

Dynamic Agency and the q Theory of Investment*

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Abstract

We introduce dynamic agency into the neoclassical q theory of investment. Costly external financing arises endogenously from dynamic agency, and influences firm value and investment. Agency conflicts drive a history-dependent wedge between average q and marginal q , and make the firm's investment policy dependent on realized profits. A larger realized profit induces higher investment, and hence a larger firm. Investment is relatively insensitive to average q when the firm is “financially constrained” (i.e. has low financial slack). Conversely, investment is sensitive to average q when the firm is relatively “financially unconstrained,” (i.e. has high financial slack). Moreover, the agent's optimal compensation is in the form of future claims on the firm's cash flows when the firm's past profits are relatively low and the firm has less financial slack, whereas cash compensation is preferred when the firm has been profitable, agency concerns are less severe, and the firm is growing rapidly. To study the effect of serial correlation of productivity shocks on investment and firm dynamics, we extend our model to allow the firm's output price to be stochastic. We show that, in contrast to static agency models, the agent's compensation in the optimal dynamic contract will depend not only on the firm's past performance, but also on output prices, even though they are beyond the agent's control. This dependence of the agent's compensation on exogenous output prices (for incentive reasons) further feeds back on the firm's investment, and provides a channel to amplify and propagate the response of investment to output price shocks via dynamic agency.

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1 Introduction

This paper integrates dynamic agency theory into the neoclassical q theory of investment. The objective is to examine the effects of financing frictions (costly external financing) on the relation between firms' investment decisions and Tobin's q , where the cost of external financing endogenously arises from agency problems. We consider a dynamic setting with optimal contracts.

We choose the modeling ingredients so that the predictions on investment and firm value under the first-best setting (with no agency conflicts) are intuitive and analytically tractable. Following the classic investment literature, e.g. Hayashi (1982), we endow the firm with constant-returns-to-scale production technology, so that output is proportional to the firm's capital stock but is subject to independently and identically distributed productivity shocks. The firm can invest/disinvest to alter its capital stock, but this investment entails a quadratic adjustment cost which is homogenous of degree one in investment and capital stock. Under these conditions, with no agency problem, we have the standard predictions that the investment-capital ratio is linear in average q , and that average q equals marginal q (Hayashi (1982)).¹

Our model differs from the neoclassical setting due to a dynamic agency problem. At each point in time, the agent chooses an action and this action together with the (unobservable) productivity shock determines output. Our agency model can be interpreted as a standard principal-agent setting in which the agent's action is unobserved costly effort, and this effort affects the mean rate of production. Alternatively, we can also interpret the agency problem to be one in which the agent can divert output for his private benefit. The agency side of our model builds on the discrete-time models of DeMarzo and Fishman (2007a, b) and the continuous-time formulation of DeMarzo and Sannikov (2006).

¹Abel and Eberly (1994) extend neoclassical investment theory to allow for various other forms of adjustment costs such as a wedge between the purchase and sale prices of capital, and fixed lumpy costs.

The optimal contract in this setting specifies, as a function of the history of the firm's profits, the agent's compensation and the level of investment in the firm. We solve for the optimal contract using a recursive, dynamic programming approach. Under this approach, the firm's history of past profitability determines (i) the agent's current discounted expected payoff, which we refer to as the agent's "continuation payoff," W ; and (ii) current investment which in turn determines the current capital stock, K . These two state variables, W and K , completely summarize the contract-relevant history of the firm. Moreover, because of the size-homogeneity of our model, the analysis simplifies even further as the contract need only specify the agent's compensation and the level of investment per unit of capital. Consequently the agent's continuation payoff per unit of capital, $w = W/K$, becomes sufficient for the contract-relevant history of the firm.²

Because of the agency problem, investment is below the first-best level. The degree of underinvestment depends on the firm's realized past profitability, or equivalently, the agent's continuation payoff (per unit of capital), w . Specifically, investment is increasing in w , which in turn is increasing in the firm's past profitability as the agent is rewarded (penalized) for delivering high (low) profits. A higher continuation payoff for the agent relaxes the agent's incentive-compatibility constraints since the agent now has a greater stake in the firm (in the extreme, if the agent owned the entire firm there would be no agency problem). Relaxing the incentive-compatibility constraints raises the value of investing in more capital. If profitability is poor and w falls to a lower threshold, the firm is liquidated. Alternatively, if profitability is high and w attains an upper threshold, the firm makes cash payments to the agent. Importantly, as in DeMarzo and Fishman (2007a, b) and DeMarzo and Sannikov (2006), we can interpret the state variable w as a measure of the firm's financial slack.³ More precisely, w is proportional

²The early contributions that developed recursive formulations of the contracting problem include Green (1987), Spear and Srivastava (1987), Phelan and Townsend (1991), and Atkeson (1991), among others. Ljungqvist and Sargent (2004) provide in-depth coverage of these models in discrete-time settings.

³See the aforementioned papers for specific capital structure implementations of the optimal contract in

to the size of the current cash flow shock that the firm can sustain without liquidating, and so can be interpreted as a measure of the firm's liquid reserves.

Our characterization of the optimal agency contract leads to important departures from standard q theory. Because the agent and investors share in the firm's profits, the appropriate measure of the market value of the firm should include the rents to each. That is, if for a given agent continuation payoff per unit of capital, w , we let $p(w)$ denote the value per unit of capital to outside investors, then average q is represented as $p(w) + w$.

A general property of agency problems, ours included, is that increasing the agent's continuation payoff by \$1 costs investors less than \$1; that is $p(w) + w$ is (weakly) increasing in w . This property also follows from the fact that as w increases, the agent's incentive-compatibility constraints are relaxed. This relaxation of the incentive-compatibility constraints leads to greater firm value. So average q is increasing in the agent's stake in the firm. Moreover, combining this result with the fact that the agent's continuation payoff w will, for incentive reasons, be increasing in the firm's past profitability, average q , i.e., $p(w) + w$, increases with the firm's past profitability. This property of agency problems introduces a wedge between average q and marginal q , as increasing the firm's capital stock reduces the agent's effective share of the firm. The magnitude of this wedge varies depending on the firm's realized past profitability, which as we stated above is summarized by the agent's continuation payoff with our optimal contracting framework. Average q and marginal q coincide when either the agent's continuation payoff hits zero and the firm is liquidated or when the agent's continuation payoff is maximized, in which case investment is also maximized. For intermediate levels of the agent's continuation payoff, marginal q lies below average q .

Our model delivers the same linear relation between the investment-capital ratio and marginal q as in Hayashi (1982). But because of the divergence between average q and marginal q , invest-

related settings.

ment is no longer linearly related to average q . Investment is relatively insensitive to average q when average q is low, i.e., when the past profitability has been low and the firm has little financial slack. Conversely, when past profitability and financial slack are high, average q better approximates marginal q , and the sensitivity of investment to average q is high. These results imply that standard linear models of investment on average q are misspecified, and that variables such as financial slack, past profitability, and past investment will be useful predictors of current investment.

To understand the importance of output price fluctuations (an example of observable productivity shocks) on firm value and investment dynamics in the presence of agency conflicts, we extend the model by introducing a serially correlated stochastic output price (our baseline model has a constant output price). In this case, we show that in an optimal contract the agent's payoff will depend on the output price even though the output price is beyond the agent's control. When the output price increases the contract gives the agent a higher continuation payoff. This dependence is optimal because the convex nature of agency costs implies that expected agency costs are minimized by reducing the volatility of the agent's share of future profits.

This result may help to explain the empirical importance of absolute, rather than relative, performance measures for executive compensation. This result also implies that the agency problem generates an amplification of output price shocks. An increase in output price has a direct effect on investment since the higher output price makes investment more profitable. There is also an indirect effect. With a higher output price, it is optimal to offer the agent a higher continuation payoff which, as discussed above, leads to further investment.⁴

Our paper is most closely related to DeMarzo and Fishman (2007a). In the current paper, we

⁴Note that a reduced-form model in which agency costs are simply specified as some function of output price and the other state variables will not generate this amplification result. This is one advantage of fully specifying the agency problem in an investment model.

provide a closer link to the theoretical and empirical macro investment literature. Our analysis is also directly related to other analyses of agency, dynamic contracting and investment, e.g., Albuquerque and Hopenhayn (2004), Quadrini (2004) and Clementi and Hopenhayn (2005). We use the continuous-time recursive contracting methodology developed in DeMarzo and Sannikov (2006) to derive the optimal contract. Philippon and Sannikov (2007) analyze the impact of growth option exercising in a continuous-time dynamic agency environment. The continuous-time methodology allows us to derive a closed-form characterization of the investment Euler equation, optimal investment dynamics, and compensation policies.⁵

Lorenzoni and Walentin (2007) provide a discrete-time industry equilibrium analysis of the relation between investment, average q , and marginal q in the presence of agency problems. Both of our papers build on Hayashi (1982) but differ on the agency side. In Lorenzoni and Walentin (2007), the agent must be given the incentive not to default and abscond with the assets, and it is directly observable whether he complies. Our analysis involves a standard principal-agent problem and whether the agent takes appropriate action is unobservable.

A growing literature in macro and finance introduces more realistic characterizations for firm's investment and financing decisions. These papers often integrate financing frictions such as transaction costs of raising funds, financial distress costs, and tax benefits of debt, with a more realistic specification for physical production technology such as decreasing returns to scale. See Gomes (2001), Cooper and Ejarque (2003), Cooper and Haltiwanger (2006), Abel and Eberly (2005), and Hennessy and Whited (2006), among others, for more recent contributions. For a survey of earlier contributions, see Caballero (2001).

In Section 2, we specify our continuous-time model of investment in the presence of agency costs. In Section 3, we solve for the optimal contract using dynamic programming. In Section

⁵In addition, our analysis owes much to the dynamic contracting models that do not involve the determination of optimal investment, e.g., Biais, Mariotti, Plantin and Rochet (2007), DeMarzo and Fishman (2007b), Tchistyi (2005), Sannikov (2006), He (2007), and Piskorski and Tchistyi (2007).

4, we analyze the implications of this optimal contract for investment and firm value. In Section 5, we consider the impact of output price variability on investment, firm value, and the agent's compensation. Section 6 contains concluding remarks. All proofs appear in the Appendix.

2 The Model

We formulate an optimal dynamic investment problem when the firm suffers from an agency issue. First, we present the firm's production technology. Second, we introduce the agency problem between investors and the agent. Finally, we formulate the optimal contracting problem.

2.1 Firm's Production Technology

Our model is based on a neoclassical investment setting. The firm employs capital to produce output, whose price is normalized to 1 (in Section 5 we consider an extension where the output price is stochastic). Let K and I denote the level of capital stock and gross investment rate, respectively. As in the standard capital accumulation models, we assume that the firm's capital stock K evolves according to

$$dK_t = (I_t - \delta K_t) dt, \quad t \geq 0, \quad (1)$$

where δ is the rate of depreciation. We further assume that the incremental gross output over time interval dt is given by $K_t dA_t$, where A is the cumulative productivity process. We will model the instantaneous productivity dA_t in the next subsection, where we introduce the agency problem.

Investment entails physical adjustment costs. Following the neoclassical investment/adjustment costs literature, we assume that the physical adjustment cost is homogeneous of degree one in investment I and capital stock K . In the main body of this paper, we assume that the adjust-

ment cost takes the following widely used quadratic form (Hayashi (1982)):

$$G(I, K) = \frac{\theta}{2} \frac{I^2}{K}, \quad (2)$$

where the parameter θ measures the degree of adjustment costs. The firm has an “AK” production technology; that is, gross output is proportional to the capital stock K . Accounting for investment and adjustment costs, we may write the dynamics for the firm’s cumulative (gross of agent compensation) cash flow process Y as follows:

$$dY_t = K_t dA_t - I_t dt - G(I_t, K_t) dt, \quad t \geq 0, \quad (3)$$

where $K_t dA_t$ is the incremental gross output. An important focus of our paper is the impact of agency conflicts on optimal investment dynamics.

The homogeneity assumption embedded in the adjustment cost and the “AK” production technology allows us to deliver our key results in a parsimonious and analytically tractable way. We acknowledge that adjustment costs may not be convex and may take other forms, such as fixed costs, and that the production technology may have decreasing returns to scale in capital. While more sophisticated specifications of the adjustment cost and production technology are likely to enrich our analysis, the key intuition on the relation between agency conflicts and investment and firm value, the focus of our analysis, is likely robust to more general specifications of adjustment costs and production technology. We leave extensions incorporating these extensions for future research.

2.2 Agency Conflicts between Investors and the Agent

We now introduce a form of agency conflicts induced by separation of ownership and control. Investment is observable and contractible. But the firm’s investors hire an agent to operate the firm. In contrast to the neoclassical model where the productivity process A is exogenously specified, the productivity process in our model is affected by the agent’s unobservable action.

Specifically, the agent's action $a_t \in [0, 1]$ determines the *expected* changes of the cumulative productivity process A , in that

$$dA_t = a_t \mu dt + \sigma dZ_t, \quad t \geq 0, \quad (4)$$

where $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $\sigma > 0$ is the constant volatility of the cumulative productivity process A . The agent controls the drift, but not the volatility of the process A . We assume that the output (per unit capital) dA_t is observable and contractible.

When the agent takes the action a_t over dt time increment, she enjoys a private benefit $(1 - a_t) \lambda \mu dt$ per unit of the capital stock, where λ is a positive constant. The action can be interpreted as the agent's effort choice; due to the linear cost structure, our framework is equivalent to the binary effort setup where the agent can *shirk*, $a = 0$, or *work*, $a = 1$. Alternatively, as in DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006), we can interpret $1 - a_t$ as the fraction of the cash flow that the agent diverts for his own consumption, with λ equal to the agent's net consumption per dollar diverted from the firm. As we show later, λ captures the minimum level of incentives required to motivate the agent.

The firm can be liquidated at a value lK_t , where $l \geq 0$ is a constant. We assume that liquidation is sufficiently inefficient and generates deadweight losses. We may endogenize the liquidation parameter l via specifications such as costly replacement of the incumbent agent, as in DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006).

Following DeMarzo and Fishman (2007), we assume that investors are risk-neutral with discount rate $r > 0$, and the agent is also risk-neutral, but with a higher discount rate $\gamma > r$. That is, the agent is impatient relative to investors. This assumption avoids the scenario where the investors postpone payments to the agent indefinitely. In practice, the agent may be more impatient than investors for reasons such as liquidity constraint. The agent has no initial

wealth and agent has limited liability. The agent's reservation value, associated with his next best employment opportunity, is normalized to zero.

2.3 Formulating the Optimal Contracting Problem

To maximize firm value, investors specify a dynamic investment policy I and offer an employment contract Π , which contains both the cumulative agent compensation (right-continuous with left-limit) process $\{U_t : 0 \leq t \leq \tau\}$, and the endogenous liquidation time τ . Agent limited liability requires the cumulative compensation process U to be non-decreasing. Each element in the contract Π depends on the history generated by the process A (which reflects the agent's performance). We leave regularity conditions on investment and contracting policies to the appendix.

The agent faces the contract Π , follows the investment policy I , and chooses an action. A contract Π combined with an action process $\{a_t : 0 \leq t \leq \tau\}$ is incentive-compatible if the action process solves the agent's problem:

$$W_0(\Pi) = \max_{a=\{a_t \in [0,1] : 0 \leq t < \tau\}} \mathbb{E}^a \left[\int_0^\tau e^{-\gamma t} (dU_t + (1 - a_t) \lambda \mu K_t dt) \right], \quad (5)$$

where $\mathbb{E}^a(\cdot)$ is the expectation operator under the probability measure that is induced by any action process $a = \{a_t \in [0, 1] : 0 \leq t < \tau\}$. Note that the agent's objective function includes both the present discounted value of compensation (the first term in (5)) and also the potential private benefits from taking action $a_t < 1$ (the second term in (5)).

We focus on the case where it is optimal for investors to implement $a_t = 1$ all the time and provide a sufficient condition for the optimality of implementing this upper-bound action in the appendix. For the remainder of this paper, the expectation operator $\mathbb{E}(\cdot)$ is under the measure induced by $\{a_t = 1 : 0 \leq t < \tau\}$, unless otherwise stated.

At the time the agent is hired, investors have K_0 in capital. The investors' optimization

problem is

$$\begin{aligned} \max_{\Pi \text{ is incentive-compatible, } I} \quad & \mathbb{E} \left[\int_0^\tau e^{-rt} dY_t + e^{-r\tau} lK_\tau - \int_0^\tau e^{-rt} dU_t \right] \\ \text{s.t. } & W_0(\Pi) \geq 0. \end{aligned} \quad (6)$$

The objective is the expected present value of the firm's gross cash flow plus liquidation value less the agent's compensation. The constraint is the agent's participation constraint. In our model, as the agent enjoys a positive rent, the participation constraint will not bind in non-trivial solutions.

3 Model Solution

In this section we solve for the optimal contract and optimal investment policy. As standard in the dynamic agency literature, e.g., Spear and Srivastava (1987), we use dynamic programming to derive the optimal contract. The key state variable in the optimal contract is the agent's continuation payoff. We then utilize the model's scale invariance to solve the investors' problem stated in the previous section.

3.1 The Agent's Continuation Payoff and Incentive Compatibility

First, we introduce the agent's continuation payoff, and provide a key result for any incentive-compatible contract Π . Fix the action process $a = \{a_t = 1 : 0 \leq t < \tau\}$. For any contract Π , define the agent's time- t continuation payoff, which equals the discounted expected value of future compensation:

$$W_t(\Pi) \equiv \mathbb{E}_t \left[\int_t^\tau e^{-\gamma(s-t)} dU_s \right], \quad (7)$$

where τ is the (stochastic) liquidation time.

The following proposition provides the dynamic evolution of the agent's continuation payoff W in terms of the observable incremental productivity performance dA , and supplies the necessary and sufficient condition for any contract Π to be incentive compatible.

Proposition 1 *For any contract $\Pi = \{U, \tau\}$, there exists a progressively measurable process $\{\beta_t : 0 \leq t < \tau\}$ such that the agent's continuation value W_t evolves according to*

$$dW_t = \gamma W_t dt - dU_t + \beta_t K_t (dA_t - \mu dt) \quad (8)$$

under $a_t = 1$ always. The contract Π is incentive-compatible, if and only if $\beta_t \geq \lambda$ for $t \in [0, \tau)$.

Proposition 1 gives a “differential” version of the dynamics for the agent's continuation payoff. Equation (8) is analogous to the equilibrium valuation equation in asset pricing, with the “asset” to be valued being the agent's continuation payoff. To be specific, (8) states that the total (instantaneous) payoff, which includes both the agent's compensation dU_t and the change of the agent's continuation payoff dW_t , is equal to the sum of the predetermined drift part $\gamma W_t dt$, and the diffusion part

$$\beta_t K_t (dA_t - \mu dt) = \beta_t K_t \sigma dZ_t. \quad (9)$$

First, the drift component $\gamma W_t dt$ in (8) reflects that the *expected* (instantaneous) return on the agent's continuation payoff W equals the agent's subjective discount rate γ ; this respects the so-called promise-keeping condition. Second, the diffusion component of the agent's continuation payoff $\beta_t K_t (dA_t - \mu dt)$ links to the action choice, and provides incentives for the agent. Take the interpretation of “shirking-working.” Suppose the agent shirks, $a = 0$. On the one hand, she gains a private benefit $\lambda \mu K_t dt$ per time increment dt . On the other hand, she loses $\mu K_t \beta_t dt$ in W because the productivity process A becomes driftless under shirking. Therefore, she will work, $a = 1$, if and only if the benefit of working exceeds the cost, that is, $\beta_t \geq \lambda$. As a result, for the optimal contract to have provide sufficient incentives, (9) implies that the volatility of the agent's continuation payoff must be sufficiently large and exceed the threshold $\lambda \sigma K_t$.

We will verify later that in the optimal contract $\beta_t = \lambda$. The economics behind this binding result is as follows. The volatility (diffusion) term in the dynamics for the continuation

payoff W implies a positive probability of future inefficient liquidation, which will be triggered once the continuation payoff W hits zero. In the absence of agency conflicts, investors prefer avoiding inefficient liquidation, thus zero volatility in W . However, the presence of agency conflicts requires necessary incentive provision, or large enough volatility in (8). To minimize the probability of future liquidation, while still meet the agent's incentive constraint required by Proposition 1, the optimal contract sets $\beta_t = \lambda$. Intuitively, incentive provisions are costly, and investors should provide just enough incentives to motivate the agent.

Next, we exploit the scale invariance feature of our model to derive the ordinary differential equation (ODE) and associated boundary conditions; they jointly characterize the investors' value function in terms of the agent's continuation payoff.

3.2 Deriving the Optimal Contract using Dynamic Programming

We have two state variables in this problem: the capital stock K and the agent's continuation payoff W . Write the investors' value function as $P(K, W)$, where capital accumulation dynamics are given by (1), and the evolution of the continuation payoff W is

$$dW_t = \gamma W_t dt - dU_t + \lambda \sigma K_t dZ_t \quad (10)$$

(note that we have set $\beta_t = \lambda$ in (8) as we discussed at the end of Section 3.1). Our analysis will heavily rely on the scale invariance property of the investors' value function $P(K, W)$ (homogenous of degree one in K). DeMarzo and Fishman (2007a) and He (2007) have also exploited these features in contracting settings. In the macro literature, the scale invariance property has played an important role. For example, in a seminal contribution, Hayashi (1982) provides conditions under which Tobin's q is equal to the marginal q .

3.2.1 Value Function $P(K, W)$ and the Hamilton-Jacobi-Bellman Equation

We characterize some properties for the investors' value function $P(K, W)$, which is the investors' highest expected future payoff given these two state variables. Note that for any given K and W , it is optimal to maximize the investors' continuation payoff (this would not necessarily be the case if investors were also subject to a moral hazard problem).

First we show that $P_W(K, W) \geq -1$. The intuition is as follows. Investors always can fulfill the agent's continuation payoff by paying the agent with cash. Given $P(K, W)$, paying the agent $\varepsilon > 0$ in cash leaves investors with $P(K, W - \varepsilon) - \varepsilon$. Therefore, investors' value function $P(K, W)$ must satisfy

$$P(K, W) \geq P(K, W - \varepsilon) - \varepsilon,$$

where the inequality describes the implication of the optimality condition. Assuming differentiability, we have $P_W(K, W) \geq -1$. In other words, the marginal cost of compensating the agent must be less than unity, which is the marginal cost of an immediate cash transfer.

Let $\bar{W}(K)$ denote the continuation payoff level that solves

$$P_W(K, \bar{W}(K)) = -1. \quad (11)$$

The above argument implies that it is optimal to pay the agent with cash in the amount of

$$dU = \max(W - \bar{W}(K), 0), \quad (12)$$

where $\bar{W}(K)$ is the optimal cash payment boundary. This standard “bang-bang” control stems from the risk-neutrality of both parties. We call the region where $W > \bar{W}(K)$ as the cash-payment region.

We now turn to the interior “continuation-payoff region” without cash payment, i.e., $dU_t = 0$ when $P_W(K, W) > -1$. Using Ito's lemma,

$$rP(K, W) = \sup_I (\mu K - I - G(I, K)) + (I - \delta K) P_K + \gamma W P_W + \frac{\lambda^2 \sigma^2 K^2}{2} P_{WW}. \quad (13)$$

Intuitively, the right side is given by the sum of instantaneous expected profit (the first term into the bracket), plus the expected change of the instantaneous profit due to capital accumulation (the second term), and the expected change of the instantaneous profit due to the drift and volatility terms in the dynamics for the agent's continuation payoff W . Investment I is optimally chosen to set the right side to equal $rP(K, W)$.

3.2.2 Scale Invariance and Scaled Value Function $p(\cdot)$

The scale invariance property implies that both the optimal investment policy I and the investors' value function $P(K, W)$ are homogeneous of degree one in capital stock K . Based on this fact, we reduce our optimal contracting problem from a two-dimensional free-boundary problem to a one-dimensional problem. Specifically, we conjecture that the investors' value function $P(K, W)$ may be written as

$$P(K, W) = K \cdot p(w), \quad (14)$$

where $w = W/K$ is the agent's scaled continuation payoff, which is the only relevant state variable in our problem. We call the smooth uni-variate function $p(\cdot)$ the investors' scaled value function. Note that $P_K(K, W) = p(w) - wp'(w)$, $P_W(K, W) = p'(w)$, and $KP_{WW} = p''(w)$. Let $i \equiv I/K$ denote the investment capital ratio.

It remains to characterize the scaled investor's value function $p(w)$ and the investment-capital ratio $i(w)$. The first-order condition (FOC) for (13) with respect to I gives $I^* = i(w)K$, where

$$i(w) = \frac{P_K(K, W) - 1}{\theta} = \frac{p(w) - wp'(w) - 1}{\theta}. \quad (15)$$

The above equation states that the marginal cost of investing equals the marginal value of investing from the investors' perspective. Substituting the investment-capital ratio i given in (15) into (13), and utilizing the scale invariance, we obtain the following second-order ODE for

$p(w)$ in the continuation-payoff region (where $p'(w) > -1$):

$$(r + \delta)p(w) = \mu + \frac{(p(w) - wp'(w) - 1)^2}{2\theta} + p'(w)(\gamma + \delta)w + \frac{\lambda^2\sigma^2}{2}p''(w), \quad 0 \leq w \leq \bar{w}. \quad (16)$$

To solve for the scaled investors' value function $p(\cdot)$, we need certain boundary conditions to which we next turn.

Due to scale invariance, the optimal cash payment boundary $\bar{W}(K)$ is linear in capital stock K , that is $\bar{W}(K) = \bar{w}K$, where $\bar{w} > 0$ is to be determined shortly. We have seen the *smooth pasting* condition $P_W(K, \bar{w}K) = -1$ in (11). Because paying cash to reduce W involves a linear cost, we have the standard *super contact* condition $P_{WW}(K, \bar{w}K) = 0$ for the optimality of the boundary control (A. Dixit (1993)). Applying these two conditions to the scaled investors' value function $p(w)$, we obtain

$$p'(\bar{w}) = -1, \quad (17)$$

$$p''(\bar{w}) = 0. \quad (18)$$

And, when $W > \bar{w}K$ so that we are in the cash-payment region, the optimal cash payment policy in (12) states that investors simply pay cash $dU = W - \bar{w}K > 0$ to the agent, i.e.,

$$P(W, K) = P(\bar{w}K, K) - (W - \bar{w}K) \text{ if } W > \bar{w}K.$$

This implies that

$$p(w) = p(\bar{w}) - (w - \bar{w}) \text{ if } w > \bar{w}.$$

Now consider the lower boundary of the agent's continuation payoff. When $W = 0$, the employment relationship terminates, and the firm is liquidated. Therefore, we have $P(K, 0) = lK$, which implies

$$p(0) = l. \quad (19)$$

We now summarize our main results on the optimal contract and optimal investment policy in the following proposition.

Proposition 2 *The investors' value function $P(K, W)$ is proportional to capital stock K , in that $P(K, W) = K \cdot p(w)$, where $p(w)$ is the scaled investor's value function. For $0 \leq w \leq \bar{w}$ (the continuation-payoff region), $p(w)$ and the optimal payment threshold $\bar{w}$ solve the ODE (16), with boundary conditions (17), (18), and (19). For $w > \bar{w}$ (the cash-payment region), $p(w) = p(\bar{w}) - (w - \bar{w})$.*

Under the optimal contract, the agent's scaled continuation payoff w evolves according to

$$dw_t = (\gamma + \delta - i(w_t)) w_t dt + \lambda \sigma (dA_t - \mu dt) - du_t, \quad (20)$$

where the optimal investment policy $i(w)$ is defined in (15), the optimal scaled wage payment $du_t = dU_t/K_t$ reflects w_t back to $\bar{w}$, and the endogenous liquidation time $\tau = \inf \{t \geq 0 : w_t = 0\}$. The capital stock K_t follows $dK_t = (i(w_t) - \delta) K_t dt$, where the optimal investment rate is given by $I_t = i(w_t) K_t$ and $i(w)$ is given in (15).

We provide necessary technical conditions and present a formal verification argument for the optimal policy in the appendix.

4 Model Implications and Analysis

Having characterized the solution, we next analyze the implications of our model. Before analyzing the agency effect, we first provide the solution to the neoclassical investment problem without agency conflicts. We use this neoclassical model as the benchmark to highlight the effects of agency conflicts on optimal investment and firm value.

4.1 Neoclassical Benchmark

In the absence of agency conflicts, i.e., when $\lambda \sigma = 0$, our model specializes to the continuous-time counterpart of Hayashi (1982). This neoclassical investment setting is a widely used benchmark in the literature. The following proposition summarizes the main results on investment and Tobin's q in the neoclassical setting. To ensure that the first-best investment policy

is well defined, we assume the following parametric condition

$$(r + \delta)^2 - 2\frac{\mu - (r + \delta)}{\theta} > 0.$$

Proposition 3 *In the neoclassical setting without agency conflicts, the firm's first-best investment policy is given by $I^{FB} = i^{FB}K$, where*

$$i^{FB} = r + \delta - \sqrt{(r + \delta)^2 - 2\frac{\mu - (r + \delta)}{\theta}}.$$

Firm's value function is $q^{FB}K$, where q^{FB} is Tobin's q and is given by

$$q^{FB} = 1 + \theta i^{FB}.$$

First, the neoclassical model has the certainty equivalence result, in that the volatility of the output process has no impact on the firm's investment decision and firm value under the assumption of risk neutrality. As we will show, agency conflicts invalidate the certainty equivalence result. Second, because of the homogeneity of the production technology ("AK" technology specification and the homogeneity of the adjustment cost function $G(I, K)$ in I and K), marginal q is equal to average (Tobin's) q , satisfying the Hayashi (1982) condition. Third, gross investment I is positive if and only if the marginal productivity μ is higher than $r + \delta$, the marginal cost of investing, in that $\mu > r + \delta$. Whenever investment is positive, Tobin's q is greater than unity in the benchmark model. Intuitively, when the firm is sufficiently productive ($\mu > r + \delta$), the installed capital is more valuable than newly purchased capital. As is standard in the literature, the wedge between installed capital and the newly purchased capital is driven by the adjustment cost.

Now consider the situation in which the firm is run by an agent, but without agency conflicts.

Suppose investors have promised the agent a payoff W in present value,⁶ which is equivalent

⁶When the agent has the same discount rate as investors, the payment timing to deliver W is irrelevant. When the agent is (strictly) more impatient than investors, the optimal way to deliver W is to pay the agent immediately.

to w per unit of capital stock ($W = wK$). Then, the investors' scaled value function is simply given by $p^{FB}(w) = q^{FB} - w$. That is, average q under the first-best benchmark equals the sum of the investors' scaled value function $p^{FB}(w)$ and the agent's scaled continuation payoff w . To be consistent with Hayashi (1982) and our Proposition 3, we include the agent's scaled continuation payoff w in the calculation of average q . Figure 1 plots $p^{FB}(w)$ as the linear decreasing function of the agent's scaled continuation payoff w .

Next, we next analyze the effects of agency conflicts on firm value and investment.

4.2 Investors' Scaled Value Function $p(w)$, Average q , and Marginal q

Financial Slack w The agent's scaled continuation payoff w , the key state variable in our model, reflects the severity of agency conflicts. Intuitively, the higher the value of w , the greater the agent's stake in the firm, and the less severe the incentive misalignment between investors and the agent.

By appealing to DeMarzo and Fishman (2007b) and DeMarzo and Sannikov (2006), we may interpret the agent's scaled continuation payoff w as the firm's financial slack per unit of capital stock, because it reflects the firm's distance to liquidation. That is, as implied by the evolution equation (20) for the agent's scaled continuation payoff w , the firm is more likely to survive a sequence of negative shocks and to avoid eventual liquidation if the current value w is higher, *ceteris paribus*. Therefore, we can view w as the firm's financial slack or liquid reserves (per unit of capital stock), which may be used to buffer a sequence of adverse productivity shocks. For an empirical proxy, financial slack may include the firm's cash balance, line of credit, and other liquid holdings.

Intuitively, the agent receives compensation via cash payments when his (scaled) continuation payoff w , or equivalently interpreted, the firm's (scaled) financial slack, is sufficiently high (greater than the upper-payment boundary $\bar{w}$). On the other hand, when the firm has

less financial slack, the agent's optimal compensation takes the form of deferred payment (via promises to pay in the future).

Investors' Scaled Value Function $p(w)$ Next, we establish the concavity of the scaled investors' value function $p(w)$.

Proposition 4 *The scaled investors' value function $p(w)$ is concave on $[0, \bar{w}]$.*

Figure 1 plots $p(w)$ as a function of the agent's scaled continuation payoff (financial slack) w . The gap between $p(w)$ and $p^{FB}(w) = q^{FB} - w$ reflects the impact of agency conflicts on the loss of investors' value. From Figure 1, we see that the loss of investors' value $p^{FB}(w) - p(w)$ is greater when financial slack w is lower.

[Insert Figure 1 Here]

The concavity of $p(w)$ confirms the intuition that providing incentives is costly, and in the optimal contract the agent has a binding incentive constraint. Interestingly, although investors are risk neutral, they behave effectively in a risk-averse manner even towards idiosyncratic risks due to the agency friction. This property fundamentally differentiates our agency model from the neoclassical (certainty equivalence) result. The dependence of investment and firm value on idiosyncratic volatility in the presence of agency conflicts arise from the investors' inability to fully separate out the agent's action from luck.

While $p(w)$ is concave, it is not monotonic in w , as seen from Figure 1. The intuition is as follows. There are two effects that drive the shape of $p(w)$. First, as illustrated in Section 4.1 where the first-best case is discussed, by holding the total surplus fixed, the higher the agent's claim w , the lower the investors' value $p(w)$. We dub this the wealth transfer effect. Second, incentive alignments from optimal contracting create wealth and hence raise the total surplus available for distribution to both the agent and the investors. Let $\hat{w} = \arg \max p(w)$

for $0 \leq w \leq \bar{w}$. When the agent's continuation payoff w is sufficiently high ($w > \hat{w}$), $p(w)$ is decreasing in w . This corresponds to the situation where the wealth transfer effect dominates the wealth creation effect. However, when the agent's continuation payoff w is sufficiently low ($w < \hat{w}$), the investors' scaled value function $p(w)$ is increasing in w . This maps to the case where the wealth creation effect is stronger than the wealth transfer effect. When the prospect of liquidation is more likely (a lower w), the incremental benefit from incentive alignment becomes larger.

The above wealth creation effect also indicates that liquidation at $w = 0$ serves as an *ex post* inefficient “money burning” mechanism for the purpose of providing better incentives *ex ante*. However, *ex post* inefficient liquidation provides room for renegotiation, as both parties will have incentives to renegotiate to achieve an *ex post* Pareto-improving allocation. This suggests that the optimal contract depicted in Figure 1 is not renegotiation-proof. Later in the Section 4.5, we extend our model to allow for the contract to be renegotiation-proof and discuss the corresponding economic implications.

Average q and Marginal q Firm value, including the claim held by the agent, is $P(K, W) + W$ (recall the discussion in Section 4.1). Therefore, average q , defined as the ratio between firm value and capital stock, is given by

$$q_a(w) = \frac{P(K, W) + W}{K} = p(w) + w.$$

An alternative definition for average q is $p(w)$, excluding the agent's scaled continuation payoff w . However, this definition does not give the prediction that Tobin's q equals average q even in the neoclassical benchmark (Hayashi (1982)) setting. For this reason, we do not use the latter definition.

It is worth pointing out that the above two definitions raise an important implication on the empirical measurement of Tobin's q . Typically, Tobin's q is calculated based on the market

value of the firm, which may partially include the agent's future rent. For instance, firm value includes the manager's equity holding, but excludes the manager's salaries and bonuses, and possibly even executive stock options. Therefore, empirical measures of Tobin's q may typically lie between $p(w) + w$ and $p(w)$. As we will see, even though the relation between the investment-capital ratio $i(w)$ and $p(w) + w$ is increasing, the one between $i(w)$ and $p(w)$ is not. This suggests that different empirical measures of Tobin's q may have a significant impact on the results.

In determining the firm's investment level, the key concept is marginal q , which is the marginal impact of additional capital on firm value:

$$q_m(w) = \frac{\partial (P(K, W) + W)}{\partial K} = P_K(K, W) = p(w) - wp'(w). \quad (21)$$

Naturally, both average q and marginal q are functions of financial slack w . In Figure 2 we plot average q_a , marginal q_m , and the first-best average (also marginal) q^{FB} . Clearly, the average q is always above the marginal q .

[Insert Figure 2 Here]

One of the most well-known results in Hayashi (1982) is that marginal q is equal to average q under a set of conditions (most importantly, the homogeneity assumptions). While our model features homogeneity properties on the production side as in Hayashi (1982), the marginal value of investing differs from the average value of capital stock for investors in our model. To be more precise, using the concavity of $p(w)$, we have

$$q_m(w) = p(w) - wp'(w) \leq p(w) + w = q_a(w).$$

Note that marginal q_m is no greater than average q_a . They coincide only at liquidation ($w = 0$) and at the upper payment boundary $w = \bar{w}$. The intuition for $q_m \leq q_a$ is as follows. An increase of capital stock K lowers the scaled agent's continuation payoff w for a given level of W . In other words, installing an additional unit of capital reduces the agent's effective share of

the firm, which leads to a more severe agency problem. This creates a negative wedge between marginal q_m and average q_a . Lorenzoni and Walentin (2007) derive similar results.

Next, we analyze the effects of agency conflicts on investment-capital ratio $i(w) = I/K$, and highlight the relationship of $i(w)$ with marginal q , average q , and financial slack w . We also provide some linkage of our model's prediction to the empirical literature.

4.3 Investment, Average q , Marginal q , and Financial Slack w

First, note that the investment-capital ratio $i(w)$ under agency depends on financial slack w . We may rewrite (15), the FOC with respect to investment, as follows:

$$1 + \theta i(w) = q_m(w) = p(w) - wp'(w), \quad (22)$$

where the left side is the marginal cost of investing—capital price and adjustment cost—for investors, and the right side is q_m , marginal q defined in (21). The optimal investment policy equates the marginal cost with marginal benefit.

More interestingly, in our model, the investment-capital ratio $i(w)$ increases with financial slack w . This follows from the concavity of $p(w)$ and the FOC (15), in that

$$i'(w) = -\frac{1}{\theta}wp''(w) \geq 0.$$

When financial slack is lower, future inefficient liquidation becomes more likely. Hence, investors optimally adjust the level of investment downward. In one limiting case ($w \rightarrow 0$) where liquidation is immediate, the marginal benefit of investing is just $p(0) = l$. Suppose that $l < 1$, i.e., the liquidation is sufficiently costly. Because the marginal cost of investing when $i = 0$ is 1 (see equation (22)), to balance the marginal benefit with marginal cost, investors will choose to disinvest, i.e., $i(0) = (l - 1)/\theta < 0$.

Now consider the other limiting case, when the financial slack w reaches its upper endoge-

nous payout boundary $\bar{w}$. We have

$$i(\bar{w}) = \frac{p(\bar{w}) - \bar{w}p'(\bar{w}) - 1}{\theta} = \frac{p(\bar{w}) + \bar{w} - 1}{\theta}.$$

Even at this upper boundary, we can show that $i(\bar{w}) < i^{FB}$, which is the first-best investment-capital ratio in the neoclassical setting.⁷ The reason is that the strict relative impatience of the agent creates a strictly positive wedge between our solution and the first-best result. In the limit, when γ is sufficiently close to r , the difference between $i(\bar{w})$ and i^{FB} approaches zero.

Therefore, in addition to costly liquidation as a form of underinvestment, the investment/capital ratio is always lower than i^{FB} . That is, our model features underinvestment at all times. Figure 3 shows the monotonically increasing relationship between investment-capital ratio $i(w)$ and financial slack w , with $i(w)$ staying below the first-best level i^{FB} always.

[Insert Figure 3 here]

Our model's prediction that investment increases with financial slack w is consistent with the prediction based on static models with exogenously specified financing constraints, such as the one proposed in FHP (1988), and summarized by Hubbard (1998) and Stein (2003). It is worth pointing out that our dynamic agency model does not yield sharp prediction on the sensitivity of di/dw with respect to financial slack w . That is, it is very difficult to sign d^2i/dw^2 and other higher-order sensitivity measures, consistent with predictions based on static models with exogenously specified financing constraints (Kaplan and Zingales (1997)).

As in the standard q theory of investment, in our model the marginal cost of investing is equated to the marginal benefit of investing, marginal q ; and as in models with quadratic

⁷At the boundary $\bar{w}$, we may write (16) as follows:

$$(r + \delta)\bar{q} = \mu + \frac{(\bar{q} - 1)^2}{2\theta} - (\gamma - r)\bar{w},$$

where we denote $\bar{q} = p(\bar{w}) + \bar{w}$. Comparing with the quadratic equation for q^{FB} , the first-best Tobin's q , $(r + \delta)q^{FB} = \mu + \frac{(q^{FB} - 1)^2}{2\theta}$, and $\gamma > r$ and $\bar{w} > 0$, we conclude $\bar{q} < q^{FB}$, and hence $i(\bar{w}) < i^{FB}$.

adjustment costs, the marginal cost of investing is linear in $i(w)$ here (see equation (22)). In other words, by invoking the definition of marginal q in (21), we can rewrite equation (22) as follows:

$$i(w) = \frac{q_m(w) - 1}{\theta}. \quad (23)$$

Therefore, as in Hayashi (1982), our model predicts a linear relationship between investment and marginal q . The top panel in Figure 4 plots this linear relationship.

Because marginal q is hardly measurable in practice, empiricists often use average q as a proxy for marginal q . In the second panel of Figure 4, we plot the investment-capital ratio as a function of average q . Recall that both investment-capital ratio $i(w)$ and average q , defined as $q_a = p(w) + w$, increase with financial slack w . Hence, investment-capital ratio is also monotonically increasing in average q . However, even though the relationship between investment-capital ratio and marginal q is linear, due to the state-contingent wedge between average q and marginal q illustrated in Figure 2, the relationship between investment-capital ratio and average q is no longer linear. Moreover, investment is much less sensitive to average q , when financial slack is low and the firm is “financially constrained.” Conversely, when the firm has a high level of financial slack and is relatively “financially unconstrained,” investment is more sensitive to average q .

In the empirical literature, researchers often regress investment-capital ratio on average q and empirical proxies for financial slack, such as cash flow or cash holdings. Next, we study our model’s implication on the impact of financial slack w on investment-capital ratio *after controlling for average q* . Our control for average q is based on the neoclassical analysis of Hayashi (1982). That is, under the neoclassical setting, the part of investment not explained by average q , is

$$\hat{i}(w) = i(w) - \frac{q_a(w) - 1}{\theta}. \quad (24)$$

In our model, this residual part $\hat{i}(w)$ is correlated with financial slack w , since average q serves as a potentially poor proxy for marginal q in our model. In the bottom panel of Figure 4, we plot $\hat{i}(w)$ defined in (24) as a function of financial slack w . Interestingly, we find that the relationship between $\hat{i}(w)$ —the part of investment unexplained by average q —and financial slack w is not even monotonic. When financial slack is high, investment-capital ratio increases with financial slack, after controlling for average q . In contrast, investment-capital ratio decreases with financial slack when financial slack is low, after controlling for average q . This result suggests that it is very difficult, if not impossible, to interpret regression coefficients for various proxies of financial slack in the investment-cash flow sensitivity analysis. Interestingly, in Table X in Kaplan and Zingales (1997), the authors report that after controlling for average q , the coefficient for cash holdings is negative for “financially constrained” firms, and is positive for “likely not financially constrained” firms. These results are in line with our model’s predictions.

[Insert Figure 4 here]

Next, we perform some comparative static analysis of $p(w)$ and $i(w)$ with respect to volatility and agency parameters in the model.

4.4 Comparative Static Analysis

We focus on the comparative static results with respect to two key parameters regarding agency frictions: λ and σ . A higher value of λ implies more private benefits for the agent to misbehave ($a < 1$), which suggests a more severe agency problem. A more volatile project makes the agent’s performance less informative, and the incentive provision becomes more difficult, which in turn leads to lower value for more volatile projects. That is, both λ and σ have implications on the severity of agency conflicts.⁸ The top left and top right graphs in Figure 2 confirm

⁸In fact, the agent’s incentive loadings are $\lambda\sigma$ in the ODE (16), which immediately implies that the comparative static analyses with respect to λ and σ have the same directional results.

the intuition that the investors' scaled value $p(w)$ decreases with both volatility σ and agency parameter λ .

The lower left and lower right graphs show that the underinvestment problem is more severe when λ is higher or σ is larger. This is consistent with the intuition that the incentive to underinvest is greater when agency frictions are larger (i.e. when λ or σ is larger), because the marginal benefit of investing is lower.

[Insert Figure 5 Here]

4.5 Renegotiation-proof Contract

In this section, we analyze the impact of renegotiation in our model. As we have indicated in Section 4.2, our contract is not renegotiation-proof. Intuitively, whenever $p'(w) > 0$, both parties may achieve an *ex post* Pareto-improving allocation by renegotiating the contract. Therefore, the value function $p(w)$ that is renegotiation-proof must be weakly decreasing in the agent's scaled continuation payoff w .⁹

We construct the renegotiation-proof contract using some insights similar to those from DeMarzo and Fishman (2007b) and DeMarzo and Sannikov (2006). The investors' renegotiation-proof scaled value function $p^{RP}(w)$ is non-increasing and concave. Moreover, $p^{RP}(w)$ has an (endogenous) renegotiation boundary $\underline{w}^{RP}$, where the scaled investors' value function $p^{RP}(w)$ has the following boundary conditions:

$$p^{RP}(\underline{w}^{RP}) = l, \tag{25}$$

$$p^{RP'}(\underline{w}^{RP}) = 0. \tag{26}$$

Specifically, $\underline{w}^{RP}$ (rather than $w = 0$ in the baseline dynamic agency model) becomes the lower bound for the agent's scaled continuation payoff w during the equilibrium employment path.

⁹Note that the renegotiation-proofness requires $P_W(W, K) \leq 0$; but due to the scale invariance $p'(w) = P_W(W, K)$, it is equivalent to require $p(w) \leq 0$.

The scaled investors' value function $p^{RP}(w)$ solves the ODE (16) for $w \in [\underline{w}^{RP}, \bar{w}^{RP}]$, with two sets of free-boundary conditions: one is the boundary conditions (17) and (18) at the payout boundary $\bar{w}^{RP}$, and the other is the boundary conditions (25) and (26) at the renegotiation boundary $\underline{w}^{RP}$.

The dynamics of the scaled agent's payoff w takes the following form:

$$dw_t = (\gamma + \delta - i(w_t)) w_t dt + \lambda \sigma dZ_t - du_t + (dv_t - \underline{w}^{RP} dM_t), \quad (27)$$

where the first (drift) term implies that the expected rate of change for the agent's scaled continuation payoff w is $(\gamma + \delta - i(w))$, the second (diffusion) term captures incentive provisions in the continuation-payoff region (away from the boundaries), and the (third) nondecreasing process u captures the reflection of the process w at the upper payment boundary $\bar{w}^{RP}$. Unlike the dynamics (20) for the agent's scaled payoff process w without renegotiation, the last term $dv_t - \underline{w}^{RP} dM_t$ in dynamics (27) captures the effect at the renegotiation boundary. The nondecreasing process v reflects w at the renegotiation boundary $\underline{w}^{RP}$. The intensity of the counting process dM is $dv_t/\underline{w}^{RP}$; and once $dM = 1$, w becomes 0, and the firm is liquidated.¹⁰ Note that the additional term $dv_t - \underline{w}^{RP} dM_t$ is a compensated Poisson process, and hence a martingale increment.

We illustrate the contracting behavior at the renegotiation boundary through the following intuitive way. When the agent's poor performance drives w down to $\underline{w}^{RP}$, the two parties run a lottery. With a probability of $dv_t/\underline{w}^{RP}$, the firm is liquidated. If the firm is not liquidated, the agent stays at the renegotiation boundary $\underline{w}^{RP}$. Here, the stochastic liquidation is to achieve the "promise-keeping" constraint so that w is indeed the scaled continuation payoff with expected growth rate $\gamma + \delta - i(w)$ as specified in Proposition 2. To see this, by running this lottery, the agent could potentially lose $(dv_t/\underline{w}^{RP}) \cdot \underline{w}^{RP} = dv_t$, which just compensates the reflection gain

¹⁰Technically speaking, the counting process has a survival probability $\Pr(M_t = 0) = \exp(-v_t/\underline{w}^{RP})$.

dv_t if the firm is not liquidated.

Compared with the value function $p(w)$ where investors can commit not to renegotiate, the renegotiation-proof contract delivers a lower value, as Figure 6 shows. This is the standard result that the investors' inability to commit not to renegotiate lowers their value. Since renegotiation further worsens the agency conflict, intuitively we expect not only a greater value reduction for investors, but also a stronger underinvestment distortion. The right panel in Figure 6 shows the impact of renegotiation on underinvestment is greater, consistent with our intuition.

[Insert Figure 6 Here]

Next, we extend our baseline model of Section 2 to a setting where output price is stochastic. This generalization allows us to analyze the interaction effect of incentive provision and the firm's investment opportunities.

5 A Generalized Model with Stochastic Output Price

5.1 Model Setup

For analytical tractability reasons, we choose a two-state regime-switching process to model the output price V_t .¹¹ Let $S_t \in \{1, 2\}$ denote the regime at time t . In each regime, the corresponding output price V_t can be either high or low, in that $V_t \in \{v_1, v_2\}$ with $v_2 > v_1$. Let ξ_n denote the transition intensity out of regime $n = 1, 2$. For example, the conditional probability that the price changes from v_1 to v_2 over a small time interval dt , is $\xi_1 dt$. Let $P(K, W, n)$ denote the investors' value function, given the capital stock K and the agent's continuation payoff W , when the output price V_t is v_n with $n = 1, 2$.

The firm's operating profit is given by the following dynamics:

$$dY_t = V_t K_t dA_t - I_t dt - G(I_t, K_t) dt, \quad t \geq 0, \quad (28)$$

¹¹Hamilton (1989) uses regime switching models to model business cycle effects, an early application of regime switching models in economics. Piskorski and Tchisty (2007) use this process to model the discount rate in studying mortgage design.

Let N_t denote the cumulative number of regime changes up to time t . For expositional purpose, suppose that the current output price is v_1 . Based on a martingale representation argument, the dynamics for the agent's continuation payoff W in regime 1 is then given by

$$dW_t = \gamma W_t dt - dU_t + \lambda K_t (dA_t - \mu dt) + \Psi(K_t, W_t, 1) (dN_t - \xi_1 dt), \quad (29)$$

where $\Psi(K_t, W_t, 1)$ will be endogenously determined in the optimal contract. As in the baseline model, the diffusion martingale term $\lambda K_t (dA_t - \mu dt)$ describes the agent's binding incentive constraint, implied by the concavity of investors' scaled value functions in both regimes (see the Appendix).¹²

Unlike our baseline dynamic agency model, the compensated Poisson martingale process $\Psi(K_t, W_t, 1) (dN_t - \xi_1 dt)$ captures the impact of exogenous price shocks on the agent's continuation payoff W . Recall that we are in regime 1 with output price v_1 . When the price exogenously switches to $v_2 > v_1$, the agent's continuation payoff W also changes by a discrete amount $\Psi(K_t, W_t, 1)$. Naturally, we may write down a dynamic evolution equation similar to (29) for the agent's continuation payoff W in regime 2.

Importantly, the optimal contract assigns non-zero value to $\Psi(K, W, 1)$ in general (and similar mechanisms hold for regime 2). In our model, the marginal impact of compensating the agent on the investors' value—which is $\partial P / \partial W$ —will depend on the output price. In the appendix, we show that, via the discrete changes Ψ 's, it is optimal to adjust the agent's continuation payoff in such a way that, when the price regime switches, the marginal impacts ($\frac{\partial P}{\partial W}$'s) across two regimes are equated (if possible). That is, in the “interior” region, investors choose the discrete change $\Psi(K, W, 1)$ so that the marginal impacts of compensating the agent—

¹²The incentive provision $\lambda K_t (dA_t - \mu dt)$ does not scale with output price V_t . This treatment is consistent with our current interpretation of moral hazard, as the agent's shirking benefit is assumed to be independent of the output price.

before and after the regime changes—are equalized:

$$P_W(K, W, 1) = P_W(K, W + \Psi(K, W, 1), 2).$$

This implies that in the optimal contract, the agent’s compensation will depend on the exogenous price shocks, indicating that *absolute*, rather than *relative*, performance evaluation might be optimal. In the next section we will come back to this point under a more concrete example after Figure 7.

The solution technique is similar to the one for the baseline dynamic agency model. The scale invariance remains here: with $w = W/K$, we let $p_n(w) = P(K, W, n)/K$, $i_n(w) = I(K, W, n)/K$, $\psi_n(w) = \Psi(K, W, n)/K$, and upper payment boundary $\bar{w}_n = \bar{W}(K, n)/K$. That is, $p_n(w)$ is the scaled investors’ value function in regime n , $i_n(w)$ is the investment-capital ratio in regime n , $\psi_n(w)$ is the scaled “additional” compensation (per unit of capital stock) when the output price switches out of regime n , and $\bar{w}_n$ is the scaled upper payment boundary. In the appendix, we provide a formal characterization of the optimal contract.

5.2 Model Implications

We now illustrate our model’s economic implications. For expositional purposes, we set the conditional transition probability from one regime to the other to be equal (i.e., $\xi_1 = \xi_2 = 0.1$); and the liquidation value in both regimes to be the same (i.e., $l_1 = l_2 = 0$).

Investors’ value function The upper panel of Figure 7 plots the investors’ scaled value functions $p_n(w)$ in both regimes. As the liquidation values are the same in these two regimes (by assumption $l_1 = l_2$), the scaled investors’ value functions are equal at liquidation, i.e., $p_1(0) = p_2(0)$. Second, the investors’ scaled value function is higher when output price is higher (regime 2), for all levels of w , in that $p_2(w) \geq p_1(w)$. This result holds in general, provided that $l_2 \geq l_1$. The firm is at least as productive in regime 2 as in regime 1 at all times (including the

liquidation scenario); and therefore investors' value will be higher in regime 2, all else equal. Third, both $p_1(w)$ and $p_2(w)$ are concave in w , as in our baseline model. The intuition for the concavity of $p_n(w)$ is essentially the same as in our baseline dynamic agency model.

[Insert Figure 7 Here]

Discrete change of the agent's continuation payoff upon regime switch The lower panel of Figure 7 plots $\psi_1(w)$ and $\psi_2(w)$, the discrete changes of the agent's continuation payoff upon regime switching. We find that the discrete change $\psi_1(w)$ of the scaled agent's continuation payoff is positive for all levels of w . That is, the agent will be rewarded when the output price increases from low to high. Note that the output price change is exogenous and beyond the agent's control.

The intuition behind this property of the compensation policy is as follows. In designing the optimal contract, investors have the discretion to make the agent's continuation payoff regime dependent, if doing so is cheaper for investors. Then we may ask the following question: Given that investors have to deliver one dollar of continuation payoff, what is their marginal cost to do so? Up to a minus sign, this marginal cost is exactly captured by the marginal impact of W to the investors' value function; or, it is $-\partial P(K, W, n)/\partial W = -p'_n(w)$ due to scale invariance. This quantity might be positive, as investors can actually gain by raising w for small w (recall the wealth-creation effect discussed in Section 4.2).

Under this interpretation, intuitively, it is cheaper to compensate the agent in regime 2 ($-p'_2(w) \leq -p'_1(w)$), as a higher output price implies a higher productivity. As a result, if output price increases (switching into regime 2), investors adjust *upward* the agent's scaled continuation payoff to the level at which the marginal cost of delivering compensation are equated before and after the regime switch, in that $p'_1(w) = p'_2(w + \psi_1(w))$. Based on a similar reason, $\psi_2(w)$, i.e., the discrete change of w when the output price decreases from high to low,

is negative. Finally, when w is low and the output price is high, the output price drop might trigger an immediate liquidation. In the figure, we see this result at the left end of $\psi_2(w)$.¹³

That the agent's optimal compensation may depend on the exogenous output price is opposite to the conventional wisdom about relative performance evaluation for executive compensation. In our model, it is cheaper to compensate the agent in the high-price state; therefore the optimal contract gives more compensation to the agent when the output price increases. As a result, a certain degree of *absolute* performance evaluation becomes optimal, which might help explain the empirical observation that absolute performance measures are sometimes used in executive compensation.

Interaction between Compensation and Investment upon Regime Switch Now we explore the interesting interaction between compensation and investment policy at the moment when the output price changes. The left and right graphs in Figure 8 plot the corresponding changes of investment-capital ratio when output price increases from v_1 to v_2 and decreases from v_2 to v_1 , respectively.

[Insert Figure 8 Here]

First, consider the left panel. The solid line corresponds to the total change of investment-capital ratio when output price drops, i.e., $i_2(w + \psi_1(w)) - i_1(w)$. To understand the impact of regime change on investment, we decompose the total change $i_2(w + \psi_1(w)) - i_1(w)$ into two components: the direct and the indirect effects. Holding w fixed when the regime changes, $i_2(w) - i_1(w)$ measures the *direct* effect of regime switch on the investment-capital ratio. The

¹³That the liquidation values in the two regimes are equal ($l_1 = l_2$) plays a role in this result. If the liquidation value is much higher in the high output price state than the low output price state, i.e., $l_2 = p_2(0) \gg l_1 = p_1(0)$, liquidation in the high output state recovers much greater value than in the low output state. Therefore, it is possible to have $p'_2(0) < p'_1(0)$. In this case, when w is close to zero, an immediate liquidation may occur when the output price switches from low to high. To understand this, when liquidation in the high output price state is less costly (high l_2), investors are less averse to liquidating the firm in the high price state.

dashed line depicts this direct effect. However, the optimal compensation policy specifies that the agent's scaled continuation payoff will change by a discrete amount $\psi_1(w)$ when the output price increases from v_1 to v_2 . Therefore, $i_2(w + \psi_1(w)) - i_2(w)$ measures the *indirect* effect of output price change on the investment-capital ratio due to the upward adjustment of the agent's continuation payoff. Adding these two components gives the total effect of regime change on investment.

Interestingly, when the output price jumps up, the “direct effect” understates the impact of regime switch on investment-capital ratio, because the additional upward adjustment of compensation ($\psi_1(w) > 0$) further enlarges the size of incremental investment. Put differently, the indirect effect of changes in the agent's continuation payoff (for incentive reasons due to regime change) further enhances investment, in that $i_2(w + \psi_1(w)) > i_2(w)$. Note that we use the positive relation between investment-capital ratio $i_2(w)$ and w .

Similarly, a drop in output price has both the direct and indirect effects on the investment-capital ratio. The investment-capital ratio decreases when the output price decreases from v_2 to v_1 , as the solid line in the right graph shows. Again, here the “indirect” effect magnifies the *negative* impact on investment, as investors reduce investment even further when they optimally lower the agent's continuation payoff after the output price drops (i.e. $\psi_2(w) < 0$) again for incentive reasons. The indirect effect vanishes when the agent's continuation payoff is high, as we see from these two panels. Intuitively, the impact of financial slack on investment decreases when financial slack is high.

This graph shows that dynamic agency amplifies the response of investment to output price fluctuations. Intuitively, when the output price increases, agency conflicts become less severe, and hence the agent's compensation is increased, which lowers the cost of external financing. As a result, investment increases for both enhanced productivity and also reduced agency conflicts. This additional agency channel may potentially play an important role in amplifying and

propagating output price shocks and contributing to the business cycle fluctuations. Bernanke and Gertler (1989), and Kiyotaki and Moore (1997) have used different agency frictions in their study of equilibrium propagation and amplification mechanisms.

Investment and financial slack for a given average q In Section 4.3, we have studied the relation among investment, Tobin’s q , and financial slack. In the baseline model, the only heterogeneity across firms is caused by agency issues, which is summarized by the firm’s financial slack w . However, another potentially important dimension of heterogeneity comes from the firm’s profitabilities. Our extension captures this feature by allowing for stochastic output prices. In fact, this two-factor setup allows us to investigate the relation between investment and financial slack, after controlling for firm value (recall that in the baseline model without output price heterogeneity, once q is fixed, investment and financial slack are determined.)

Consider two firms with the same average q , but facing different (high and low) output prices and with different degrees of financial slack. To have the same values of q for the two firms, the firm facing the higher output price will necessarily has less financial slack. Let δw denote the corresponding difference of financial slack between the two firms given the value of average q . We calculate δi , the implied difference between the investment-capital ratios for the two firms. In Figure 9, we plot both δi and δw for a given value of average q . Our model predicts that the firm with more financial slack will have a higher investment-capital ratio, holding average q fixed.

6 Conclusions

This paper integrates the impact of dynamic agency into a neoclassical model of investment (Hayashi (1982)). Using continuous-time recursive contracting methodology, we characterize the impact of dynamic agency on firm value and the optimal investment dynamics. Agency

costs introduce a history-dependent wedge between marginal q and average q . Even under the assumptions which imply homogeneity (e.g. constant returns to scale and quadratic adjustment costs of Hayashi (1982)), investment is no longer linearly related to average q . Investment is relatively insensitive to average q when the firm is “financially constrained.” Conversely, investment is sensitive to average q when the firm is relatively “financially unconstrained.” Moreover, the agent’s optimal compensation takes the form of future claims on the firm’s cash flows when the firm has less financial slack, whereas cash compensation is preferred when the firm has been profitable and the firm is growing rapidly.

To understand the potential importance of output price fluctuations on firm value and investment dynamics in the presence of agency conflicts, we further extend our model to allow for the output price to vary stochastically over time. We find that investment increases with financial slack after controlling for average q . The agent’s compensation will depend not only on the firm’s realized productivity, but also on realized output prices, even though output prices are beyond the agent’s control. This result may help to explain the empirical relevance of absolute performance evaluation. Moreover, this result on compensation also suggests that the agency problem provides a channel through which the response of investment to output price shocks is amplified and propagated. A higher output price encourages investment for two reasons. First, investment becomes more profitable. Second, the optimal compensation contract rewards the agent with a higher continuation payoff, which in turn relaxes the agent’s incentive constraints and hence further raises investment.

Appendices

A Proof of Proposition 1

We impose the usual regularity condition on the payment policy

$$\mathbb{E} \left(\int_0^\tau e^{-\gamma s} dU_s \right)^2 < \infty. \quad (\text{A.1})$$

Then, given any contract $\Pi = \{U, \tau\}$, define the process $V_t \equiv \mathbb{E}_t \left[\int_0^\tau e^{-\gamma s} dU_s \right]$ for $t \in [0, \tau)$ as the agent's value process. Under (A.1), $\{V_t : 0 \leq t < \tau\}$ forms a square-integrable martingale until τ . According to the *Martingale Representation Theorem*, there exists a progressively measurable process $\{\beta_t : 0 \leq t < \tau\}$ s.t. $V_t = V_0 + \int_0^t e^{-\gamma s} K_s \beta_s \sigma dZ_s$ for $\forall t \in [0, \tau)$. Hence under the presumption $\{a_t = \mu : 0 \leq t < \tau\}$ we have

$$V_t = V_0 + \int_0^t e^{-\gamma s} K_s \beta_s (dA_t - \mu dt) \text{ for } \forall t \in [0, \tau),$$

by replacing the Brownian increment dZ_s with $\frac{1}{\sigma} (dA_t - \mu dt)$. Now due to the definition of W , $V_t = \int_0^t e^{-\gamma s} dU_s + e^{-\gamma t} W_t$. By taking derivative on both sides, we obtain W 's evolution.

We show that Π is incentive-compatible if and only if $\beta_t \geq \lambda$ a.e.. Consider any action policy $a = \{a_t \in \{0, \mu\} : 0 \leq t < \tau\}$. For $t < \tau$ her associated value process is

$$V_t(a) = V_0 + \int_0^t e^{-\gamma s} K_s \beta_s (dA_s(a) - \mu ds) + \int_0^t e^{-\gamma s} \lambda K_s (\mu - a_s) ds.$$

We have,

$$\begin{aligned} dV_t(a) &= e^{-\gamma t} K_t \beta_t ((a_t - \mu) dt + \sigma dZ_t) + e^{-\gamma t} \lambda K_t (\mu - a_t) dt \\ &= e^{-\gamma t} K_t (\beta_t - \lambda) (a_t - \mu) dt + e^{-\gamma t} K_t \beta_t \sigma dZ_t. \end{aligned}$$

If $\beta_t \geq \lambda$, then it has a non-positive drift, and is a martingale if $\{a_t = \mu : 0 \leq t < \tau\}$. If there is a positive probability event that $\beta_t \geq \lambda$ during $[0, \tau)$, the agent will deviate to $a_t = 0$, and $\{a_t = \mu : 0 \leq t < \tau\}$ is suboptimal. Therefore Π is incentive-compatible if and only if $\beta_t \geq \lambda$ a.e. Q.E.D.

B Proof of Proposition 2

The evolution of $w = \frac{W}{K}$ follows easily from the evolutions of W and K . Here we verify that the contract and the associated investment policy derived from the Hamilton-Jacobi-Bellman equation are indeed optimal. Similar to technical conditions in dynamic portfolio theory, certain conditions are placed for the well-behavedness of the problem. In addition to (A.1), we require that

$$\mathbb{E} \left[\int_0^T (e^{-rt} K_t)^2 dt \right] < \infty \text{ for all } T > 0. \quad (\text{B.1})$$

and

$$\lim_{T \rightarrow \infty} \mathbb{E} (e^{-rT} K_T) = 0. \quad (\text{B.2})$$

Both regularity conditions place certain restrictions on the investment policies.¹⁴ Since the project is terminated at τ , throughout we take the convention that $M_T \mathbf{1}_{\{T > \tau\}} = M_\tau$ for any random process M .

Take any incentive-compatible contract $\Pi = \{U, \tau\}$ and some investment policy. For any $t \leq \tau$, define its auxiliary gain process $\{G\}$ as

$$\begin{aligned} G_t(\Pi) &= \int_0^t e^{-rs} (dY_s - dU_s) + e^{-rt} P(K_t, W_t) \\ &= \int_0^t e^{-rs} \left(K_s dA_s - I_s ds - \frac{\theta I_s^2}{2K_s} - dU_s \right) + e^{-rt} P(K_t, W_t), \end{aligned} \quad (\text{B.3})$$

where the agent's continuation payoff W_t evolves according to (8). Under the optimal contract Π^* , the associated optimal continuation payoff W_t^* has a volatility $\lambda \sigma K_t$, and $\{U^*\}$ reflects W_t^* at $\bar{W}_t^* = \bar{w} K_t$.

¹⁴Note that under our optimal policy,

$$\frac{dK}{K} = (i(w) - \delta) dt < (i^{FB} - \delta) dt$$

and $K_T < K_0 e^{(i^{FB} - \delta)T}$ for $T < \tau$. But since $i^{FB} < r + \delta$, the above two conditions hold.

Recall that $w_t = W_t/K_t$ and $P(K_t, W_t) = K_t p(w_t)$. Ito's lemma implies that, for $t < \tau$,

$$e^{rt} dG_t = K_t \left\{ \left[\begin{aligned} & -rp(w_t) + \mu - I_t/K_t - \frac{\theta}{2} (I_t/K_t)^2 + (I_t/K_t - \delta) (p(w_t) - w_t p'(w_t)) \\ & + \gamma w_t p'(w_t) + \frac{\beta_t^2 \sigma^2}{2} p''(w_t) \\ & + [-1 - p'(w_t)] dU_t/K_t + \sigma [1 + \beta_t p'(w_t)] dZ_t \end{aligned} \right] dt \right\}.$$

Now, let us verify that, under any incentive-compatible contract Π , $e^{rt} dG_t(\Pi)$ has a non-positive drift, and zero drift for the optimal contract and its associated optimal investment policy. Focus on the first piece. Optimization with respect to I_t/K_t gives the investment policy stated in (15); and because $p''(w_t) \leq 0$, setting $\beta_t = \lambda$ maximizes the objective given the restriction that Π is incentive-compatible. Under these two optimal policies, the first piece—which is just our (16)—stays at zero always; and other investment policies and incentives provision will make this term nonpositive. The second piece captures the optimality of the cash payment policy. It is nonpositive since $p'(w_t) \geq -1$, but equals zero under the optimal contract.

Therefore, for the auxiliary gain process we have

$$dG_t(\Pi) = \mu_G(t) dt + e^{-rt} K_t \sigma [1 + \beta_t p'(w_t)] dZ_t,$$

where $\mu_G(t) \leq 0$. Let $\varphi_t \equiv e^{-rt} K_t \sigma [1 + \beta_t p'(w_t)]$. The condition (A.1) and the related argument in the proof for Proposition 1—combining with the condition B.1—imply that $\mathbb{E} \left[\int_0^T \varphi_t dZ_t \right] = 0$ for $\forall T > 0$ (note that p' is bounded). And, under Π the investors' expected payoff is

$$\tilde{G}(\Pi) \equiv \mathbb{E} \left[\int_0^\tau e^{-rs} dY_s - \int_0^\tau e^{-rs} dU_s + e^{-r\tau} l K_\tau \right].$$

Then, given any $t < \infty$,

$$\begin{aligned} \tilde{G}(\Pi) &= \mathbb{E}[G_\tau(\Pi)] \\ &= \mathbb{E} \left[G_{t \wedge \tau}(\tilde{\Pi}) + \mathbf{1}_{t \leq \tau} \left[\int_t^\tau e^{-rs} dY_s - e^{-rs} dU_s \right] + e^{-r\tau} l K_\tau - e^{-rt} P(K_t, W_t) \right] \\ &= \mathbb{E} \left[G_{t \wedge \tau}(\tilde{\Pi}) \right] + e^{-rt} \mathbb{E} \left\{ \left[\int_t^\tau e^{-r(s-t)} (dY_s - dU_s) + e^{-r(\tau-t)} l K_\tau - P(K_t, W_t) \right] \mathbf{1}_{t \leq \tau} \right\} \\ &\leq G_0 + (q^{FB} - l) \mathbb{E}[e^{-rt} K_t]. \end{aligned}$$

The first term of third inequality follows from the negative drift of $dG_t(\Pi)$ and martingale property of $\int_0^{t \wedge \tau} \varphi_s dZ_s$. The second term is due to the fact that

$$\mathbb{E}_t \left[\int_t^\tau e^{-r(s-t)} (dY_s - dU_s) + e^{-r(\tau-t)} l K_\tau \right] \leq q^{FB} K_t - w_t K_t$$

which is the first-best result, and

$$q^{FB} K_t - w_t K_t - P(K_t, W_t) < (q^{FB} - l) K_t$$

as $w + p(w)$ is increasing ($p' \geq -1$). But due to (B.2), we have $\tilde{G} \leq G_0$ for all incentive-compatible contract. On the other hand, under the optimal contract Π^* the investors' payoff $\tilde{G}(\Pi^*)$ achieves G_0 because the above weak inequality holds in equality when $t \rightarrow \infty$. Q.E.D.

Finally, we require that the agent's shirking benefit $\phi \equiv \lambda\mu$ be sufficiently small to ensure the optimality of $a = 1$ (working) all the time. Similar to DeMarzo and Sannikov (2007) and He (2007), there is a sufficient condition for the optimality of $a = \{\mu\}$ against $a_t = 0$ for some t (shirking sometimes). Let $\hat{w} = \arg \max_w p(w)$, and we require that

$$\frac{(p(w) - wp'(w) - 1)^2}{2\theta} \leq (r + \delta) p(w) - p'(w) [(\gamma + \delta)w - \phi] \text{ for all } w$$

Since the left side is increasing in w , and right side dominates $p\left(\frac{\phi}{\gamma + \delta}\right) - \frac{\gamma - r}{r + \delta} \left(p(\hat{w}) - p\left(\frac{\phi}{\gamma + \delta}\right)\right)$ (see the proof in DeMarzo and Sannikov (2006)), a sufficient condition is

$$\frac{(p(\bar{w}) + \bar{w} - 1)^2}{2\theta} \leq p\left(\frac{\phi}{\gamma + \delta}\right) - \frac{\gamma - r}{r + \delta} \left(p(\hat{w}) - p\left(\frac{\phi}{\gamma + \delta}\right)\right).$$

C Proof of Proposition 4

First of all, by differentiating (16) we obtain

$$(r + \delta) p' = -\frac{(p - wp' - 1)wp''}{\theta} + p''(w)(\gamma + \delta)w + p'(\gamma + \delta) + \frac{\lambda^2 \sigma^2}{2} p'''. \quad (\text{C.1})$$

Evaluating (C.1) at the upper-boundary $\bar{w}$, and using $p'(\bar{w}) = -1$ and $p''(\bar{w}) = 0$, we find

$$\frac{\lambda^2 \sigma^2}{2} p'''(\bar{w}) = \gamma - r > 0;$$

therefore $p''(\bar{w} - \epsilon) < 0$.

Now let $q(w) = p(w) - wp'(w)$, and we have

$$(r + \delta)q(w) = \mu + \frac{(q(w) - 1)^2}{2\theta} + (\gamma - r)wp'(w) + \frac{\lambda^2\sigma^2}{2}p''.$$

Suppose that there exists some $\tilde{w} < \bar{w}$ such that $p''(\tilde{w}) = 0$; then without loss of generality assume that $p''(w) < 0$ for $w \in (\tilde{w}, \bar{w})$. Evaluating the above equation at $\tilde{w}$, we have

$$(r + \delta)q(\tilde{w}) = \mu + \frac{(q(\tilde{w}) - 1)^2}{2\theta} + (\gamma - r)\tilde{w}p'(\tilde{w}).$$

Since $q(\tilde{w}) < q^{FB}$, and $(r + \delta)q^{FB} = \mu + \frac{(q^{FB} - 1)^2}{2\theta}$, it implies $p'(\tilde{w}) < 0$. Therefore evaluating (C.1) at point $\tilde{w}$, we obtain

$$(r + \delta)p'(\tilde{w}) = p'(\tilde{w})(\gamma + \delta) + \frac{\lambda^2\sigma^2}{2}p'''(\tilde{w}),$$

which implies that $p'''(\tilde{w}) = \frac{2(r-\gamma)}{\lambda^2\sigma^2}p'(\tilde{w}) > 0$. It is inconsistent with the choice of $\tilde{w}$ where $p''(\tilde{w}) = 0$ but $p''(\tilde{w} + \epsilon) < 0$. Therefore $p(\cdot)$ is strictly concave over the whole domain $[0, \bar{w}]$.

D Appendix for Section 5

Fix regime 1 as the current regime (similar results hold for regime 2 upon necessary relabelling.)

Based on (28) and (29) in Section 5, the following Bellman equation holds for $P(K, W, 1)$:

$$\begin{aligned} rP(K, W, 1) = \sup_{I, \Psi} & (\mu v_1 K - I - G(I, K)) + (I - \delta K)P_K + (\gamma W - \Psi(K, W, 1)\xi_1)P_W \\ & + \frac{\lambda^2\sigma^2 K^2}{2}P_{WW} + \xi_1(P(K, W + \Psi(K, W, 1), 2) - P(K, W, 1)), \end{aligned} \quad (\text{D.1})$$

where $I(K, W, 1)$ and $\Psi(K, W, 1)$ are state-dependent controls.

The first-order condition (FOC) for optimal $\Psi(K, W, 1)$, given that the solution takes an interior solution, yields that

$$P_W(K, W, 1) = P_W(K, W + \Psi(K, W, 1), 2), \quad (\text{D.2})$$

As discussed in the main text, to provide compensation effectively, the optimal contract equates the marginal cost of delivering compensation, i.e., $-P_W$, across different Markov states at any time. However, in general, the solution of $\Psi(K, W, n)$ might be binding (corner solution), as the agent's continuation payoff after the regime change has to be positive. Therefore, along the equilibrium path the optimal $\Psi(K, W, n)$'s might bind, i.e., $\Psi(K, W, n) + W \geq 0$ holds with equality.

Investment policy $I(K, W, n)$, by taking a FOC condition, is similar to the baseline case. We will solve $\Psi(K, W, n)$ and $I(K, W, n)$ jointly with the investors' value functions $P(K, W, n)$'s.

We now exploit scale invariance feature of the problem. As discussed in the text, denote the scaled (by K) version of $P(K, W, n)$, $\Psi(K, W, n)$, and $I(K, W, n)$ as $p_n(w)$, $\psi_n(w)$ and $i_n(w)$, where $w = W/K$ is the agent's scaled continuation payoff. Similar to equation (15),

$$i_n(w) = \frac{P_K(K, W, n) - 1}{\theta} = \frac{p_n(w) - wp'_n(w) - 1}{\theta}. \quad (\text{D.3})$$

Combining this result with the above analysis regarding $\psi_n(w)$'s (notice that $P_W(K, W, n) = p'_n(w)$), the following proposition characterizes the ODE system $\{p_n\}$ when the output price is stochastic.

Proposition 5 *For $0 \leq w \leq \bar{w}_n$ (the continuation-payoff region for regime n), the scaled investor's value function $p_n(w)$ and the optimal payment threshold $\bar{w}_n$ solve the following coupled ODEs:*

$$(r + \delta) p_1(w) = \mu_1 + \frac{(p_1(w) - wp'_1(w) - 1)^2}{2\theta} + p'_1(w) [(\gamma + \delta)w - \xi_1 \psi_1(w)] + \frac{\lambda^2 \sigma^2}{2} p''_1(w) \\ + \xi_1 (p_2(w + \psi_1(w)) - p_1(w)), \quad 0 \leq w \leq \bar{w}_1, \quad (\text{D.4})$$

$$(r + \delta) p_2(w) = \mu_2 + \frac{(p_2(w) - wp'_2(w) - 1)^2}{2\theta} + p'_2(w) [(\gamma + \delta)w - \xi_2 \psi_2(w)] + \frac{\lambda^2 \sigma^2}{2} p''_2(w) \\ + \xi_2 (p_1(w + \psi_2(w)) - p_2(w)), \quad 0 \leq w \leq \bar{w}_2, \quad (\text{D.5})$$

subject to the following boundary conditions at the upper boundary $\bar{w}_n$:

$$p'_n(\bar{w}_n) = -1, \quad (\text{D.6})$$

$$p''_n(\bar{w}_n) = 0, \quad (\text{D.7})$$

and the left boundary conditions at liquidation:

$$p_n(0) = l_n, \quad n = 1, 2.$$

The scaled endogenous jump-size functions $\psi_n(w)$ satisfy:

$$p'_1(w) = p'_2(w + \psi_1(w))$$

$$p'_2(w) = p'_1(w + \psi_2(w))$$

if $w + \psi_n(w) > 0$ (interior solution); otherwise $\psi_n(w) = -w$. For $w > \bar{w}_n$ (cash-payment regions), $p_n(w) = p_n(\bar{w}_n) - (w - \bar{w}_n)$.

Now we show that p_n 's are concave functions. Denote two states as n, m (here q_m is not the marginal q as used in the main text.) By differentiating (D.5) we obtain

$$\begin{aligned} (r + \delta) p'_n &= -\frac{(p_n - wp'_n - 1)wp''_n}{\theta} + p''_n \cdot [(\gamma + \delta)w - \xi_n \psi_n(w)] + p'_n (\gamma + \delta - \xi_n \psi'_n(w)) + \frac{\lambda^2 \sigma^2}{2} p'''_n \\ &\quad + \xi_n (p'_m(w + \psi_n(w)) (1 + \psi'_n(w)) - p'_n). \end{aligned}$$

Notice that when $\psi_n(w)$ takes an interior solution, $p'_m(w + \psi_n(w)) = p'_n(w)$; and otherwise $\psi'_n(w) = -1$. Either condition implies that

$$(r + \delta) p'_n = -\frac{(p_n - wp'_n - 1)wp''_n}{\theta} + p''_n \cdot (\gamma + \delta)w + p'_n (\gamma + \delta) + \frac{\lambda^2 \sigma^2}{2} p'''_n, \quad (\text{D.8})$$

which takes the exact same form as in (C.1).

Now let $q_n(w) = p_n(w) - wp'_n(w)$, i.e., the marginal q that captures the investment benefit.

We have

$$\begin{aligned}
(r + \delta + \xi_n) q_n(w) &= \mu_n + \frac{(q_n(w) - 1)^2}{2\theta} + \xi_m q_m(w + \psi_n(w)) + (\gamma - r) w p'_n(w) + \frac{\lambda^2 \sigma^2}{2} p''_n \\
(r + \delta + \xi_m) q_m(w + \psi_n(w)) &= \mu_m + \frac{(q_m(w + \psi_n(w)) - 1)^2}{2\theta} + \xi_n q_n(w) + \\
&\quad (\gamma - r) (w + \psi_n(w)) p'_m(w + \psi_n(w)) + \frac{\lambda^2 \sigma^2}{2} p''_m(w + \psi_n(w))
\end{aligned}$$

Recall that the first-best pair (q_n^{FB}, q_m^{FB}) solves the system

$$\begin{cases} (r + \delta + \xi_n) q_n^{FB} = \mu_n + \frac{(q_n^{FB} - 1)^2}{2\theta} + \xi_m q_m^{FB} \\ (r + \delta + \xi_m) q_m^{FB} = \mu_m + \frac{(q_m^{FB} - 1)^2}{2\theta} + \xi_n q_n^{FB} \end{cases}.$$

Suppose that there exists some points so that p_n is convex. Pick the largest $\tilde{w}$ such that $p''_n(\tilde{w}) = 0$ but $p''_n(\tilde{w}-) < 0$, and $p''(w) \leq 0$ for $w \in (\tilde{w}, \bar{w})$. If $\psi_n(\tilde{w})$ is interior, then

$$k = p'_n(\tilde{w}) = p'_m(\tilde{w} + \psi_n(\tilde{w})), p''_n(\tilde{w}) = p''_m(w + \psi_n(w)) (1 + \psi'_n(\tilde{w})) = 0,$$

Clearly, if $p''_m(w + \psi_n(w)) = 0$, then

$$\begin{aligned}
(r + \delta + \xi_n) q_n(\tilde{w}) &= \mu_n + \frac{(q_n(\tilde{w}) - 1)^2}{2\theta} + \xi_m q_m(\tilde{w} + \psi_n(\tilde{w})) + (\gamma - r) \tilde{w} k \\
(r + \delta + \xi_m) q_m(\tilde{w} + \psi_n(\tilde{w})) &= \mu_m + \frac{(q_m(\tilde{w} + \psi_n(\tilde{w})) - 1)^2}{2\theta} + \xi_n q_n(\tilde{w}) + (\gamma - r) (\tilde{w} + \psi_n(\tilde{w})) k
\end{aligned}$$

Since a positive k will imply that $q_n > q_n^{FB}$ and $q_m > q_m^{FB}$, we must have $k < 0$. Then evaluating (C.1) at the point $\tilde{w}$, we obtain

$$\frac{\lambda^2 \sigma^2}{2} p'''_n(\tilde{w}) = (r - \gamma) p'_n(\tilde{w}) = (r - \gamma) k > 0.$$

This is inconsistent with the choice of $\tilde{w}$ where $p''(\tilde{w}) = 0$ but $p''(\tilde{w} + \epsilon) < 0$. Notice that the above argument applies to the case $p''_m(w + \psi_n(w)) > 0$.

Now we consider the case $1 + \psi'_n(\tilde{w}) = 0$ but $p''_m(w + \psi_n(w)) < 0$. We first rule out the case of $p'_n(\tilde{w}) = 0$. Otherwise, given $p'_n(\tilde{w}) = 0$ and $p''_n(\tilde{w}) = 0$, $q_n(w)$ admits a constant solution q_n which solves the quadratic equation

$$(r + \delta + \xi_n) q_n = \mu_n + \frac{(q_n - 1)^2}{2\theta} + \xi_m q_m(\tilde{w} + \psi_n(\tilde{w}));$$

notice that for all w at state n , the constant solution implies that after the regime switching q_m takes the constant value $q_m(\tilde{w} + \psi_n(\tilde{w}))$. This is inconsistent with our upper boundary conditions.

Therefore we must have $p'_n(\tilde{w}) > 0$ and $p''_n(\tilde{w}) < 0$ according to the above argument. In the neighborhood of $\tilde{w}$ find two points $\tilde{w} - \epsilon < \tilde{w} < \tilde{w} + \eta\epsilon$ where ϵ, η are positive, such that $p''_n(\tilde{w} - \epsilon) > p''_n(\tilde{w}) = 0 > p''_n(\tilde{w} + \eta\epsilon)$, but $(\tilde{w} - \epsilon)p'_n(\tilde{w} - \epsilon) = (\tilde{w} + \eta\epsilon)p'_n(\tilde{w} + \eta\epsilon) = k > 0$. Therefore,

$$\begin{aligned} (r + \delta + \xi_n)q_n(\tilde{w} - \epsilon) &= \mu_n + \frac{(q_n(\tilde{w} - \epsilon) - 1)^2}{2\theta} + \\ &\quad \xi_n q_m(\tilde{w} - \epsilon + \psi_n(\tilde{w} - \epsilon)) + (\gamma - r)k + \frac{\lambda^2 \sigma^2}{2} p''_n(\tilde{w} - \epsilon), \text{ and} \\ (r + \delta + \xi_n)q_n(\tilde{w} + \eta\epsilon) &= \mu_n + \frac{(q_n(\tilde{w} + \eta\epsilon) - 1)^2}{2\theta} + \\ &\quad \xi_n q_m(\tilde{w} + \eta\epsilon + \psi_n(\tilde{w} + \eta\epsilon)) + (\gamma - r)k + \frac{\lambda^2 \sigma^2}{2} p''_n(\tilde{w} + \eta\epsilon). \end{aligned}$$

Because $1 + \psi'_n(\tilde{w}) = 0$, the difference in q_m will be dominated (since it is in a lower order) by the difference in p''_n 's. Now since $p''_n(\tilde{w} - \epsilon) > p''_n(\tilde{w} + \eta\epsilon)$, it implies that $q_n(\tilde{w} - \epsilon) > q_n(\tilde{w} + \eta\epsilon)$. But because $q_n(\tilde{w} - \epsilon) - q_n(\tilde{w} + \eta\epsilon) = p_n(\tilde{w} - \epsilon) - p_n(\tilde{w} + \eta\epsilon) < 0$, contradiction.

Now consider the case where $\psi_n(\tilde{w})$ is binding at $-w$. Take the same approach; notice that in this case the points after regime switching are exactly 0. Therefore the same argument applies, and $p_n(\cdot)$ is strictly concave over the whole domain $[0, \bar{w}_n]$. Q.E.D.

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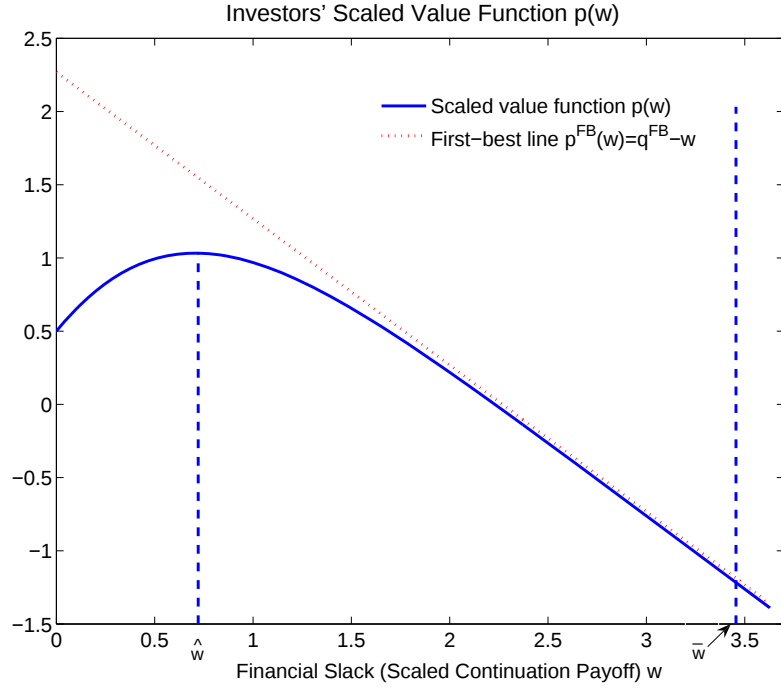


Figure 1: **The investors' scaled value function $p(w)$.** The solid line is the concave investors' scaled value function $p(w)$. The dotted line corresponds to the first-best neoclassical setting: $p^{FB}(w) = q^{FB} - w$, where the corresponding Tobin's q is $q^{FB} = 2.27$. The baseline parameters are $r = 0.1$, $\gamma = 0.101$, $\mu = 0.4$, $\sigma = 0.6$, $\lambda = 0.8$, $l = 0.5$, and $\theta = 15$. The payment boundary is $\bar{w} = 3.46$. and $p(w)$ is maximized at $\hat{w} = \arg \max p(w) = 0.72$.

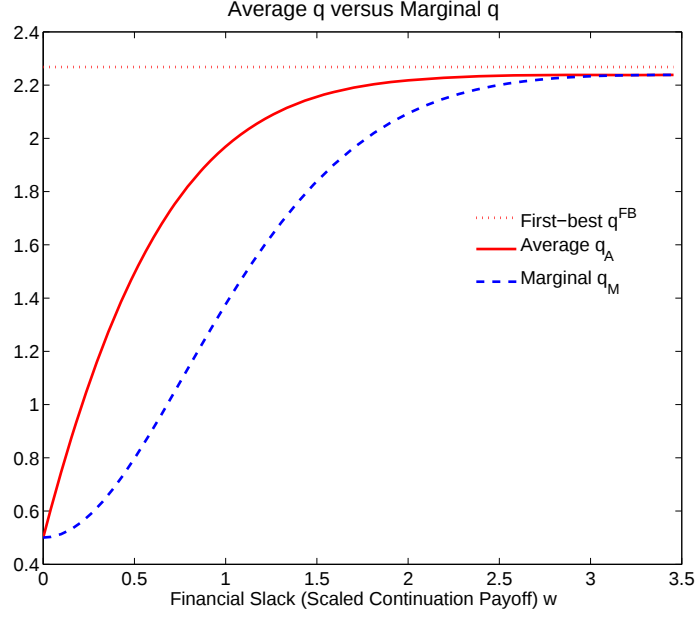


Figure 2: **Marginal q versus Average q .** The solid line corresponds to the average $q_a(w) = p(w) + w$. The dashed line gives the marginal $q_m(w) = p(w) - wp'(w)$. Note that average q is greater or equal to marginal q . The dotted line gives the benchmark (Hayashi) result, where average q and marginal q are equal and independent of financial slack w . The baseline parameters are $r = 0.1$, $\gamma = 0.101$, $\mu = 0.4$, $\sigma = 0.6$, $\lambda = 0.8$, $l = 0.5$, and $\theta = 15$.

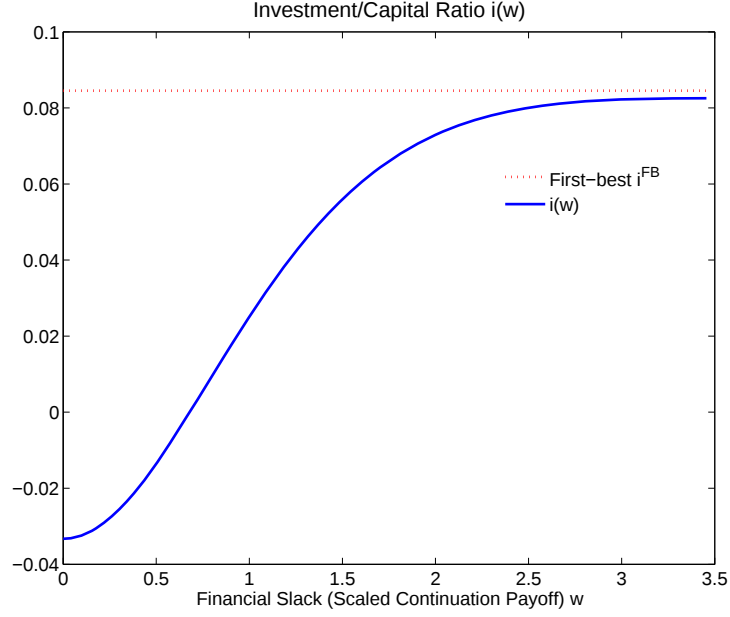


Figure 3: **Investment-capital ratio.** The solid line plots the investment-capital ratio $i(w)$ as a function of financial slack w . The dotted line is the first-best investment-capital ratio i^{FB} . Investment-capital ratio is lower than i^{FB} at all levels of w . The degree of underinvestment is lower for higher level of w . The baseline parameters are $r = 0.1$, $\gamma = 0.101$, $\mu = 0.4$, $\sigma = 0.6$, $\lambda = 0.8$, $l = 0.5$, and $\theta = 15$.

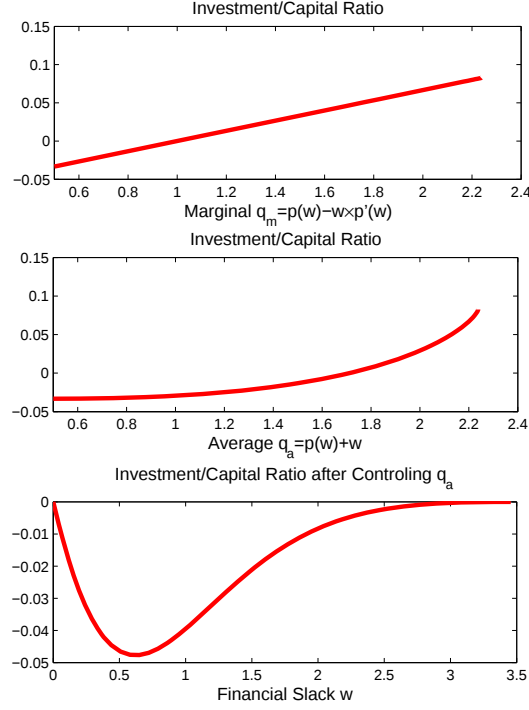


Figure 4: **Investment-capital ratio, marginal q , average q , and financial slack w .** The top panel shows the linear relationship between $i(w)$ and marginal q . The mid panel shows the monotonically increasing relationship between $i(w)$ and average q . Investment is more sensitive to average q when financial slack is higher. The bottom panel plots the relationship between $\hat{i}(w)$, investment-capital ratio after controlling for average q , and financial slack w . Investment responds negatively with increase in financial slack when financial slack is low, and responds positively with increase in financial slack when financial slack is high. The baseline parameters are $r = 0.1$, $\gamma = 0.101$, $\mu = 0.4$, $\sigma = 0.6$, $\lambda = 0.8$, $l = 0.5$, and $\theta = 15$.

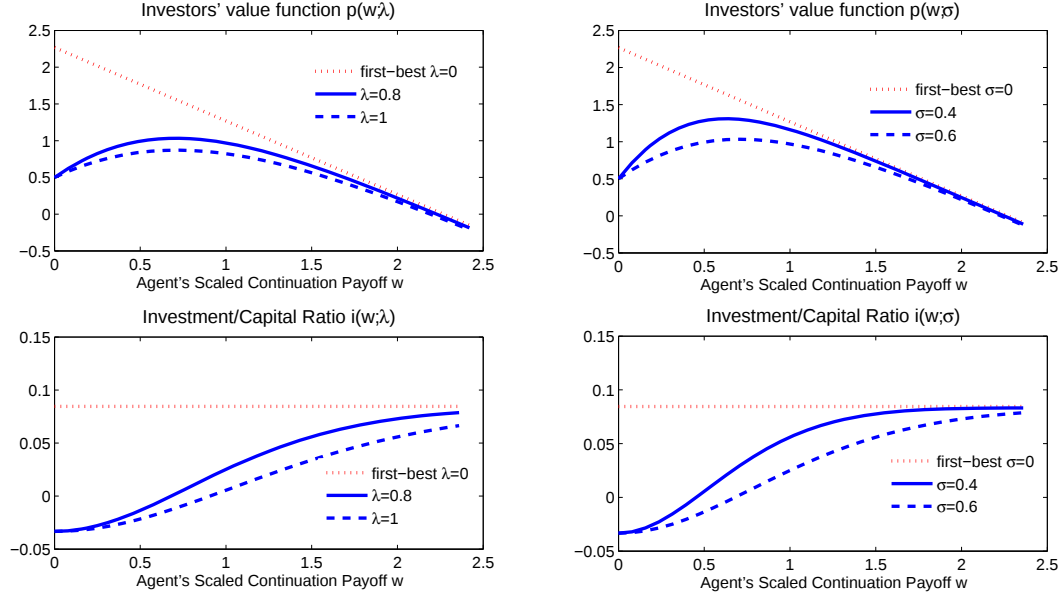


Figure 5: **Comparative statics.** The left column gives the comparative static results with respect to λ , and the right column provides the comparative static results with respect to the volatility parameter σ . Within each column, the first and the second row correspond to the analysis for the investors' scaled value function $p(w)$ and the optimal investment-capital ratio $i(w)$, respectively. The baseline parameters are $r = 0.1$, $\gamma = 0.101$, $\mu = 0.4$, $\sigma = 0.6$, $\lambda = 0.8$, $l = 0.5$, and $\theta = 15$.

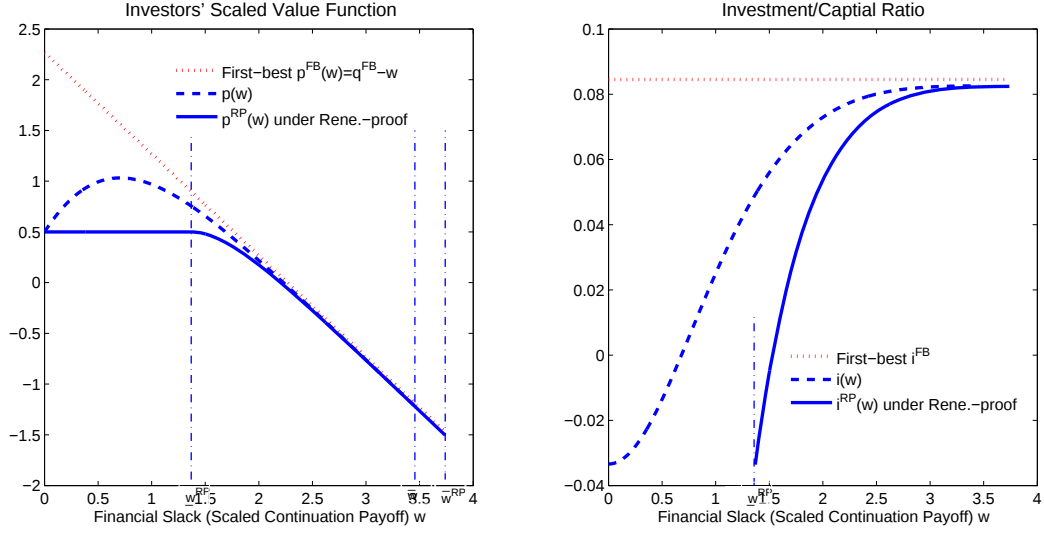


Figure 6: **Renegotiation proofness.** The baseline parameters are $r = 0.1$, $\gamma = 0.101$, $\mu = 0.4$, $\sigma = 0.6$, $\lambda = 0.8$, $l = 0.5$, and $\theta = 15$. The original scaled value function $p(w)$ is not renegotiation-proof, because $p'(0) > 0$. For the renegotiation-proof contract, $\underline{w}^{RP}$ is the lower bound for the agent's scaled continuation payoff w , with the following properties: $p(\underline{w}^{RP}) = p(0) = l$, and $p'(\underline{w}^{RP}) = 0$. The value function $p^{RP}(w)$ solves the ODE (16) subject to the boundary conditions (17)-(18) and the above stated conditions at $\underline{w}^{RP}$.

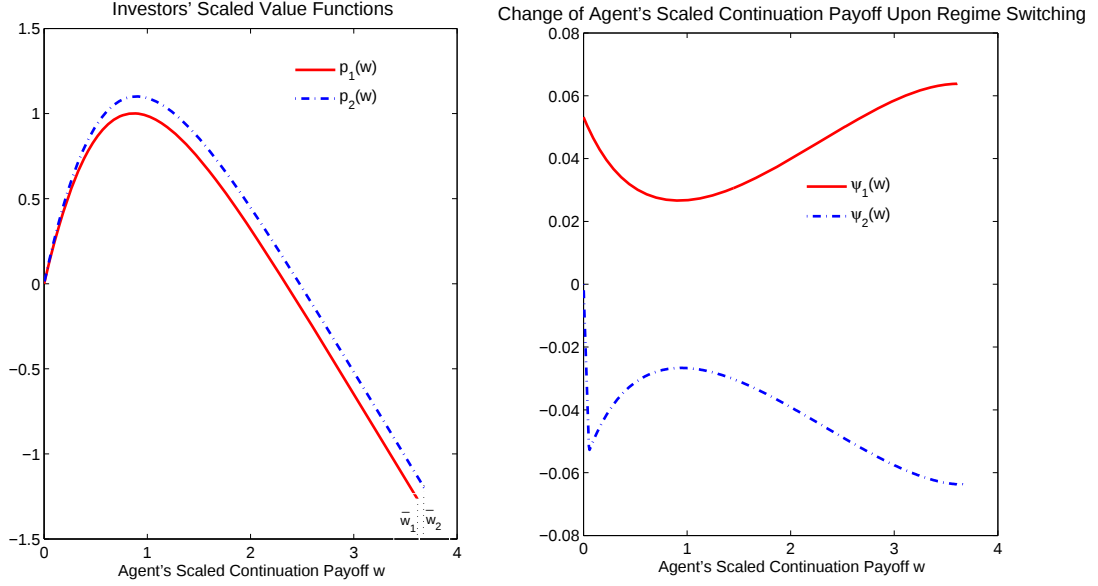


Figure 7: **Scaled investors' value functions $p_n(w)$ and change of the agent's scaled continuation payoff upon regime switching ($l_1 = l_2$).** The upper panel plots scaled investors' value function $p_1(w)$ and $p_2(w)$ when the output price is low and high, respectively. The lower panel plots $\psi_1(w)$ and $\psi_2(w)$, the discrete changes of the agent's scaled continuation payoff when output prices changes. The parameters are $r = 0.1$, $\gamma = 0.101$, $\theta = 15$, $\mu_1 = 1$, $\mu_2 = 1.1$, $\sigma = 0.6$, $\lambda = 0.8$, $l_1 = l_2 = 0$, $\xi_1 = \xi_2 = .1$. In the first-best case (with the same output price process), $q_1^{FB} = 2.39$ and $q_2^{FB} = 2.52$. The upper payment boundaries are $\bar{w}_1 = 3.62$, and $\bar{w}_2 = 3.68$, in the low and high output price state, respectively. When w is sufficiently low ($w \leq .0532$ in this example) and output price is high, a drop in output price from v_2 to v_1 leads to immediate liquidation. This corresponds to $\psi_2(w) = -w$ for $w \leq .0532$.

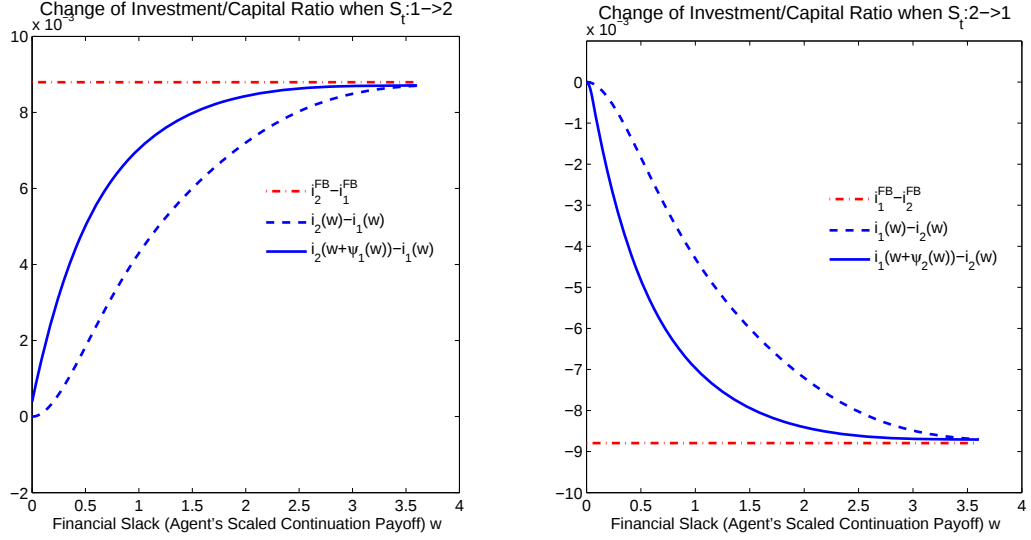


Figure 8: **Change of the investment-capital ratio when output price switches.** The left and the right panel plot the change of the investment-capital ratio, when output price increases from low to high, and decreases from high to low, respectively. The solid lines give the total change due to change of output price. The dashed lines plot the change holding w fixed. The dash-dotted lines are the changes of investment-capital ratio when output price changes in the neoclassical benchmark (without agency). The parameters are $r = 0.1$, $\gamma = 0.101$, $\theta = 15$, $\mu_1 = 1$, $\mu_2 = 1.1$, $\sigma = 0.6$, $\lambda = 0.8$, $l_1 = l_2 = 0$, $\xi_1 = \xi_2 = .1$.