

Maturity, Indebtedness and Default Risk ¹

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Abstract

Most existing models of unsecured debt (both consumer and sovereign) assume that all debt matures in one-period. In reality both consumers and countries borrow long term. We present a model in which borrowers are required to repay a constant fraction of the existing debt each period and if they choose to not repay, they must stop repayment (i.e. default) on all existing debt (i.e., all debt is equally senior). We find that incorporating long-term debt increases both spreads (or equivalently default probabilities) and debt level (relative to income) substantially, bringing the model more in line with the data on consumer and sovereign debt. In addition, we find that when output volatility is sufficiently high, having access to long-term bond is welfare improving. This gain in welfare comes about because long maturity bonds provide insurance against fluctuations in the market interest rates of new debt (because the borrower is not required to re-finance the entire stock of existing debt each period). This insurance benefit can dominate the well-known cost of long maturity unsecured debt that comes from the borrowers inability to commit to not borrowing in the future (so-called debt dilution problem).

Key Words: Unsecured Debt, Maturity, Debt Dilution, Default Risk

JEL:

1 Introduction

We study an equilibrium model of unsecured debt and default in which borrowers can issue long-term debt. The existing literature on this subject – both the consumer debt and sovereign debt parts – has mostly considered one-period debt. In reality, both consumers and countries can and do borrow long-term. For instance, in the US, the standard credit card contract requires borrowers to pay only a portion of their debt each period. Incorporating long-term debt brings these models closer to reality and, we argue in this paper, enhances their ability to account for observed high levels of household and sovereign debt and high default rates (spreads).

It is well known that existing models of one-period unsecured debt – both the consumer and the sovereign debt varieties – imply that default spreads rise very sharply with debt. Consequently these models have difficulty matching observed high levels of indebtedness and spreads since borrowers in the model simply do not wish to borrow much. We show that lengthening the maturity level of debt (i.e., going beyond one period debt) can decrease the elasticity of spreads with respect to debt and potentially increase equilibrium borrowing as well equilibrium spreads.

The main reason why this is possible is as follows. In our model, borrowers are required to repay a constant fraction of the existing debt each period and if they choose to not repay, they must stop repayment (i.e. default) on all existing debt (i.e., all debt is equally senior). Consider the situation of a borrower who is contemplating issuing an additional unit of debt. If income declines next period and the default risk on the additional new unit of debt rises, servicing the additional unit will not require the borrower to refinance the *entire stock* of debt tomorrow at the higher interest rate. Thus the decline in consumption from servicing the additional unit of debt when income is low tomorrow is not as sharp as in the case of one-period debt. This attenuates the incentive to default on the additional unit of debt relative to the one-period debt case and accounts for the decreased elasticity of spreads with

respect to debt.

However the decreased elasticity of spreads comes at a cost to the lender. Because all debt is not completely redeemed each period there is now a possibility of default on even very low levels of debt. This happens because of the well-known “debt dilution” problem. The borrower cannot commit to not issue additional debt in the future and the lender must recognize that even if the current level of debt is very low, future levels of debt could be high. Thus (given the assumption that all debt is equally senior regardless of when it was issued) there is a positive probability of default even if total debt issued is small.

These two effects of longer-term debt tend to offset each other in different ways for different levels of debt. For low levels of debt, the cost of borrowing is actually higher for long maturity debt compared to one-period debt. But because the elasticity of spreads with respect to debt is low for long maturity debt, the interest rates for long maturity debt at high debt levels can be lower compared to one-period debt. If income is sufficiently volatile, the average level of equilibrium debt and equilibrium default risk (or spreads) can be considerably higher for long maturity debt compared to one-period debt. In these circumstances it is also the case that consumer welfare is higher with long maturity debt compared to one period debt.

2 Modeling Long-Term Debt

A natural way to introduce long-term debt is to assume that debt issued in period t is due for repayment in period $t + T$. For simplicity assume that only zero coupon debt can be issued so that the only payment on debt being issued today is to be made in T periods in the future. Unfortunately, this approach poses challenging computational problems having to do with computing the probability of default. These difficulties motivate a novel approach to long-term debt. Before we describe that approach, it is useful to understand the computational difficulties involved in the more natural approach to modeling long-term debt.

Suppose, then, that individuals can issue zero-coupon debt that matures period T periods in the future. To be concrete suppose also that if the individual defaults then she will be permanently excluded from the credit market. Since new debt can be issued each period, and assuming that individuals have no interest in saving, the state vector at the start of period t can be described by $(b_0, b_1, b_2, \dots, b_{T-1}, s, h = 0)$, where b_τ is the quantity of bonds due for repayment τ periods in the future, s are shocks impinging on the household in the current period and $h = 0$ is flag that indicates the the household has not defaulted in any previous period. If $b_t < 0$, the individual has a default decision to make in period t provided she has not defaulted in any earlier period. The decision can to default on debt currently due can be captured by the default decision rule $d(b_0, b_1, b_2, \dots, b_{T-1}, s, h = 0)$. Then, the probability of default on a bond due for repayment in period τ is the sum of the probability of default in the current period plus the probability of repayment in the current period but default in the next period plus the probability of repayment in the next two periods followed by default in the third period ... plus probability of repayment in the next $\tau - 1$ periods followed by default in period τ . Unfortunately, even for modest values of T (such as 3 or 4), these calculations can become quite demanding because we will have at least T state variables (more if s is itself a vector) each of which can potentially take many, many values.

In the credit-risk modeling literature, this difficulty is side-stepped by *assuming* that the probability of repayment over any arbitrary number of periods into the future is some simple decreasing function of the number of periods. Since we wish to relate the per-period default probability to the maturity of the debt itself (or more generally, to the individual's decision problem) this strategy does not appear to be available to us (i.e., we do not know of conditions on the primitives of the discrete-time decision problem that will deliver a simple "survival probability" functions that depend on the maturity of the debt itself).

Our approach, instead, is to simplify the maturity structure of debt in a way that calculation of default probabilities from the individual's decision problem becomes easier. Specifically, we propose to analyze debt contracts with payoffs governed by a 3-state Markov chain. For 1

unit of debt issued in period t , the individual promises to pay, conditional on not defaulting, z units in period $t + 1$ with probability $(1 - \lambda)$ and 1 unit with probability λ . For $\tau > t + 1$ payments are as follows: if $m_{\tau-1} = z$ then conditional on not defaulting in period τ , $m_\tau = z$ with probability $(1 - \lambda)$ and 1 with probability λ , if $m_{\tau-1} = 1$ then regardless of whether the person defaults or not $m_\tau = 0$ with probability 1 and if $m_{\tau-1} = 0$ then regardless of whether the person defaults or not $m_\tau = 0$ with probability 1. Observe that if $\lambda = 1$ then the bond is a one-period discount bond and if $z > 0$ and $\lambda = 0$ then the bond is a perpetuity promising to pay z units each period. For intermediate values of λ we have a bond which makes coupon payments of z units for, on average, $1/(1 - \lambda)$ periods followed by repayment of principle. In what follows we will assume that all bonds are unit bonds and we will distinguish one unit bond from another by its coupon payment and probabilistic maturity (z, λ) .

Why is this probabilistic maturity structure easier to analyze? The benefit comes from the “memory-less” nature of the bond. Going forward, a unit bond (z, λ) issued $k \geq 1$ periods in the past has exactly the same payoff structure as another (z, λ) unit bond issued $k' > k$ periods in the past. This means we can aggregate all outstanding (z, λ) unit bonds regardless of the date of issue and thereby cut down on the number of state variables relevant to the individual’s decision problem. This reduction in turn reduces the computational burden of computing default probabilities.

In what follows, we will also make the calculations easier by assuming that unit bonds are infinitesimally small – meaning that if b unit bonds of type (z, λ) are outstanding at the start of next period, the issuer’s coupon payments next period will be $z \cdot (1 - \lambda)b$ for sure, her payment of principle will be λb for sure and, if no bonds are issued or redeemed next period, $(1 - \lambda)b$ unit bonds will be outstanding for sure at the start of the following period.

3 Environment

For expositional ease, we will describe the environment in terms of a model of unsecured consumer credit with a large number of borrowers and lenders interacting in a competitive credit market. However, we can think of individuals as small open economies interacting in an international credit market.

3.1 Preferences and Endowments

Time is discrete and denoted $t = 0, 1, 2, \dots$. There is a unit measure of individuals. Each individual maximizes expected utility over consumption sequences, where the utility from any given sequence c_t is given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \beta < 1 \tag{1}$$

The momentary utility function $u(c) : [0, \infty) \rightarrow \mathbb{R}$ is strictly increasing and strictly concave.

Individuals receive an endowment each period denoted y_t . These endowments are drawn in an iid fashion from a first order Markov process whose transition probabilities are given by

$$Pr\{y_{t+1} \in A | y_t = y\} = F(y, A) \tag{2}$$

3.2 Option to Default and the Market Arrangement

Borrowers have the option to default on a loan. If they default they are excluded from the credit market (go into financial autarky) for some random length of time with average duration given by $1/\xi$ – that is, if a person defaults she is excluded from the credit market with probability $(1 - \xi)$. As long as they remain excluded they suffer a loss in output $\phi(y)$.

A bond in this economy is denoted by a pair of numbers (z, λ) . For $b > 0$ it will be assumed that $z = 0$ and $\lambda = 1$ (the bond is a one-period bond). Under competition, the unit price of a savings bond of size $b > 0$ will be $q^* = 1/(1 + r_f)$. For $b < 0$ the unit price of a bond of size b is given by $q_{z,\lambda}(y, b)$. Because of the possibility of default $q_{z,\lambda}(y, b) < q^*$.

3.3 Decision Problems

Consider the decision problem of an individual with bonds $b < 0$ and income y . Because the individual is indebted, she has the option to default. If she defaults, she goes into the default state where she is excluded from financial markets for some random length of time. If she does not default, she must repay the portion of the debt that is come due in the current period.

Denote her lifetime utility in the default state by $X(y)$ and the repayment state by $V(b, y)$.

Then we have:

$$X(y) = U((1 - \phi(y)) \cdot y) + \beta\{(1 - \xi)E_{y'|y}X(y') + \xi E_{y'|y}V(0, y')\}, \quad (3)$$

and

$$V(y, b) = \max_{b'} \{U(y + [\lambda + (1 - \lambda)z]b - q_{z,\lambda}(y, b') [b' - (1 - \lambda)b]) + E_{y'|y} [\beta \max \{V(y', b'), X(y')\}]\}. \quad (4)$$

These decision problems imply a default decision rule $d(y, b)$ and, conditional on not defaulting, an asset choice rule $h(y, b)$.

3.4 Equilibrium

Since the risk-free rate is taken as exogenous the only non-trivial equilibrium conditions in the model is the price of debt – i.e., the value of $q_{z,\lambda}(y, b')$ when $b' < 0$. Given a competitive market in debt, these prices must be consistent with zero profits adjusting for the probability of default. That is:

$$q(y, b') = E_{y'|y} \left[(1 - d(y', b')) \frac{\lambda + (1 - \lambda)[z + q(y', h(y', b'))]}{1 + r_f} \right] \quad (5)$$

Typically, q is increasing in b' so borrowers can inflict capital losses on existing lenders by taking on additional debt. This is referred to as the debt dilution problem in the unsecured debt literature.

4 Quantitative Exercise

We will analyze this model with help of a numerical example. We will make the following assumptions on primitives, roughly resembling the ones in Arellano (2007).

- Endowment process:

$$\ln(y_{t+1}) = \rho \ln(y_t) + \epsilon_t, \epsilon_t \sim N(0, \sigma_\epsilon^2)$$

It will be convenient to define:

$$\sigma_y = \left[\frac{\sigma_\epsilon^2}{1 - \rho^2} \right]^{0.5}.$$

- Default punishment:

$$\phi(y) = \begin{cases} 0 & \text{if } y \leq \bar{y} \\ y - \bar{y} & \text{if } y > \bar{y} \end{cases}$$

- Parameter Choices:

Parameter	Description	Experiment 1	Experiment 2
β	discount factor	0.95	same
γ	risk aversion	2	same
σ_ϵ	standard deviation of innovation	0.1	0.025
ρ	autocorrelation	0.98	same
$\bar{y}$	default punishment	$E(y) - 0.55\sigma_y$	same
ξ	probability of reentry	0.25	same
r_f	risk free return	0.01	same

We analyze two economies that differ only in terms of the volatility of the endowment shock. We will focus on the first experiment (the one with the high volatility) and then discuss how the results change with low volatility. For each experiment the main focus will be on how equilibrium pricing functions (and therefore equilibrium debt and default) change with the length of maturity of the bond. Specifically, we will examine cases where $\lambda \in \{0.05, 0.15, 1\}$. Lower values of λ correspond to longer maturities. Of course, $\lambda = 1$ is one period debt.

The best way to understand the significance of the length of maturity of the debt is to begin with Figure 1. The figure plots the default probability for different levels of debt holdings next period given today's endowment. At any debt level, default probability is lower with long-term bonds. In addition, default probability increases at a slower rate with long-term bonds. The lower default probability results because it is easier to service long-term debt when a bad endowment shock occurs next period. With short-term debt, the country or individual needs to borrow at a higher interest rates to refinance its entire stock of debt.

With a low productivity shock and persistence in income shocks, spreads increase a lot and refinancing the entire stock of debt becomes very expensive. This triggers default. In contrast, if the debt is long-term, debt servicing requires the country to pay back a small portion of its debt. This reduces the burden of debt servicing greatly and thereby reduces the incentives to default.

Next, we will examine the interest rate (or equivalently, default spread) in each of the three cases. However, we need to explain how the default spread is calculated in the case of long-term debt (it is better to look at the implied spreads because the risk-free price of one unit of bond of different maturities differ). Given the equilibrium price of the bond q and maturity λ , we find the average return r that satisfies the below equation:

$$\begin{aligned} q &= \frac{\lambda}{1+r_b} + \frac{\lambda}{1+r_b} \left(\frac{1-\lambda}{1+r_b} \right) + \frac{\lambda}{1+r_b} \left(\frac{1-\lambda}{1+r_b} \right)^2 + \dots \\ &= \frac{\lambda}{\lambda + r_b}. \end{aligned}$$

Then the implied average interest rate is given by

$$r_b = \frac{\lambda}{q} - \lambda,$$

and the implied average spread is

$$s_b = r_b - r_f.$$

Figure 2 plots the spreads calculated as above for the three cases and for different levels of debt. The short-term bond starts at the risk-free interest rate r_f (spreads are zero at low choice levels of debt because there is no risk of non-payment for those debt levels) but the spreads increase very fast. In contrast, for long-term bonds, the spread is positive even at very low levels of debt. This is because of the debt-dilution effect: even if the default probability is zero next period for small levels of debt, the lender today predicts that the

country will borrow more in the future and enter the default regions. Thus there is positive probability of default on any debt being issued today. Given this expectation of future behavior, the spreads start at a positive level even for low-levels of debt. Other important thing to notice is that the spread is less elastic with respect to the debt for long-term bonds. Explain? Figure 3 plots the prices underlying the spreads in Figure 2. Consistent with the discussion concerning spreads, the price of long-term is less elastic with respect to changes in the level of debt.

The following table presents the operating characteristics of these different economies.

	$V(E(Y), 0)$	Def Prob	Spreads	Debt at Mkt Prices	Debt at r_f
$\lambda = 0.05$	-21.2118	4.90	9.325	-0.817	-1.333
$\lambda = 0.15$	-21.2442	3.75	6.875	-0.371	-0.436
$\lambda = 1$	-21.2784	0.85	0.850	-0.095	-0.095

The first column are the three maturity levels. The second column reports the life-time utility of a country/individual who starts life with no assets and the mean level of endowment. The second column reports the average default probability in percentage terms. The third column reports the average spreads in percentage terms. The fourth column reports the market value of debt where mean endowment has been normalized to 1. The final column is the value of debt where future cash flows have been discounted by the risk-free rate. We see that the average debt level is higher for long-term bonds as are spreads. The next item explains these facts.

The equation that determines the equilibrium borrowing:

$$\left(-q(y, b') - \frac{\partial q(y, b')}{\partial b'} [b' - (1 - \lambda)b] \right) U'(y + \lambda b - q(y, b') [b' - (1 - \lambda)b]) = \beta E_{y'|y} V'_2(y', b')$$

To understand the equation, it is easier to think of the country as the "monopolistic" supplier of bonds. When the country issues an extra unit of bond, it gets revenue from that extra unit sold, but at the same time the decrease in the price of the bond (because of increased

default probability) decreases the revenue on all bonds that are currently issued. The country borrows more with long-term bond because both the price schedule is flatter with long-term bond (as seen in graph), and also because the decrease in price of the bond by extra borrowing only affects the currently issued bonds, and with long-term bonds, the currently issued bonds is just a small portion of the outstanding debt.

The average spreads are much higher with long-term bond. There are two reasons for this. One is, as we see in the above equation, the country more readily enters these high probability of default regions since it does not internalize the effect increasing the level of debt has on decreasing the value of bonds issued in previous periods (which explains why the lenders take into account that the country will increase its debt in upcoming periods and charges high spreads from the start). The second reason is that, with long-term bonds, just before defaulting, the country issues high levels of debt at very high spreads. Typically, with long-term bond, it is not very costly to not default one more period (by just paying the portion λ of the debt). But by borrowing at high spreads the country can disproportionately tax the high income outcomes next period (where it won't default), and consume more today. And it does borrow at very high spreads just before defaulting with long-term bond. In contrast, with short-term debt this opportunity never arises, since raising enough money at very high spreads to refinance all existing debt becomes impossible.

Welfare is higher for long-term bonds. As noted in the introduction, there are two effects of longer-term debt that tend to offset each other. With high endowment volatility, the default probability is more volatile and this implies more volatile interest rates. Long-term bonds provide insurance with respect to volatile interest rates, which in turn decreases the probability of default and helps the country to borrow disproportionately more with long-term bonds. But this insurance comes at a cost to the country in that the cost of borrowing is higher with long-term bonds because the country does not internalize the fact that higher borrowing in the future inflicts capital losses on previous lenders. Anticipating this, lenders charge higher interest rates right off the bat. When the volatility of output is high, the

insurance effect dominates the debt dilution effect and welfare is increasing in maturity of the debt. This logic is confirmed when we examine the operating characteristics of an economy with lower endowment volatility (experiment 2). The following table shows what happens when output volatility is lower.

	$V(E(Y), 0)$	Def Prob	Spreads	Debt at Mkt Prices	Debt at r_f
$\lambda = 0.05$	-20.0575	4.25	7.45	-0.215	-0.383
$\lambda = 0.15$	-20.0483	3.42	5.50	-0.212	-0.249
$\lambda = 1$	-20.0424	0.83	0.88	-0.158	-0.156

Observe that welfare is decreasing in maturity. Notice too that average debt level does not change very much with maturity. Thus, the insurance effect of longer maturities is not important and the debt dilution effect dominates.

5 The Role of Commitment

What happens if lenders insist that borrowers protect them from changes in the market value of debt? We can think of this case as representing the possibility of borrowers re-financing their debt when the market value improves or as lenders asking for more collateral if market value deteriorates. We will not distinguish between different reasons for improvement and deterioration – that is, even if (say) the deterioration is due to worse fundamentals and not greater borrowing, the value of existing obligations is protected.

The value of default:

$$X(y) = U((1 - \phi(y)) \cdot y) + \beta\{(1 - \xi)E_{y'|y}X(y') + \xi E_{y'|y}V(0, y')\}.$$

The value of no default:

$$V(y_{-1}, y, b) = \max_{b'} \left\{ \begin{aligned} &U(y + [\lambda + (1 - \lambda)(z + q_{z,\lambda}(y_{-1}, b))]b - q_{z,\lambda}(y, b')b') \\ &+ E_{y'|y} [\beta \max \{V(y, y', b'), X(y')\}] \end{aligned} \right\} \quad (6)$$

The decision rules are now $d(y_{-1}, y, b)$ and $h(y_{-1}, y, b)$. Then equilibrium price of a unit bond is given by:

$$q(y, b') = [\lambda + (1 - \lambda)(c + q(y, b'))] \left(\frac{E_{y'|y}(1 - d(y, y', b'))}{1 + r} \right) \quad (7)$$

We can do a change of variables that will allows us to re-write the above problem in terms of only two state variables. The key insight here is that the default decision depends only on y and the total obligation of the borrower which is given by $[\lambda + (1 - \lambda)(c + q(y_{-1}, b))]b$. Therefore, we should be able to re-write the problem in terms of these two state variables. With this in mind, define the total obligation at the start of the period by $A = [\lambda + (1 - \lambda)(c + q(y_{-1}, b))]b$. Next, notice that multiplying both sides of (7) with b' gives:

$$\begin{aligned} q(y, b')b' &= [\lambda + (1 - \lambda)(c + q(y, b'))]b' \left(\frac{E_{y'|y}(1 - d(y, y', b'))}{1 + r} \right) \\ q(y, b')b' &= A' \left(\frac{E_{y'|y}(1 - d(A', y'))}{1 + r} \right). \end{aligned}$$

Then , we can re-write the dynamic program as follows:

$$V(y, A) = \max_{A'} \left\{ U \left(y + A - A' \left(\frac{E_{y'|y}(1 - d(A', y'))}{1 + r} \right) \right) + E_{y'|y} [\beta \max \{V(y', A'), X(y')\}] \right\} \quad (8)$$

$$d(y, A) = \begin{cases} 1 & \text{if } V_{c,\lambda}(y, A) < X_\phi(y) \\ 0 & \text{if } V_{c,\lambda}(y, A) \geq X_\phi(y) \end{cases}$$

We can now recover the function $q(y, b')$ by solving the following two equations for q and b' given A' and y .

$$q(y, b')b' = A' \left(\frac{E_{y'|y}(1 - d(y', A'))}{1 + r} \right) \quad (9)$$

$$A' = [(1 - \lambda) + \lambda(c + q(y, b'))] b' \quad (10)$$

In particular, using the first equation, we can solve for b' in terms of A' and y .

$$A' = [(1 - \lambda) + \lambda c] b' + A' \left(\frac{E_{y'|y}(1 - d(y', A'))}{1 + r} \right).$$

Thus, with this market arrangement, long-term bonds act like one period bonds and none of the effects noted in the previous sections will occur.

6 Conclusion

To be added

7 References

To be added

Figure 1: Default probability next period

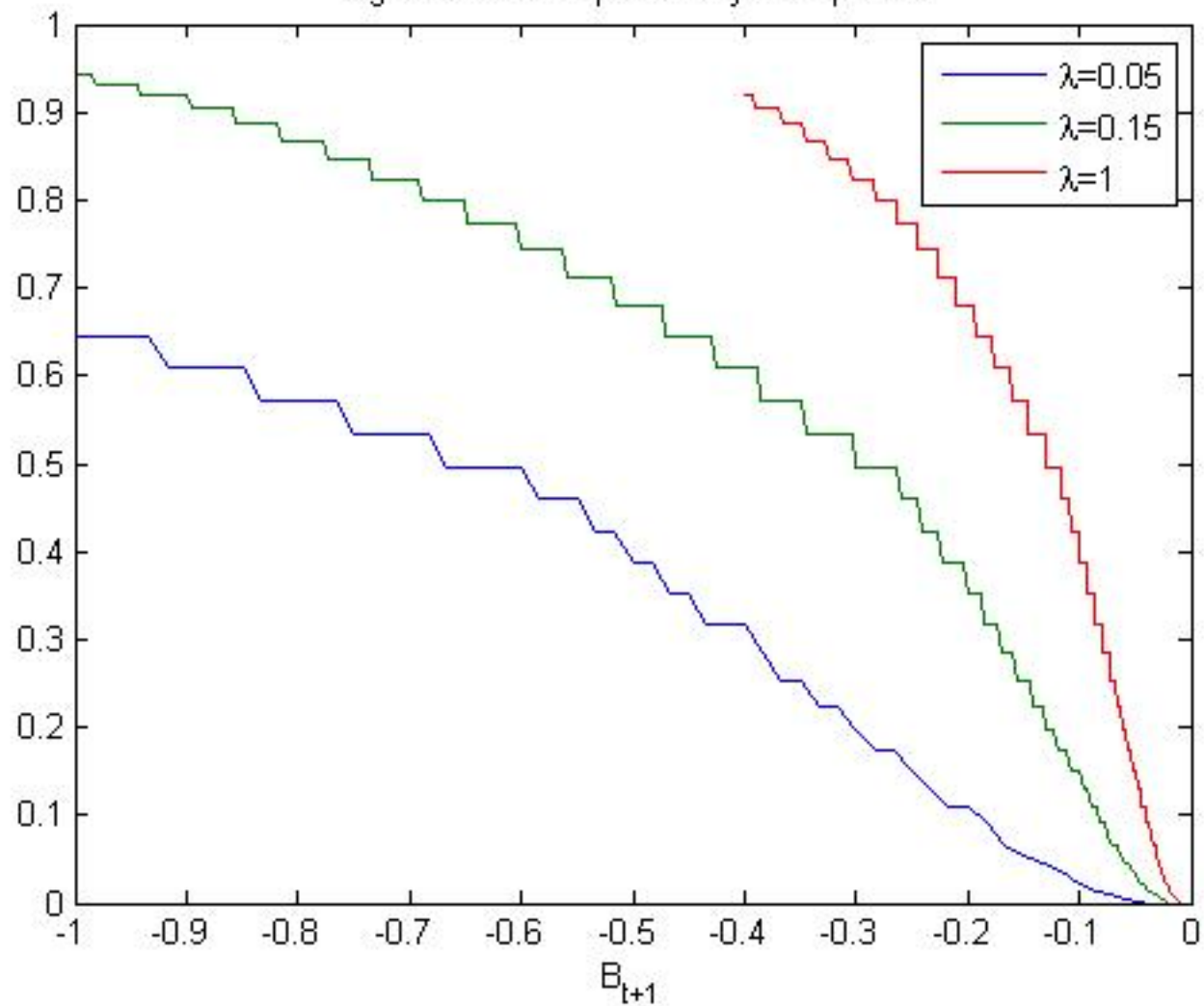


Figure 2: Spreads

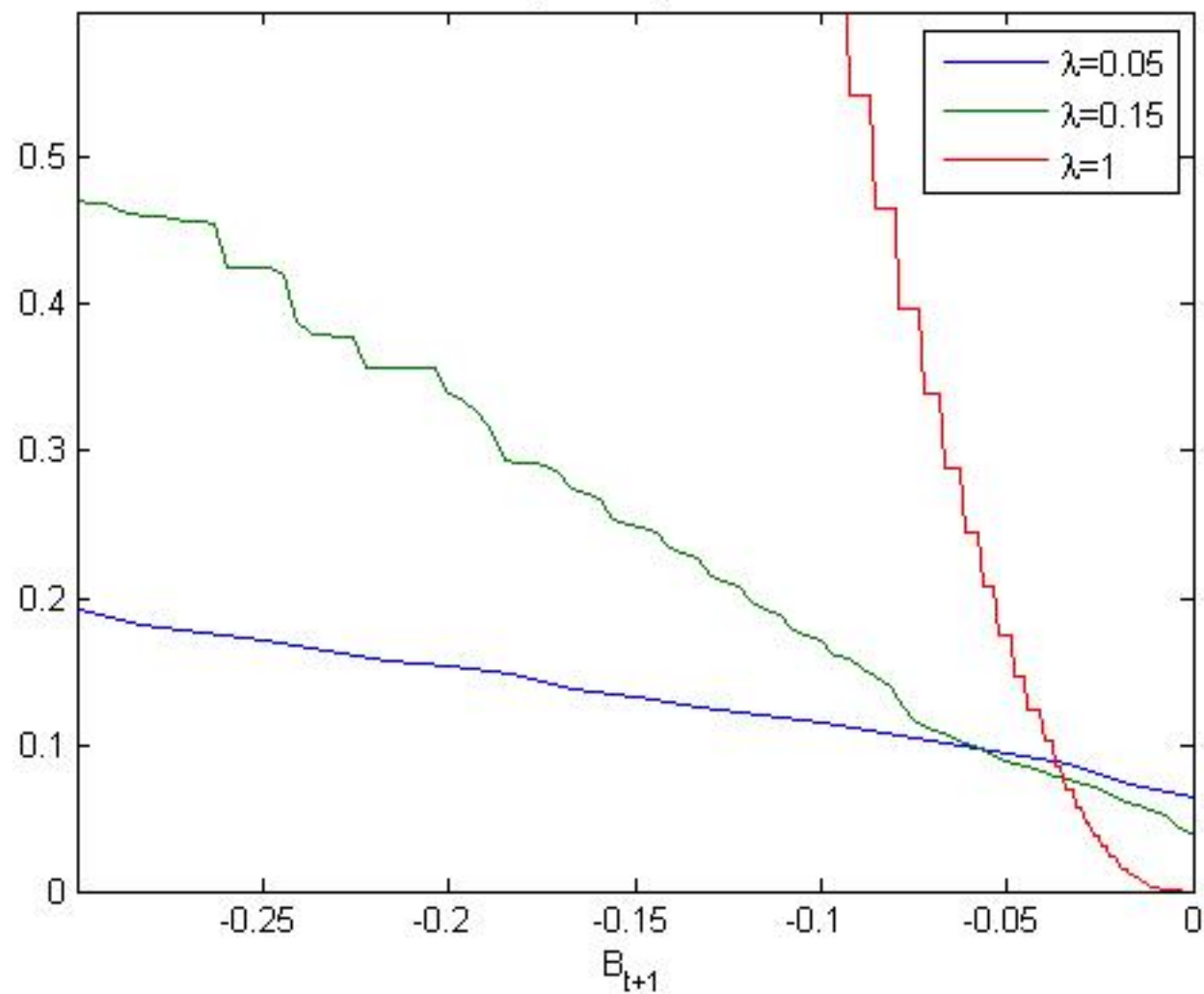


Figure 3: $q(y_{\text{fix}}, B_{t+1})$

