarrow\Re$, we can find a sequence of continuous non-decreasing functions $\{g_m\}_m$ which converge pointwise to $g(.)$. In order to do so, we first use the following result, which proof can be found in Rosenlicht (1968, pages 237 and 238)

Lemma 5 *For a given N, consider the partition of $[0,1]$ given by $\{A_i\}_i$, where $A_i = \left[\frac{1}{2} + \frac{i}{2N}, \frac{1}{2} + \frac{i+1}{2N}\right)$ whenever $i \in \{-N, .., N-2\}$, and $A_{N-1} = \left[\frac{1}{2} + \frac{N-1}{2N}, 1\right]$. Consider a non-decreasing simple function $g : [0,1]^2 \to \Re$; that is, a function*

$$g(\theta) = c_{ij} \in \Re \text{ whenever } \theta \in A_i \times A_j.$$

with $c_{i+1j+1} \geq c_{i+1j+1} \geq c_{ij}$.

One can then find a sequence of non-decreasing continuous functions that converge to $g(.)$ pointwise.

With the above result in hands, we are now ready to prove

Lemma 6 *Let $g : [0,1]^2 \to \Re$ be a non-decreasing function. Then there is a sequence of non-decreasing, continuous functions that converge pointwise to g.*

Proof. For an integer N, consider the partition of $[0,1]$ given by $\{A_i\}_i$, where $A_i = \left[\frac{1}{2} + \frac{i}{2N}, \frac{1}{2} + \frac{i+1}{2N}\right)$ whenever $i \in \{-N, .., N-2\}$, and $A_{N-1} = \left[\frac{1}{2} + \frac{N-1}{2N}, 1\right]$. Define $g_N(.)$ as follows:

$$g_N(\theta) = E_\theta\left[g(\theta) | A_i \times A_j\right] \text{ if } \theta \in A_i \times A_j.$$

Clearly, for all θ,

$$||g_N(\theta) - g(\theta)|| \to 0$$

as $N \to \infty$.

Now, fixing a N, one has that $g_N(\theta)$ is a non-decreasing simple function. Hence, by 5, one can find, for each N, a sequence of non-decreasing continuous functions $\left\{g_m^N(.)\right\}_m$ so that

$$\left|\left|g_m^N(\theta) - g_N(\theta)\right|\right| \to 0$$

as $m \to \infty$. Since

$$\left|\left|g_m^N(\theta) - g(\theta)\right|\right| \leq \left|\left|g_m^N(\theta) - g_N(\theta)\right|\right| + ||g_N(\theta) - g(\theta)||,$$

we have that for all θ

$$\left|\left|g_m^N(\theta) - g(\theta)\right|\right| \to 0$$

as $m, N \to \infty$. ■

APPENDIX C: Folk Theorem and the Dictatorship Result

In this appendix, we show that the first best payoffs can be arbitrarily approximated, and also that, in the long-run, the optimal provision of incentives leads to a dictatorial mechanism.

Appendix C.1: Approximate Efficiency

Proof of Proposition 3. We prove that, for any $\epsilon > 0$, there exists $\delta^* \in (0,1)$ such that for $\delta > \delta^*$, the sum of the players equilibrium payoff is within ϵ of the first best payoff.

Given equal weights, the optimal allocation would be

$$a_t^*(\theta) = \frac{\theta_{1t} + \theta_{2t}}{2}, \ \ for\ all\ t.$$

Simple calculations show that the discounted expected value for player i under this allocation is $-\left[\frac{1}{24}\right]$, so that the sum of the players' payoff is $-\frac{1}{12}$

Consider the following candidates for continuation values

$$\begin{aligned}
v_1(\theta_1,\theta_2) &= \begin{pmatrix} -\left(\frac{1-\delta}{\delta}\right) E_{\theta_2}\left[(a^*(\theta_1,\theta_2)-\theta_2)^2\right] \\ +\frac{1-\delta}{\delta} E_{\theta_1}\left[(a^*(\theta_1,\theta_2)-\theta_1)^2\right] - \left[\frac{1}{24}\right] - \frac{\epsilon(\delta)}{2} \end{pmatrix}, \\
v_2(\theta_1,\theta_2) &= \begin{pmatrix} -\left(\frac{1-\delta}{\delta}\right) E_{\theta_1}\left[(a^*(\theta_1,\theta_2)-\theta_1)^2\right] \\ +\frac{1-\delta}{\delta} E_{\theta_2}\left[(a^*(\theta_1,\theta_2)-\theta_2)^2\right] \\ -\left[\frac{1}{24}\right] - \frac{\epsilon(\delta)}{2} \end{pmatrix}.
\end{aligned}$$

where $\epsilon(\delta)$ is the constant that guarantees that, for all θ, $(v_i(\theta_1,\theta_2))_{i=1}^2$ is an pair of Equilibrium Payoffs.

It is easy to see that $v_1(\theta_1,\theta_2) + v_2(\theta_1,\theta_2) = -\frac{1}{12} - \epsilon(\delta)$ for all (θ_1,θ_2). We now show that, with these continuation values, one can implement $a^*(\theta_1,\theta_2)$ in an interim incentive compatible way. Toward that, note that, if player 2 is being truthful, Player 1's problem if a^* is implemented, and if he faces $v_1(\theta_1,\theta_2)$ as a continuation value is

$$\max_{\widehat{\theta}_1} -(1-\delta) E_{\theta_2}\left(a^*(\widehat{\theta}_1,\theta_2)-\theta_1\right)^2 + \delta E_{\theta_2} v_1(\widehat{\theta}_1,\theta_2)$$

which has the same solution as the one associated with the program[18]

$$\max_{\widehat{\theta}_1} -(1-\delta) E_{\theta_2}\left(a^*(\widehat{\theta}_1,\theta_2)-\theta_1\right)^2 - (1-\delta) E_{\theta_2}\left[\left(a^*(\widehat{\theta}_1,\theta_2)-\theta_2\right)^2\right].$$

Since

$$a^*(\theta_1,\theta_2) = \arg\max_a -(a-\theta_1)^2 - (a-\theta_2)^2,$$

the announcement $\widehat{\theta}_1 = \theta_1$ is optimal. An analogous reasoning applies to player 2.

We show that $\epsilon(\delta)$ goes to zero as $\delta \to 1$.

The magnitude of $\epsilon(\delta)$ depends on the variability of the promised continuation values. This variability in turn is bounded by

$$d(\delta) = \max_{1,2}\{d_1(\delta), d_2(\delta)\},$$

[18] The term

$$\frac{1-\delta}{\delta} E_{\theta_1}\left[(a^*(\theta_1,\theta_2)-\theta_1)^2\right] - \left[\frac{1}{24}\right] - \frac{\epsilon(\delta)}{2}$$

does not affect incentives.

where

$$d_i(\delta) = \left(\frac{1-\delta}{\delta}\right)\left[\begin{array}{c} \max_\theta \left[-E_{\theta_{-i}}\left[(a^*(\theta_1,\theta_2)-\theta_i)^2\right] + E_{\theta_i}\left[(a^*(\theta_1,\theta_2)-\theta_i)^2\right]\right] \\ -\min_\theta \left[-E_{\theta_{-i}}\left[(a^*(\theta_1,\theta_2)-\theta_i)^2\right] + E_{\theta_i}\left[(a^*(\theta_1,\theta_2)-\theta_i)^2\right]\right] \end{array}\right].$$

Clearly, $d_i(\delta) \to 0$ as $\delta \to 1$. Hence, $\epsilon(\delta)$ also goes to zero as $\delta \to 1$ ■

Appendix C.2: Dictatorship in the Long-Run

Now, we proof the dictatorship result. We do so in a sequence of lemmas. Before proving the lemmas, however, we define the relevant maximization problem and its first order conditions.

If one assigns multipliers $\{\lambda_i(\theta_i)\}_{i=1,2}$ to the local incentive constraints, multiplier γ to the Promise Keeping Constraints, and ignores Monotonicity, the program of interest can be written as

$$V(w) = \max_{\{a(\theta),w(\theta)\}_\theta} \left\{\begin{array}{c} E_\theta\left[-(1-\delta)(a(\theta)-\theta_1)^2 + \delta V(w(\theta))\right] \\ +\int_0^1 \lambda_1(\theta_1)\left[-(1-\delta)2E_{\theta_2}\left(\left[(a(\theta)-\theta_1)\frac{da(\theta)}{d\theta_1}\right] + \delta\frac{d}{d\theta_1}\left[E_{\theta_2}V(w(\theta))\right]\right)\right]d\theta_1 \\ +\int_0^1 \lambda_2(\theta_2)\left[-(1-\delta)2E_{\theta_1}\left(\left[(a(\theta)-\theta_2)\frac{da(\theta)}{d\theta_2}\right] + \delta\frac{d}{d\theta_2}\left[E_{\theta_1}\left[w(\theta)\right]\right]\right)\right]d\theta_1 \\ +\gamma\left(-E_\theta\left[(1-\delta)(a(\theta)-\theta_2)^2 + \delta w(\theta)\right] - w\right) \end{array}\right\}$$

Integration by parts allows us to re-write $V(w)$ as

$$V(w) = \max_{\{a(\theta),w(\theta)\}_\theta} \left\{\begin{array}{c} E_\theta\left[-(1-\delta)(a(\theta)-\theta_1)^2 + \delta V(w(\theta))\right] + \\ \left[\begin{array}{c} +E_\theta\left[\left(\frac{d\lambda_1(\theta_1)}{d\theta_1}\right)(1-\delta)\left[a(\theta)^2 - 2a(\theta)\theta_1\right] - 2(1-\delta)\lambda_1(\theta_1)a(\theta)\right] \\ -\delta E_\theta\left[\frac{d\lambda_1(\theta_1)}{d\theta_1}V(w(\theta))\right] \end{array}\right] \\ +\left[\begin{array}{c} +E_\theta\left[\left(\frac{d\lambda_2(\theta_2)}{d\theta_2}\right)(1-\delta)\left[a(\theta)^2 - 2a(\theta)\theta_2\right] - 2\lambda_2(\theta_2)(1-\delta)a(\theta)\right] \\ -\delta E_\theta\left[\frac{d\lambda_2(\theta_2)}{d\theta_2}w(\theta)\right] \end{array}\right] \\ +\gamma\left(-E_\theta\left[(1-\delta)(a(\theta)-\theta_2)^2 + \delta w(\theta)\right] - w\right) \\ +(1-\delta)\left[\lambda_1(1) + \lambda_2(1)\right] \end{array}\right\}.$$

Using that we can prove:

Lemma 7 *The marginal value of the agents' values follows a martingale:*

$$V'(w) = E\left[V'(w(\theta))\right]$$

Proof. Maximizing the objective pointwise with respect to $w(\theta)$ one has

$$V'(w(\theta)) - \frac{d\lambda_1(\theta_1)}{d\theta_1}V'(w(\theta)) - \frac{d\lambda_2(\theta_2)}{d\theta_2} = -\gamma$$

By the Envelope Theorem,

$$V'(w) = -\gamma,$$

so

$$V'(w(\theta)) - \frac{d\lambda_1(\theta_1)}{d\theta_1} V'(w(\theta)) - \frac{d\lambda_2(\theta_2)}{d\theta_2} = V'(w).$$

Integrating both sides with respect to θ_2, and using the fact that $\lambda_2(1) = \lambda_2(0)$, we get

$$E_{\theta_2}[V'(w(\theta))] - \frac{d\lambda_1(\theta_1)}{d\theta_1} E_{\theta_2}[V'(w(\theta))] = V'(w).$$

Integrating once more with respect to θ_1, and using the fact that

$$\lambda_1(1) E_{\theta_2}[V'(w(1,\theta_2))] = \lambda_1(0) E_{\theta_2}[V'(w(0,\theta_2))],$$

one has

$$E_\theta[V'(w(\theta))] + \int_0^1 \lambda_1(\theta_1) \frac{dE_{\theta_2}[V'(w(\theta))]}{d\theta_1} d\theta_1 = V'(w)$$

Now symmetry around $\frac{1}{2}$ implies that for all $x \in \left[0, \frac{1}{2}\right]$

$$E_{\theta_2}\left[V'\left(w\left(\frac{1}{2} - x, \theta_2\right)\right)\right] = E_{\theta_2}\left[V'\left(w\left(\frac{1}{2} + x, \theta_2\right)\right)\right]$$

Hence,

$$\frac{d}{d\theta_1} E_{\theta_2}\left[V'\left(w\left(\frac{1}{2} - x, \theta_2\right)\right)\right] = -\frac{d}{d\theta_1} E_{\theta_2}\left[V'\left(w\left(\frac{1}{2} + x, \theta_2\right)\right)\right].$$

This along with

$$\lambda_1\left(\frac{1}{2} - x\right) = \lambda_1\left(\frac{1}{2} + x\right)$$

implies that

$$\int_0^1 \lambda_1(\theta_1) \frac{dE_{\theta_2}[V'(w(\theta))]}{d\theta_1} d\theta_1 = 0.$$

It then follows

$$E_\theta[V'(w(\theta))] = V'(w),$$

as claimed. ■

Lemma 8 $V(.)$ *is strictly concave*

Proof. Let $(a_1(\theta), w_1(\theta))$ and $(a_2(\theta), w_2(\theta))$ be, respectively, solutions of the problem when the promise keeping constraint is indexed by w_1 and w_2.

If it were feasible to implement $a^\alpha(\theta) = \alpha a_1(\theta) + (1-\alpha) a_2(\theta)$, where $\alpha \in (0,1)$, with continuation values $w^\alpha(\theta) = \alpha w_1(\theta) + (1-\alpha) w_2(\theta)$, we would have

$$E_\theta\left[-(1-\delta)(a^\alpha(\theta) - \theta_1)^2 + \delta w^\alpha(\theta)\right] > \alpha w_1 + (1-\alpha) w_2.$$

Now, since the inequality above is strict, we can find, for some $\overline{w} \geq w^\alpha$, a non-negative function $g(\theta) \equiv g_1(\theta_1) + g_2(\theta_2)$ such that

$$E_\theta\left[-(1-\delta)(a^\alpha(\theta) - \theta_1)^2 + \delta[w^\alpha(\theta) - g(\theta)]\right] = \overline{w}, \tag{1}$$

and at the same time

$$\begin{aligned} & \delta E_{\theta_1}\left[w^{\alpha}(\theta)-g(\theta)\right] \\ = \quad & E_{\theta_1}\left[(1-\delta)\left(a^{\alpha}(\theta)-\theta_2\right)^2\right]+\left[\begin{array}{c} E_{\theta_1}\left[-(1-\delta)\left(a\alpha\left(0,\theta_1\right)\right)^2+\delta\left[w^{\alpha}\left(0,\theta_2\right)-g\left(0,\theta_2\right)\right]\right] \\ +(1-\delta)\int_0^{\theta_2} 2E_{\theta_1}\left[a^{\alpha}\left(\tau,\theta_1\right)-\tau\right]d\tau \end{array}\right] \end{aligned} \tag{2}$$

and

$$\begin{aligned} & \delta E_{\theta_2}\left[V\left(w^{\alpha}(\theta)-g(\theta)\right)\right] \\ = \quad & E_{\theta_2}\left[(1-\delta)\left(a^{\alpha}(\theta)-\theta_1\right)^2\right]+\left[\begin{array}{c} E_{\theta_2}\left[-(1-\delta)\left(a^{\alpha}\left(0,\theta_2\right)\right)^2+\delta V\left(w\left(0,\theta_2\right)-g\left(0,\theta_2\right)\right)\right] \\ +(1-\delta)\int_0^{\theta_1} 2E_{\theta_2}\left[a^{\alpha}\left(\tau,\theta_2\right)-\tau\right]d\tau \end{array}\right] \end{aligned} \tag{3}$$

Conditions 2 and 3 guarantee Incentive Compatibility (indeed, with these values, the integral formula implied by the Envelope Theorem holds. Also, since both $a_1(\theta)$ and $a_2(\theta)$ satisfy expected monotonicity, so will $a^{\alpha}(\theta)$). Therefore, $a^{\alpha}(\theta)$ along with $w^{\alpha}(\theta)-g(\theta)$ is feasible when the promised value for player 2 is $\overline{w}\geq w^{\alpha}$

It then follows

$$\begin{aligned} V\left(w^{\alpha}\right) &= V\left(\alpha w_1+(1-\alpha)w_2\right)\geq V\left(\overline{w}\right) \\ &\geq E_{\theta}\left[-(1-\delta)\left(a^{\alpha}(\theta)-\theta_1\right)^2+\delta V\left(w^{\alpha}(\theta)-g(\theta)\right)\right] \\ &> \alpha\left[E_{\theta}\left[-(1-\delta)\left(a_1(\theta)-\theta_1\right)^2\right]\right]+(1-\alpha)\left[E_{\theta}\left[-(1-\delta)\left(a_2(\theta)-\theta_1\right)^2\right]\right]+\delta V\left(w^{\alpha}(\theta)\right) \\ &\geq \left(\begin{array}{c} \alpha\left[E_{\theta}\left[-(1-\delta)\left(a_1(\theta)-\theta_1\right)^2\right]+\delta V\left(w_1(\theta)\right)\right] \\ +(1-\alpha)\left[E_{\theta}\left[-(1-\delta)\left(a_2(\theta)-\theta_1\right)^2\right]+\delta V\left(w_2(\theta)\right)\right] \end{array}\right)+ \\ &= \alpha V\left(w_1\right)+(1-\alpha)V\left(w_2\right), \end{aligned}$$

(where the first inequality follow from the fact that $V(.)$ is decreasing, the second inequality follows from the fact that $a^{\alpha}(\theta)$ along with $w^{\alpha}(\theta)-g(\theta)$ is feasible when the promised value for player 2 is $\overline{w}$, the third inequality follow from strict concavity of Player 1 instantaneous payoff, and the fact that $V(.)$ is decreasing), which proves the result. ∎

Lemma 9 *$V'(w)$ is continuous.*

Proof. Since $V(.)$ is strictly concave, $V'(w)$ is strictly decreasing, and, therefore, continuous except possibly at a countable set. If a point of discontinuity existed, more than a solution for the maximization problem would exist.

Using the same steps as in the above Lemma, one can show that there will always be a unique solution, so $V'(w)$ must be continuous everywhere. ∎

Lemma 10 (Spreading of Values) *For all $w\in\left(-\frac{1}{6},0\right)$, there is positive probability of both $w(\theta)>w$, and $w(\theta)<w$.*

Proof. We prove that $w(\theta) > w$ with positive probability. The proof of the other case follows analogous reasoning. If

$$w(\theta) \leq w$$

for almost all θ, one has, from the concavity of $V(.)$, that

$$V'(w(\theta)) \geq V'(w) \text{ for almost all } \theta.$$

From Lemma 1, this can hold only if

$$w(\theta) = w \text{ for almost all } \theta.$$

Plugging this in the first order condition for $w(.)$, we get

$$V'(w) - \frac{d\lambda_1(\theta_1)}{d\theta_1} V'(w) - \frac{d\lambda_2(\theta_2)}{d\theta_2} = V'(w) \text{ a.e.}$$

Hence,

$$\frac{d\lambda_2(\theta_2)}{d\theta_2} = -\frac{d\lambda_1(\theta_1)}{d\theta_1} V'(w) \text{ a.e.,}$$

which, in turn, calls for

$$\frac{d\lambda_2(\theta_2)}{d\theta_2} = \frac{d\lambda_1(\theta_1)}{d\theta_1} = 0 \text{ a.e.}$$

so that $\lambda_1(\theta_1)$, and $\lambda_2(\theta_2)$ are constant. The FOC with respect to $a(.)$ is

$$\begin{aligned} & -[a(\theta) - \theta_1] + \frac{d\lambda_1(\theta_1)}{d\theta_1}[a(\theta) - \theta_1] - \lambda_1(\theta_1) \\ & -[a(\theta) - \theta_2] + \frac{d\lambda_2(\theta_2)}{d\theta_2}[a(\theta) - \theta_2] - +\lambda_2(\theta_2) \\ = \ & 0. \end{aligned}$$

Solving for $a(\theta)$, we get

$$a(\theta) = \frac{\theta_1\left(1 - \frac{d\lambda_1(\theta_1)}{d\theta_1}\right) + \theta_2\left(1 - \frac{d\lambda_2(\theta_1)}{d\theta_2}\right) - \lambda_1(\theta_1) - \lambda_2(\theta_2)}{2 - \frac{d\lambda_1(\theta_1)}{d\theta_1} - \frac{d\lambda_2(\theta_2)}{d\theta_2}}.$$

Plugging

$$\frac{d\lambda_2(\theta_2)}{d\theta_2} = \frac{d\lambda_1(\theta_1)}{d\theta_1} = 0 \text{ a.e.}$$

in the expression for $a(\theta)$, it follows that

$$a(\theta) = \frac{\theta_1 + \theta_2}{2} + c,$$

where c is a constant (almost everywhere at least). This schedule, however, is not Incentive Compatible when continuation values are so that $w(\theta) = w$ for almost all θ. A contradiction. ■

Proposition 5 (Dictatorship in the limit) *The provision of intertemporal incentives necessarily leads to a dictatorial mechanism: In the limit as $t \to \infty$ either v or $V(v)$ converge to 0 almost surely.*

Proof. Since $V'(w)$ is a non-positive martingale by Dobb's convergence Theorem (see Dobb (1953)) it converges almost surely to some random variable, R. Next we show by contradiction that R cannot have any positive likelihood for values $\in (0, \infty)$. Hence, all the probability is concentrated where $R = 0$ or $R = -\infty$.implying that one of the two players becomes a dictator in the limit. In search of a contradiction suppose there existed a positive probability of finding a path $V'(w_t)$ with the property that $lim_{t\to\infty}$ $V'(w_t) = C$ where $0 < C < \infty$. Since $V'(w)$ is continuous there must exist a convergent sequence of w_t. Call $lim_{t\to\infty}$ $w_t = \bar{w} \in \left(-\frac{1}{6}, 0\right)$ the limit of the player's continuation value. Let $W(w, \theta)$ denote the next period's continuation value given the current promised value w and reported state θ. For w_t to converge it must be that $W(\bar{w}, \theta) = \bar{w}$ for all θ. This however contradicts Lemma (10). ■