Recursive equilibrium in stochastic OLG economies*

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Abstract

We prove generic existence of recursive equilibrium for overlapping generations economies with uncertainty. Generic here means in a residual set of utilities and endowments. The result holds provided there is sufficient intragenerational household heterogeneity.

1 Introduction

The overlapping-generations (OLG) model, introduced first by Samuelson (1958), is one of the two major workhorses for macroeconomic and financial modeling of open-ended dynamic economies. Following developments in the study of two-period economies, the OLG model has been subsequently extended to cover stochastic economies with production and possibly incomplete financial markets. In such instances, the general notion of competitive equilibrium à la Arrow-Debreu has proven to be not very useful for applied, quantitative work, even in stationary Markovian environments. This is due, among other things, to the large dimensionality of the allocation and price sequences when histories of arbitrary length are allowed, which strains the ability of approximating solutions —through truncations— with present-day computers. It also strains the notion of rational expectations equilibrium because of the complexity of the forecasts involved in some of these equilibria.

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An advance in the direction of simplifying computations was provided by Duffie et al. (1994), who give a general theorem for the existence of stationary Markov equilibria for OLG economies, with associated ergodic measure.[1] While these equilibria provide the analogue in stochastic economies of steady states or low-order cycles in deterministic economies, they are still too complicated to allow for computational work. Hence, the applied literature has focused on a notion of simple time-homogeneous Markov equilibrium, also known as recursive equilibrium.[2]

In a recursive equilibrium the state space is reduced to the exogenous shocks and the initial distribution of wealth for the agents –asset portfolios from the previous period, and capital and storage levels if production is considered. A recursive equilibrium can be thought of as a time-homogeneous Markov equilibrium that is based on a minimal state space.

However, no existence theorem is available for such recursive equilibria. In fact, Kubler and Polemarchakis (2004) provide two examples of nonexistence of recursive equilibrium in OLG exchange economies. The idea that recursive equilibria may not exist is based on the observation that when there are multiple temporary equilibria the continuation of an equilibrium may depend on past economic variables other than the wealth distribution. That is, the current wealth distribution may not be enough to summarize the information contained in past equilibrium prices and marginal utilities.

While this phenomenon may occur, we prove that it is nongeneric — under some qualifying condition. The argument follows three fundamental observations.

The first is that it is always possible to find competitive equilibria which are time-homogeneous Markov over a simple state space. This is the state space made of the current exogenous state, the current wealth distribution, current commodity prices, and marginal utilities of income for all generations except for the first and the last ('newly born' and 'eldest'). This result is similar to what obtained or conjectured in Spear and Srivastava (1986) and Duffie et al. (1994).

The second observation is that in such equilibria prices and multipliers are typically a function of the current state and of the current wealth dis-

[1] See e.g. also the earlier work by Spear (1985) and Spear and Srivastava (1986), and by Cass et al. (1992) and Gottardi (1996).

[2] Examples by now abund; see, e.g., Rios Rull (1996), Constantinides, Donaldson and Mehra (2002), Geanakoplos, Magill and Quinzii (2004), Storesletten, Telmer and Yaron (2004a, 2004b).

tribution, provided that one can allow for sufficient dispersion of individual characteristics within each generation. We call these equilibria nonconfounding.

The trick we use here to prove the nonconfounding property is to abstract from the equilibrium equations and instead, using stationarity, focus on the optimization problem of an arbitrary individual born at any date-event. This trick allows one to bypass the infinite dimensional nature of the equilibrium set, and the fact that with overlapping generations there is an infinite number of individuals and an infinite number of market clearing equations, rendering direct genericity analysis quite problematic. A simple Markov equilibrium fails to be nonconfounding essentially when in a given state, called critical, all individuals of all types spend the same amount of money, while some prices for goods or assets from the critical state onward are different. However, typically this cannot be the case, as with *enough heterogeneity* in preferences and endowments there is almost always going to be an individual who will spend differently at different prices, no matter what prices are at all the remaining states. Since we need to check this for all admissible, and not just equilibrium, prices, the degree of heterogeneity we use must be large —it should be noted in passing that this is not at odds with the notion of price-taking behavior which is assumed in competitive models such as ours.[3]

Aside from the use of Sard's theorem, the nonconfounding property so obtained follows from a perturbation analysis of a family of individual dynamic programming problems. We show that when enough individuals face different prices, it is possible to control either their expenditure at a critical state, or some of their marginal utilities through wealth or utility changes at some point in the future, which then trickle down to induce different expenditures at the critical state for some agent.

The third and final step is to show that nonconfounding simple time-homogeneous Markov equilibria are indeed recursive equilibria.

[3] While the heterogeneity condition does not encompass the economies of the examples constructed by Kubler and Polemarchakis (2004), in a previous paper (Citanna and Siconolfi 2006) we show that such examples are nonrobust. That is, we show existence of recursive equilibria for an open and dense set within a class of exchange economies. In that class, individuals live for two periods –'young' and 'old' age–, restrictions apply to their preferences when young, but we allow for any degree of heterogeneity. Our main idea there is to prove existence of recursive equilibria via short-memory equilibria. Hence, that result complements the one of this paper in showing the generic character of recursive equilibria.

The notion of genericity we will use will rely on utility perturbations, and therefore will only be topological. In fact, due to the infinite dimension of the equilibrium set, we will not be able to establish local uniqueness of competitive equilibria, whether or not time-homogeneous. Without this prerequisite, the argument essentially showing some one-to-oneness property of prices will have to be made without knowing whether such prices are or not 'critical' —in fact, without knowing if they are equilibrium prices. Therefore, we will resort to an argument reminiscent of Mas-Colell and Nachbar (1991), and we will show the existence of recursive equilibria for a *residual* or *nonmeager* subset of parameters, i.e., a set of stationary utilities and endowments which is dense and is the countable intersection of open and dense sets. This is a well-established notion of genericity for dynamic systems.

Our class of OLG economies has multiple goods, generations and types within each generation; it allows for incomplete markets, with short-lived real —namely, numéraire— assets in zero net supply. While complete markets,[4] long-lived assets with nonzero real payoffs and in positive net supply, or production (à la Rios Rull 1996), can be added, we present the model and results without such features to simplify the exposition.

Our result suggests that the notion of recursive equilibrium is computationally useful as well as coherent as an exact concept, adding robustness to its interpretation and quantitative use.[5] The heterogeneity level we use in our theorem allows for the state space dimension in the recursive equilibrium to be drastically reduced, relative to the dimension that truncation techniques require when applied to a path-dependent competitive equilibrium. If we denote by $G+1$ the number of periods an individual lives, i.e., the number of generations present in an economy, this heterogeneity is proportional to the dimension of a tree of length $G+1$, and this is the minimal length of a truncated economy. Besides, while the number of agents must be large, their characteristics can be taken to be very similar for the underlying economy to have exact recursive equilibria.

[4] When markets are complete, the gist of the argument goes through, but density proofs are of a different nature. In a companion paper we provide the argument for the case of complete markets.

[5] For their interpretation as mere computing devices, see See Kubler and Schmedders (2005). When feedback policies are examined in a recursive formulation, the interpretation is of course strengthened by the exact nature of recursivity. The focus on these equilibria is also justified by their 'simplicity'.

2 The model

We consider standard stochastic OLG economies where time and uncertainty are represented by date-events, and a tree structure $(\widetilde{S}, \prec)$. Here $\widetilde{S}$ is a set and $\prec$ is a precedence relation: $\prec$ is irreflexive and transitive, and partially orders the set S. If s, s' are two elements of a chain in $\widetilde{S}$, with $s' \prec s$, and there is no other $s'' \in \widetilde{S}$ with $s' \prec s'' \prec s$, s' is the (unique) immediate predecessor of s, also denoted s_-. Let $\{s\}_+ \subset \widetilde{S}$ denote the set of *immediate successors* of s. We assume that $\{s\}_+ \neq \emptyset$ and $\#\{s\}_+ < \infty$ for all $s \in \widetilde{S}$. Let s_0 be the root of the tree, i.e., the unique element of $\widetilde{S}$ with no predecessor.

A notion of time is imposed as follows. We denote by $t(s) = t$ the length of the chain between s_0 and s. Note that $t(s_0) = 0$. The tree is then partitioned into 'dates', or equivalence classes of length t, with $t = 0, 1, ...$ representing such dates. Time is discrete, the horizon is infinite, but at each date t, $S_t < \infty$ events or states can be realized, and $\widetilde{S} = \cup_t S_t$ is countable. Each element of S_t is denoted by s_t. To each s_t, $\{s_t\}_+ \subset S_{t+1}$. We also define a *history* up to and including date t as an array of states, one for each date $\tau \leq t$, and is denoted by $s^t = (s_0, s_1, ..., s_t) \in S_0 \times ... \times S_t$, the set of all such histories; we also write $s^t = (s^{t-1}, s_t)$. A proper history s^t leading to a date-event s_t is the (finite) sequence of date-events, with $s_{\tau-1} = s_{\tau-}$ for all $\tau = 1, .., t$. From now on, when we refer to date event as histories, we always mean proper histories.

We are going to consider trees which are generated by a (finite) set S, i.e., where $\{s\}_+ = S$ for all $s \in \widetilde{S}$; for all $t > 0$, $s^t = (s_0, s_1, ..., s_t)$ where $s_\tau \in S$, all $\tau \leq t$. That is, an underlying first–order Markov stochastic process generates the tree, with time–invariant transition $\pi(s_{t+1}|s_t)$, all t. We assume that $\pi(s_{t+1}|s_t) > 0$ for all $s_t, s_{t+1} \in S \times S$ (full support of the transition).

At each s^t, there are $C \geq 1$ physical commodities, and the demographic structure is that of an overlapping generations economy. At each s^t, $H \geq 1$ individuals are born living $G + 1 \geq 2$ periods, or generations, indexed by $a = 0, ..., G$, from the youngest $(a = 0)$ to the oldest $(a = G)$ age. The economy starts off at s_0 with H individuals of each generation. This simple demographic structure can be generalized to any exogenous stochastic process which is a time-homogenous finite Markov chain.

The commodity space is the space of sequences $(\mathbb{R}^{(G+1)C}_{++})^{\widetilde{S}}$, $(G + 1)C$–dimensional vectors. The utility of the young agent h born at s^t is U^{h,s^t} :

$\mathbb{R}_{++}^{C(\sum_{a=0}^{G} S^{t+a})} \to \mathbb{R}$, which is time-separable and of the von Neumann - Morgenstern type:

$$U^{h,s^t}(x^{h0}(s^t), .., x^{hG}(s^{t+G}), ...) = \mathbb{E}_{s^t}\{\sum_{a=0}^{G} u^{ha}(x^{ha}(s^{t+a}), s^{t+a})\}.$$

In addition, each utility $u^{ha}(\cdot, s^{t+a}) : \mathbb{R}_{++}^{C} \to \mathbb{R}$ for $a = 0, ..., G$ is smooth, differentially strictly increasing, and strictly concave. The utility $u^{ha}(\cdot, s^{t+a})$ for $a > 0$ already includes a discount factor. Agent h, s^t is also endowed with physical goods at all ages, i.e., $e^{ha}(s^{t+a})$, all a. We assume that utilities and endowments are *stationary*, in that $u^{ha}(\cdot, s^{t+a}) = u^{ha}(\cdot, s_{t+a})$ and $e^{ha}(s^{t+a}) = e^{ha}(s_{t+a})$, for all a. That is, the economy has a first-order Markovian structure.

We also assume that $e_1^{ha}(s) > 0$ for all s, all a, all h, and that $\sum_{h,a} e_c^{ha}(s) > 0$ for $c > 1$, all s. Finally, we assume that u^{ha} satisfies the boundary condition $\lim_{n\to\infty} x_{1,n}^{ha} = 0$, then $\lim_{n\to\infty} ||D_1 u^{ha}(x_n^{ha}, s)||^{-1} x_n^{ha\prime} D_1 u^{ha}(x_n^{ha}, s) = 0$. These conditions simply state that good $c = 1$ is necessary to individuals of all ages, and that total resources in the economy are positive.

At each s^t, there are spot markets for the exchange of physical commodities. The price vector of the C commodities at s^t is $p(s^t) \in \mathbb{R}_{++}^{C}$. Commodity $c = 1$ is dubbed the numéraire commodity.

There are also $J < S$ one-period securities in zero net supply, paying in units of the numéraire commodity. Their prices are $q(s^t) \in \mathbb{R}^J$. Their payoffs at s^t are given by an $S \times J$-dimensional matrix $Y(s^t) = Y$ for all s^t, with column rank J –hence, there is no outside money. There is one asset, say asset $j = 1$, which has positive payoffs, $y_s^1 > 0$ for all $s \in S$, and Y is in general position. Agents hold $m^{ha}(s^t) \in \mathbb{R}^J$ units of these assets when $a < G$, without loss of generality.

Economies will be triples (e, π, u) of endowments, transition probabilities and utilities satisfying our assumptions. Endowments and probabilities lie in open subsets of Euclidean spaces. Utilities u^{ha} are points in the space of $\mathcal{C}^\infty(\mathbb{R}_{++}^{C}, \mathbb{R})$ functions with the topology of $\mathcal{C}^2$–uniform convergence. Namely, they belong to the G_δ subset of such space consisting of functions satisfying our maintained assumptions. Then, we let Ω be the space of such endowments, probabilities and utilities, endowed with the product topology.

For each $s^t \in \widetilde{S}$, we let $\Xi(s^t) \subset \Xi$ be the set of endogenous variables at s^t, i.e., of admissible vectors $\xi(s^t) = (x(s^t), m(s^t), p(s^t), q(s^t))$. The sequence $\xi =$

$\left(\xi(s^t), s^t \in \widetilde{S}\right) \in \times_{s^t \in \widetilde{S}} \Xi(s^t)$ represents a vector–valued stochastic process of consumption, asset holdings and prices for commodities and assets –adapted to $\widetilde{S}$. Given a tree $\widetilde{S}$ and a history $s^t \in \widetilde{S}$, we let $\widetilde{S}_{s^t}$ be the subtree starting at s^t, i.e., the set of s^τ with $\tau \geq t$ and $s^\tau = (s^t, s_{t+1}, ..., s_\tau)$. If $\xi \in \times_{s^t \in \widetilde{S}} \Xi(s^t)$, for any $s^t \in \widetilde{S}$, $\xi_{s^t} = \left(\xi(s^\tau), s^\tau \in \widetilde{S}_{s^t}\right) \in \times_{s^\tau \in \widetilde{S}_{s^t}} \Xi(s^\tau)$ is the process in the subtree starting at s^t, and we let $\xi_+(s^t) = (x^{ha}(s^t)_{a<G}, m^{ha}(s^t)_{a<G}, q(s^t), \xi_{s^{t+1}})$ be the *continuation* of the process at s^t.

2.1 Competitive equilibrium

Letting $m^{h(-1)}(s^t) \equiv 0$, $m^{hG}(s^t) \equiv 0$ for all h, s^t, a *competitive equilibrium* is a stochastic process ξ such that:

(H) agents optimize given prices, i.e., for each h, s^t,

$$\begin{gathered}
\max_{x^{ha}(s^{t+a})_{a=0}^G, m^{ha}(s^{t+a})_{a=0}^G} U^{h,s^t}(x^{h0}(s^t), .., x^{hG}(s^{t+G}), ...) \\
\text{s.t. } p(s^{t+a})[x^{ha}(s^{t+a}) - e^{ha}(s^{t+a})] + q(s^{t+a})m^{ha}(s^{t+a}) = \\
p_1(s^{t+a})y_{s_{t+a}} m^{h(a-1)}(s^{t+a-1}), \text{ for all } s^{t+a} = (s^{t+a-1}, s_{t+a}), \text{ all } a,
\end{gathered}$$

and at s^0, for $\overline{a} > 0$ agents of age $\overline{a}$ optimize given prices and initial portfolios,

$$\begin{gathered}
\max_{x^{ha}(s^{a-\overline{a}})_{a=\overline{a}}^G, m^{ha}(s^{a-\overline{a}})_{a=\overline{a}}^{G-1}} U^{h,s^0}(x^{h\overline{a}}(s^0), .., x^{hG}(s^{G-\overline{a}}), ...) \\
\text{s.t. } p(s^{a-\overline{a}})[x^{ha}(s^{a-\overline{a}}) - e^{ha}(s^{a-\overline{a}})] + q(s^{a-\overline{a}})m^{ha}(s^{a-\overline{a}}) = \\
p_1(s^{a-\overline{a}})y_{s_{a-\overline{a}}} m^{h(a-1)}(s^{a-\overline{a}-1}), \text{ for all } s^{a-\overline{a}} = (s^{a-\overline{a}-1}, s_{a-\overline{a}}), \text{ all } a \geq \overline{a},
\end{gathered}$$

with $m^{h(\overline{a}-1)}(s_{0-}) = \overline{m}^{h(\overline{a}-1)}(s_{0-})$ given;

(M) markets clear, or $\sum_{h,a}[x^{ha}(s^t) - e^{ha}(s^t)] = 0$ and $\sum_{h,a} m^{ha}(s^t) = 0$, all $s^t \in \widetilde{S}$.

Notice that, by time and state separability of the agents' utility functions, we do not need to write separately the optimization problem of agents of age $a > 0$ for histories $s^t \neq s^0$, as they are implied by their maximization when young. A competitive equilibrium exists for these economies using a standard truncation argument (see Balasko and Shell (1980)), and combining it with the argument for existence of equilibrium in a two-period economy with incomplete markets and numéraire assets (see Geanakoplos and Polemarchakis (1986)). Let $E(\omega, \overline{m})$ be the set of competitive equilibria of an economy $\omega \in \Omega$ with initial portfolio distribution $\overline{m} = \left(\overline{m}^{ha}(s_0)\right)_{h \in H, a>0}$.

As we explained in the Introduction, the focus of this paper will be on a special kind of competitive equilibrium, also known as recursive equilibrium, to which we now turn.

3 Recursive equilibrium

Let W be a subset of $\mathbb{R}^{HG}$, with element $w = (..., w^{ha}, ...)_{h \in H, a>0}$. This is the space of beginning-of-period financial wealth levels.

A *recursive equilibrium*[6] is the state space $S \times W$ together with time–invariant price functions $p(s,w), q(s,w)$, allocation functions $m^{ha}(s,w), x^{ha}(s,w)$ for all a, and transitions $T_a(s,w,s',w') = 0$, for $s, s' \in S$ and $w, w' \in W$, all $a < G$ such that:

(H') for each h, each $(s,w) \in S \times W$, individuals of age $\overline{a} \geq 0$ optimize given the price functions and $(T_a)_{G>a\geq\overline{a}}$, i.e.,

$$\begin{gathered} x^{ha}(s_a, w_a)_{a=\overline{a}}^{G}, m^{ha}(s_a, w_a)_{a=\overline{a}}^{G-1} \in \\ \arg\max_{x^{ha}(s_a,w_a)\in\mathbb{R}_+^C, m^{ha}(s_a,w_a)\in\mathbb{R}^J} U^h(x^{h\overline{a}}(s_{\overline{a}}, w_{\overline{a}}), ..., x^{hG}(s_G, w_G), ...) \\ \text{s.t. } p(s_a, w_a)[x^{ha}(s_a, w_a) - e^{hy}(s_a)] + q(s_a, w_a) m^{ha}(s_a, w_a) = \\ p_1(s_a, w_a) y_{s_a} m^{h(a-1)}(s_{a-1}, w_{a-1}), \text{ all } a \geq \overline{a} \end{gathered}$$

for all $(s_a, w_a)_{a=\overline{a}}^{G}$ s.t. $T_a(s_a, w_a, s_{a+1}, w_{a+1}) = 0$, for $a < G$, $(s_{\overline{a}}, w_{\overline{a}}) = (s,w)$, and with $m^{h(-1)}(s,w) \equiv 0$, $m^{hG}(s,w) \equiv 0$ all $(s,w) \in S \times W$;

(M') markets clear, or $\sum_{h,a}[x^{ha}(s,w) - e^{ha}(s)] = 0$ and $\sum_{h,a} m^{ha}(s,w) = 0$ for all $(s,w) \in S \times W$;

(R) the transitions T_a are consistent with optimization (H') and with nature's moves, that is, given $(s_a, w_a) \in S \times W$, then $T_a(s_a, w_a, s_{a+1}, w_{a+1}) = 0$ implies

- $s_{a+1} \in \{s_a\}_+$;
- $w_{a+1}^{h(a+1)} = (w)_{s_{a+1}}^{h(a+1)} = y_{s_{a+1}} m^{ha}(s_a, w_a)$ all $a < G$.

Note that in this definition the equilibrium process starts from a point in W. In particular, a recursive equilibrium is a competitive equilibrium only if the initial distribution of asset portfolios, $\overline{m}(s_{0-})$, multiplied by y_{s_0}, is in W.

Consumption plans for any age a only depend on the current state s, w; any more variability in consumption is not going to be optimal for the agent, given the equilibrium price functions and transition, separability and strict

[6] See, e.g., Rios Rull (1996).

concavity of the utility function. This is why feasible plans are already restricted to functions $x^{ha}(s, w)$.

3.1 Generic existence of recursive equilibrium

We want to show the existence of these equilibria in a large class of economies. In order to accomplish this, the basic idea is to show that the beginning-of-period wealth distribution is typically a sufficient statistic of the memory of the economic system at least at some competitive equilibrium.

3.1.1 Existence of simple time-homogeneous Markov equilibria

Let the initial portfolio distribution $\overline{m}$ be given. We are going to first show that there are always competitive equilibria which are time-homogenous and Markov on the state space given by p, $(\lambda^{ha})_{h\in H, 0<a<G}$, w, s, that is, the current exogenous state, current wealth distribution, current Lagrange multipliers for all individuals of all generations except for the eldest and the youngest, and current commodity prices. Denote by $E_M(\omega, \overline{m}) \subset E(\omega, \overline{m})$ the set of these Markov equilibria.

Lemma 1 *For all $\omega \in \Omega$, $E_M(\omega, \overline{m}) \neq \varnothing$.*

Proof. See the Appendix.

Let $\lambda(s) = (\lambda^{ha}(s))_{h\in H, 0<a<G}$ be the collection of marginal utilities of the numéraire commodity for all individuals $h \in H$ and cohorts a, $a = 1, ..., G-1$. Equilibria in $E_M(\omega, \overline{m})$ do not have multiplicity problems in any continuation. In particular, we have excluded situations where even if $p(s^t), \lambda(s^t)$, $w(s^t)$ and s_t are given at an arbitrary s^t, multiple equilibrium asset prices $q(s^t)$ and commodity price expectations $p(s^{t+1})$ are possible along the tree $\widetilde{S}$. Note that we do not know whether such selection process gives rise to continuous, or even measurable, transitions, but neither property will be used in our argument or is required in the definition of recursive equilibrium.

3.1.2 Nonconfounding simple time-homogeneous Markov equilibria

For the Markov equilibria in $E_M(\omega, \overline{m})$, a Markov state is defined by a current realization of $s \in S$, by commodity prices $p(s)$, by multipliers $\lambda(s)$, and by the initial wealth distribution $w(s)$. We want to show that the Markov equilibria which we constructed above, typically and under some qualifying condition on intragenerational heterogeneity, have the following injection property: if at s, two Markov states are given, $(p, \lambda, w; s)$ and $\left(\hat{p}, \hat{\lambda}, \hat{w}; s\right)$ with $w = \hat{w}$, then $(p, \lambda) = (\hat{p}, \hat{\lambda})$. We call Markov equilibria that satisfy this property *nonconfounding*, and denote the set of nonconfounding Markov equilibria $E_M^{NC}(\omega, \overline{m})$. Instead, a state s that violates this property is called *critical*, and the corresponding equilibrium *confounding*.

The qualifying condition on intragenerational heterogeneity used below is

A1 $H > 2[(C-1)\sum_{a=0}^{G} S^a + J\sum_{a=0}^{G-1} S^a]$.

We aim at proving the following theorem.

Theorem 2 *Under A1, $E_M(\omega, \overline{m}) = E_M^{NC}(\omega, \overline{m})$ in a residual subset Ω^* of Ω.*

Let s be the critical state, where after some pairs of histories $s_k^{T_k}$, $k = 1, 2$, our simple equilibrium fails to be nonconfounding. Then, in the corresponding competitive equilibrium, $w^{ha}(s_1^{T_1}) = w^{ha}(s_2^{T_2})$ for each h and any age $a > 0$, and either (i) $p(s_1^{T_1}, s) \neq p(s_2^{T_2}, s)$ or (ii) there exists some individual h^* of some age a^* such that $\lambda^{h^*a^*}(s_1^{T_1}, s) \neq \lambda^{h^*a^*}(s_2^{T_2}, s)$. Let s_s^a be an arbitrary history of length $a = 0, ..., G - a^*$ starting at s after histories $s_k^{T_k}$, i.e., with $s_s^0 = s$, and let $p(s_{k,s}^a)$ and $q(s_{k,s}^a)$, be commodity and asset prices at the history $(s_k^{T_k}, s_s^a)$, $a = 0, ..., G - a^*$, $k = 1, 2$. Given our assumptions on utilities and given wealth equality at s, if

$$(p(s_{1,s}^a), q(s_{1,s}^a)) = (p(s_{2,s}^a), q(s_{2,s}^a)), \text{ for } a = 0, ..., G{-}a^*, \text{ and } p(s_{1,s}^{G-a^*}) = p(s_{2,s}^{G-a^*}),$$

then individual (h^*, a^*)'s programming problem implies that $\lambda^{h^*a^*}(s_1^{T_1}, s) = \lambda^{h^*a^*}(s_2^{T_2}, s)$, a contradiction to (ii). Thus, (i) and (ii) require that some prices are different between the critical state $(s_k^{T_k}, s)$ and $(s_k^{T_k}, s_s^{G-a^*})$. Hence, independently of what a^* is, at a confounding price pair an individual born at $s_1^{T_1}$ faces during his lifetime a price sequence different than that faced by an individual born at $s_2^{T_2}$. Therefore, in order to rule out the existence of confounding equilibria and establish Theorem 2, very conveniently it suffices to show that typically if price sequences are different, wealths of individuals with age $a = 1$ at the critical state cannot all be equal.

At this junction, two possibilities arise: 1) one history, say $s_2^{T_2}$, is empty and s is the root of the tree $\widetilde{S}$, and the other is not; 2) neither history $s_k^{T_k}$ is empty. In case 2) is verified, we consider the demand functions of individuals of age $a = 0$ at some pair of states $(s_{01}, s_{02}) \in S^2$, for two sets of commodity prices $p(s_k^a)$, $s_k^a \in S^a$, $k = 1, 2$, and $a = 0, ..., G$, and of asset prices $q(s_k^a)$, $s_k^a \in S^a$, $k = 1, 2$, and $a = 0, ..., G - 1$, where $s_k^0 = s_{0k}$ for $k = 1, 2$. We do not need to keep track of the asset prices $q(s_k^G)$, $s_k^G \in S^G$, $k = 1, 2$, since in the absence of arbitrage the old generation G does not trade on the asset market.

Without loss of generality normalize commodity prices by setting

$$p_1(s_k^a) = 1, \text{ for all } s_k^a \in S^a, k = 1, 2,$$

To have a compact notation, let (p, q) denote the entire collection of prices over the two finite trees. Thus, $p \in \mathbb{R}_{++}^{2(C-1)\sum_{a=0}^{G} S^a}$ and $q \in \mathbb{R}^{2J\sum_{a=0}^{G-1} S^a}$. Let $N_{(s_{01}, s_{02})}$ be the set of parameters ω and variables

$$z = (\left(x^{ha}(s_k^a), \lambda^{ha}(s_k^a), m^{ha}(s_k^a)\right)_{h, s_k^a, a, k}, p, q)$$

which satisfy the Kuhn-Tucker equations

$$\begin{array}{cc}
Du^{ha}(x^{ha}(s_k^a)) - \lambda^{ha}(s_k^a) p(s_k^a) = 0, & (KT1, h; s_k^a) \\
-\lambda^{ha}(s_k^a) q(s_k^a) + \sum_s \pi(s|s^a) \lambda^{ha}(s_k^a, s) y_s = 0, \ a < G & (KT2, h; s_k^a) \\
p(s_k^a)[x^{ha}(s_k^a) - e^{ha}(s_k^a)] + q(s_k^a) m^{ha}(s_k^a) = & \\
y_s m^{h(a-1)}(s_k^{a-1}), \text{ for all } s_k^a = (s_k^{a-1}, s), \text{ all } a \geq 0 & (KT3, h; s_k^a)
\end{array}$$

with $q(s_k^G) = m^{hG}(s_k^G) \equiv 0$, for all $s_k^G \in S^G$, and $m^{h(-1)}(s_k^{-1}) = 0$, for $k = 1, 2$ and

$$\begin{array}{cc}
\sum_h x^{ha}(s_k^a) \leq M(\omega), & (i; k) \\
x^{ha}(s_k^a) \geq 0 & (ii; k)
\end{array} \qquad \text{(IN)}$$

Here $M(\omega) > 0$ is a real, with $M(\omega) = 2 \max_s \sum_{ha} e^{ha}(s)$. Note that this system has $2[(C - 1)\sum_{a=0}^{G} S^a + J\sum_{a=0}^{G-1} S^a]$ too many unknowns: $p(s_k^a)$, for $s_k^a \in S^a$, $a = 0, ..., G$, $k = 1, 2$, and $q(s_k^a)$, for $s_k^a \in S^a$, $a = 0, ..., G - 1$, $k = 1, 2$.[7]

[7]For $G > 1$; when $G = 1$, no asset prices are unmatched, and the term $J\sum_{a=1}^{G-1} S^a$ drops.

For $k = 1, 2$, define

$$\hat{p}(s_k^1) = \left(p(s_k^1), (\lambda^{h1}(s_k^1))_{h \in H}\right)$$

We are interested in the subset $N_{(s_{01},s_{02})}(s)$ of $N_{(s_{01},s_{02})}$ consisting of the solutions to equations $(KT1 - KT3)$, inequalities (IN), and the H equations

$$y_s(m^{h0}(s_{01}) - m^{h0}(s_{02})) = 0 \quad (KT4, h)$$

for all h, with

$$\left\|(\hat{p}(s_1^1) - \hat{p}(s_2^1)\right\| \neq 0 \tag{NC}$$

System $(KT1 - KT4)$ has now $H - 2[(C-1)\sum_{a=0}^{G} S^a + J\sum_{a=0}^{G-1} S^a]$ too many equations, a positive number under A1.

If instead case 1) is verified, then the maximization problem of the individual going through prices p_2, q_2 starts at $s_2^1 = (s_{02,}s)$; the individual budget constraint at $s_2^1 = (s_{02,}s)$ is $p(s_2^1)[x^{h1}(s_2^1) - e^{h1}(s)] + q(s_k^a)m^{ha}(s_k^a) = y_s\overline{m}^{h0}(s_{02})$, and it depends on $\overline{m}^{h0}$, the exogenous initial wealth of age $a = 1$ individuals. As will be evident from case 2), this wealth level, and therefore the entire initial distribution $y_{s_0}\overline{m}$, plays no role in the perturbations. Thus, it is enough to focus on case 2) without loss of generality. All the generic sets obtained hereafter must simply be considered indexed by a given initial wealth distribution $y_{s_0}\overline{m}$.

With this caveat in mind, clearly any $\xi \in E_M(\omega, \overline{m})$ 'goes through' system $(KT1 - KT3)$ and (IN), i.e., ξ intersects some $N_{(s_{01},s_{02})}$. As argued above, if $\xi \in E_M(\omega, \overline{m})$ but $\xi \notin E_M^{NC}(\omega, \overline{m})$, then $(KT4)$ must also hold when (NC) holds for some pair of histories (s_1^1, s_2^1) with current state s, or ξ intersects also some $N_{(s_{01},s_{02})}(s)$. We are then going to show that in a residual set of parameters $\Omega_{(s_{01},s_{02})}(s)$ the set $N_{(s_{01},s_{02})}(s)$ is empty. Repeating this across all finitely many pairs of initial states (s_{01}, s_{02}) and states s, we obtain a residual set

$$\Omega^* = \cap_{(s_{01},s_{02})} \cap_s \Omega_{(s_{01},s_{02})}(s)$$

where if $(p(s), \lambda(s), w(s); s)$ and $\left(\hat{p}(s), \hat{\lambda}(s), \hat{w}(s); s\right)$ are given and $(p(s), \lambda(s)) \neq \left(\hat{p}(s), \hat{\lambda}(s)\right)$, then $w(s) \neq \hat{w}(s)$, i.e., $E_M(\omega, \overline{m}) = E_M^{NC}(\omega, \overline{m})$ if $\omega \in \Omega^*$, under A1, proving Theorem 2.

To show that $N_{(s_{01},s_{02})}(s)$ is empty in a residual set $\Omega_{(s_{01},s_{02})}(s)$, we are going to restrict attention to sets $N^n_{(s_{01},s_{02})}(s)$ where $(KT1 - KT4)$ and (IN)

hold, and moreover z satisfies

$$\left\|\hat{p}(s_1^1) - \hat{p}(s_2^1)\right\| \geq 1/n, \tag{NC.n}$$

for some positive integer n. Since $N_{(s_{01},s_{02})}(s) \subset \cup_n N^n_{(s_{01},s_{02})}(s)$, emptiness of $N^n_{(s_{01},s_{02})}(s)$ for all n will imply that $N_{(s_{01},s_{02})}(s)$ is empty.

We are then left to showing the following proposition.

Proposition 3 *Given any* (s_{01}, s_{02})*,* s *and* n*, there is an open and dense set* $\Omega^n_{(s_{01},s_{02})}(s) \subset \Omega$ *such that* $N^n_{(s_{10},s_{20})}(s)$ *is empty.*

Setting

$$\Omega_{(s_{01},s_{02})}(s) = \cap_n \Omega^n_{(s_{01},s_{02})}(s)$$

will deliver the desired conclusion. For suppose not, and there is a $z \in N_{(s_{01},s_{02})}(s)$. Then there is an $\hat{n} > 0$ such that $(NC.\hat{n})$ holds true and $(KT4)$ is satisfied. But this implies $N^{\hat{n}}_{(s_{01},s_{02})}(s) \neq \varnothing$, a contradiction. Hence, in a residual set of economies, if $\xi \in E_M(\omega, \overline{m})$ and $(p(s), \lambda(s)) \neq \left(\hat{p}(s), \hat{\lambda}(s))\right)$, then $\|w(s) - \hat{w}(s)\| \neq 0$, as desired. Section 4 is devoted to proving Proposition 3.

3.1.3 Main result

We are now ready to state and prove our main result. Again, let the initial portfolio distribution $\overline{m}$ be given.

Theorem 4 *Under A1, recursive equilibria exist on the residual set* $\Omega^* \subset \Omega$*.*

Proof. See the Appendix.

Note that our construction delivers recursive equilibria which are competitive equilibria, since initial conditions can always be included in W. Remember that the set Ω^* depends on the initial wealth distribution $y_{s_0}\overline{m}$. Of course, we can say nothing about the uniqueness of such equilibria. Also, we do not know whether the function $\xi(s, w)$ is continuous or even measurable on $S \times W$.

4 Perturbation Analysis

In this technical section we are going to prove Proposition 3. We separate the openness and density parts. For density, we use transversality, i.e., Sard's and the preimage theorems, to show that $N^n_{(s_{01},s_{02})}(s)$ is empty in a dense subset of economies. In other words, we resort to a perturbation analysis of the individual demand function at all admissible prices.

4.0.4 Openness

The set of endogenous variables and parameters $N^n_{(s_{01},s_{02})}(s)$ is closed in $N_{(s_{01},s_{02})}$, since it is the preimage of continuous functions and weak inequalities. If the projection of endogenous variables and parameters satisfying $(KT1{-}KT3)$, and (IN), that is, $\Pr : N_{(s_{01},s_{02})} \to \Omega$, is proper, $\Omega \backslash \Omega^n_{(s_{01},s_{02})}(s)$, the set of parameters $\Pr(N^n_{(s_{01},s_{02})}(s))$ is closed. Clearly, its complement $\Omega^n_{(s_{01},s_{02})}(s)$ is an open set of parameters where either $(KT4)$ does not hold or $\|\hat{p}(s_1^1) - \hat{p}(s_2^1)\| < 1/n$, i.e., where $N^n_{(s_{01},s_{02})}(s)$ is empty.

Lemma 5 *The projection* $\Pr$ *is proper.*

Proof. See the Appendix.

4.0.5 Density

We are going to show that for a dense subset $\Omega^n_{(s_{01},s_{02})}(s)$, either $(KT4)$ does not hold or $\|\hat{p}(s_1^1) - \hat{p}(s_2^1)\| < 1/n$. First observe that inequalities $(IN.ii;k)$ hold strictly when $z \in N^n_{(s_{01},s_{02})}(s)$. This follows from equations $(KT1, h; s_k^a)$, $k = 1, 2$, and the boundary condition on u^{ha}. Let $F(z, \omega) = 0$ represent system $(KT1 - KT4)$. It is enough to show that $F(z, \omega) = 0$ has, typically, no solution in Ω, disregarding the inequalities $\|\hat{p}(s_1^1) - \hat{p}(s_2^1)\| \geq 1/n$ and $(IN.i;k)$ —i.e., assuming that they do not bind— since this will be *a fortiori* true if the inequalities are satisfied (note that if they are satisfied with equality, we are adding equations and so potentially reducing the dimensionality of $N^n_{(s_{01},s_{02})}(s)$ relative to $F^{-1}(0)$).

We want to show that for each $n > 0$, (s_{01}, s_{02}) and s, there is a dense set $\Omega^n_{(s_{01},s_{02})}(s)$ such that $N^n_{(s_{01},s_{02})}(s) = \varnothing$. Without loss of generality, we will show this by fixing the critical state to be $s = 1$. Let f^h represent the l.h.s. of equations $(KT1, h) - (KT3, h)$, f_k^h representing the l.h.s. of equations

$(KT1,h;s_k^a)-(KT3,h;s_k^a)$ for a pair (h,k); let W^h represent the l.h.s. of $(KT4,h)$, and $F^h=\left(f_1^h,f_2^h,W^h\right)=\left(f^h,W^h\right)$.

We start by showing that either (i) $\|p(s_{01},1)-p(s_{02},1)\|\geq 1/n$ or (ii) $\left\|\lambda^{h1}(s_{01},1)-\lambda^{h1}(s_{02},1)\right\|\geq 1/n$ for some $h\in H$ implies that F^h is transversal to zero.

Lemma 6 *(i) If $p(s_{01},1)\neq p\left(s_{02},1\right)$, then the rank of $D_{x,\lambda,m,e}F$ is full.*
(ii) If $\lambda^{h1}(s_{01},1)\neq\lambda^{h1}(s_{02},1)$, then the rank of $D_{x^h,\lambda^h,m^h,u^h}F^h$ is full.

Proof. See the Appendix.

When $G=1$, $(NC.n)$ only takes the form $\|p(s_{01},s)-p(s_{02},s)\|\geq 1/n$ for some critical state s. The argument establishing Proposition 3 must only show that equations $(KT4)$ cannot hold when $p(s_{01},s)\neq p(s_{02},s)$, hence it is concluded with Lemma 6.i. The proof of this lemma shows that in traditional overlapping generations models, with only young and old generations, the ensuing existence of recursive equilibrium can be guaranteed for *any* utility function and with a *low* degree of intragenerational heterogeneity.

Economies with $G\geq 2$ require a more elaborated argument, which is worked out below. Always under the assumption that $(NC.n)$ takes the form

$$\left\|\lambda^{h'1}(s_{01},1)-\lambda^{h'1}(s_{02},1)\right\|\geq 1/n \text{ for some } h'\in H.$$

we need to show that when $\lambda^{h1}(s_{01},1)=\lambda^{h1}(s_{02},1)$ either rank of $D_{x^h,\lambda^h,m^h,\omega^h}F^h$ is full, or else directly equations $(KT4)$ do not hold. As argued in Section 3.1.2, observe that $(NC.n)$ when $p(s_{01},1)=p\left(s_{02},1\right)$ together with equation $(KT4,h')$ implies that either $q(s_{01},1)\neq q(s_{02},1)$ or $(p(s_1^a),q(s_1^a))\neq(p(s_2^a),q(s_{2,s}^a))$ for some $s_k^a\succ(s_{0k},1)$, $k=1,2$ (remember that $\prec$ stand for 'precedes'). Hence, $(NC.n)$ implies now that

$$||(q(s_{01},1),(p(s_1^a),q(s_1^a))_{s_1^a\succ(s_{01},1)})-(q(s_{02},1),\left(p(s_2^a),q(s_{2,s}^a)\right)_{s_2^a\succ(s_{02},1)})||\geq 1/n \tag{1}$$

In order to proceed, we will take advantage of some recursive structure of wealth equalities and lambda inequalities. For each $n>0$, we embed the H equations $(KT4,h)$ into a finite family of functions of differences in multipliers, with a recursive structure dictated by two conditions on the zeros of these functions and their derivatives. We prove that under these conditions the first layer of this family —to be later identified with equations $(KT4)$— does not have a solution in a dense subset of Ω, proving our claim.

Recall that $\Omega = \times_{h\in H}\Omega^h$ is a smooth manifold of parameters, where Ω^h admits an at least locally finite parametrization. Let

$$(p_k, q_k) = \left(\left(p(s_k^a), q(s_k^a)_{s_k^a\in S^a}\right)_{a=0,...,G-1}, p(s_k^G)\right), k = 1, 2.$$

so that $(p, q) = (p_k, q_k)_{k=1,2}$. For $n > 0$, let

$$P^n = Int\{(p, q) : (1) \text{ holds}\},$$

be the $1/n$-domain of price pairs, where by Lemma 6.i $p(s_{01}, 1) = p(s_{02}, 1)$ without loss of generality. At $(p, q) \in P^n$ and ω, by regularity of demand we can solve the Kuhn-Tucker conditions for optimal asset portfolios and multipliers as functions of (p, q) and ω^h. Hereafter drop all the superscripts n and write $P^n = P$. Note that P is a smooth manifold of dimension N with $N < H$, by Assumption A1. Given any $s^a = (s^{a-2}, \bar{s}, \bar{\sigma})$, identify any other pair (s', σ') as $(s', \sigma') = (s_{j'}(\bar{s}), \sigma_l(\bar{\sigma}))$ for $j', l \in \{0, ..., S-1\}$, where $s_0(\bar{s}) = \bar{s}$, and $s_{j'}(\bar{s}) \neq \bar{s}$ otherwise, and similarly $\sigma_0(\bar{\sigma}) = \bar{\sigma}$, and $\sigma_l(\bar{\sigma}) \neq \bar{\sigma}$ for $l > 0$.

For each $h \in H$, we then define a family of Σ maps:

1) $Z^h_{s^0}(p, q, \omega) = Z^h_0(p, q, \omega) = y_1[m^{h0}(s_{01})(p_1, q_1; \omega^h) - m^{h0}(s_{02})(p_2, q_2; \omega^h)]$;

2) for $a \leq 2$, $Z^h_{s^a,0}(p, q, \omega) = \lambda^{ha}(s^{a-1}, \sigma)(p_1, q_1; \omega^h) - \lambda^{ha}(s^{a-1}, \sigma)(p_2, q_2; \omega^h)$, where $s^1 \succeq (s_0, 1)$

3) for each $a \geq 3$ and $s^a = (s^{a-2}, s, \sigma)$,

$Z^h_{s^a,j}(p, q, \omega) = \lambda^{ha}(s^{a-2}, s, \sigma_l)(p_1, q_1; \omega^h) - \lambda^{ha}(s^{a-2}, s_{j'}, \sigma_l)(p_2, q_2; \omega^h)$, for $j = lS + j'$, and $j', l \geq 0$

$Z^h_{s^a,j}(p, q, \omega) = \lambda^{ha}(s^{a-2}, s, \sigma_l)(p_k, q_k; \omega^h) - \lambda^{ha}(s^{a-2}, s_{j'}, \sigma_l)(p_k, q_k; \omega^h)$, for $j = l(S-1) + j' + S(S-1)(k-1) + S^2$, for $j' > 0, l \geq 0$.

So, for each a and s^a this is a collection of $J_a + 1$ real-valued and twice continuously differentiable maps $Z^h_{s^a,j} : P \times \Omega^h \to \mathbb{R}$, for all h, $j = 0, ..., J_a$. Then, Σ is the set of the indexes (s^a, j) of the various maps. From this collection, we will select H maps, that is: let $\mathcal{G} = \{g : g : H \to \Sigma\}$ be the set of maps that associate to each individual h a function $Z^h_{g(h)}$ in the family, and let $Z_g = (Z^h_{g(h)})_{h\in H} : P \times \Omega \to \mathbb{R}^H$, for $g \in \mathcal{G}$, with $Z_{\mathbf{0}} = (Z^h_0)_{h\in H}$.

For the sequel of the argument bear in mind that

$$D_{\omega^h} Z^{h'}_{s^a,j} = 0, \text{ for } h' \neq h.$$

Let

$$E^h_{s^a,j} = \{(p,q,\omega) \in P \times \Omega : D_{\omega^h} Z^h_{s^a,j} \neq 0\}$$

and let

$$N^h_{s^a,j} = \{(p,q,\omega) \in P \times \Omega : Z^h_{s^a,j} \neq 0\},\ 0 \leq j \leq J_a.$$

Also, for given s^a and any given subset of indexes $J' \subset J_a \backslash \{0\}$, let

$$E^{h,J'}_{s^a} = \cup_{j \in J'} E^h_{s^a,j} \text{ and } N^{h,J'}_{s^a} = \cap_{j \in J'} N^h_{s^a,j}$$

and for any nonempty subset $S' \subset S$ and $s^a = (s^{a-1}, s)$ let

$$E^{h,S'}_{s^{a-1}} = \cup_{s \in S'} E^h_{(s^{a-1},s),0} \text{ and } N^{h,S'}_{s^{a-1}} = \cap_{s \in S'} N^h_{(s^{a-1},s),0}$$

Finally, let

$$E_g = \{(p,q,\omega) : D_\omega Z_g \text{ is onto}\} \text{ and } N_g = \{(p,q,\omega) : Z_g \neq 0\}$$

Evidently, $E_g = \cap_h E^h_{g(h)}$ and $N_g = \cup_h N^h_{g(h)}$. Since Z_g is twice continuously differentiable, E_g is an open subset of $P \times \Omega$. Let $\phi_g : E_g \times \Omega \to \mathbb{R}^H \times \Omega$ be a smooth map defined as

$$\phi_g(e,\omega) \equiv [Z_g(e), proj_\Omega(e) - \omega], \text{ for } (e,\omega) \in E_g \times \Omega.$$

By construction ϕ_g is transversal to zero and hence, since $H > N$, by the transversality and preimage theorems, $\phi^{-1}_{g,\omega}(0)$ is an empty set, for $\omega \in \Omega_g$, a dense subset of $\omega \in \Omega$. Let $\Omega^* = \cap_{g \in G} \Omega_g$, a dense subset of Ω.

The argument recursive structure hinges upon the existence of a first time in history where some functions in the family have nonzero derivative, and before that point they are zero, that is on the existence of a minimal history.

History s^a *is minimal* at (p,q,ω) for $h \in H$ if $(p,q,\omega) \in E^h_{s^a,0}$, while $(p,q,\omega) \in [E^h_{s^{a'},0} \cup N^h_{s^{a'},0}]^c$ for all $a' < a$. *Selection* g *is minimal* at (p,q,ω) if for all $h \in H$, $g(h) = (s^a_h, 0)$ and s^a_h is minimal at (p,q,ω) for h.

Let's now introduce the following conditions on the family $\left(Z^h_{\sigma \in \Sigma}\right)_{h \in H}$.

Condition Univ *For each $h \in H$ and any $(p,q,\omega) \in P \times \Omega$, there exists s^a, $a \geq 1$, such that $(p,q,\omega) \in E^h_{s^a,0}$.*

Condition Nest *For any $(p,q,\omega) \in P \times \Omega$, let $s^a = (s^{a-1}, s)$, $a \geq 2$, be minimal for h. Then:*

0)

$$(p,q,\omega)\in N^{h,J_a}_{s^a}\cup E^{h,J_a}_{s^a};$$

1) for all proper subsets $J'\subset J_a$,

$$N^{h,J_a\backslash J'}_{s^a}\ \subset\ E^{h,J'}_{s^a};$$

2) for $S'\subset S$

$$N^{h,J_a}_{s^a}\cap N^{hS\backslash S'}_{s^{a-1}}\ \subset E^{h,S'}_{s^{a-1}};$$

3)

$$N^{h,J_a}_{s^a}\cap N^{hS}_{s^{a-1}}\ \subset U_{a'<a}E^{h}_{s^{a'},0}.$$

Conditions Univ and Nest have a very powerful implication. Condition Univ will imply the existence of a minimal history. Condition Nest-3 allows 'rolling back' through the system in a recursive fashion.

Proposition 7 *Under the maintained assumptions, in the dense subset $\Omega^*\subset\Omega$, $Z_{\mathbf{0}}(p,q,\omega)=0$ does not have a solution.*

The proof of the proposition is broken into two lemmas.

Lemma 8 *For $(p,q,\omega)\in P\times\Omega^*$, either (i) $Z_{\mathbf{0}}(p,q,\omega)\neq 0$; or (ii) there exists $g^*\in\mathcal{G}$ and $h^*\in H$, such that $(p,q,\omega)\in E_{g^*}$, $g^*(h^*)=(s^a,0)$, $(p,q,\omega)\in N^{h^*}_{g^*(h^*)}$, for some s^a minimal at (p,q,ω) for h^*, and $(p,q,\omega)\in[N^h_{g^*(h)}]^c$, for all $h\neq h^*$.*

Proof: By Condition Univ, $\cup_{g\in\mathcal{G}}E_g=P\times\Omega$, and for each (p,q,ω) there exists a minimal history for each h —although it may be not be unique. Then, there exists a minimal g at (p,q,ω) such that $(p,q,\omega)\in E_g$. By definition of Ω^*, $\omega\in\Omega_g$ and hence $(p,q,\omega)\in N_g\cap E_g$. We now construct g^*. Starting from $h=1$, let h^* be the first h such that $Z^{h^*}_{g(h^*)}(p,q,\omega)\neq 0$. If $g(h^*)=0$, the argument is over, establishing (i).

Hence, suppose that this is not the case and set $g^*(h^*)=g(h^*)$. Consider now $h=h^*+1$ and let $s^a=(s^{a-1},s)$ be h's minimal state. If $Z^h_{g(h)}(p,q,\omega)=0$, set $g^*(h)=g(h)$ and proceed to $h+1$. Otherwise, by Condition Nest-0 and Nest-1, either $(p,q,\omega)\in N^{h,J_a}_{s^a}$ or $(p,q,\omega)\in E^h_{s^a,j}\backslash N^h_{s^a,j}$, for some $j>0$. In the second case, set $g^*(h)=(s^a,j)$ and proceed to $h+1$. Otherwise, by Condition Nest-2, either there exists $(s^{a-1},s')\neq(s^{a-1},s)$ such that $(p,\omega)\in E^h_{(s^{a-1},s),0}\backslash N^h_{(s^{a-1},s),0}$, or $(p,\omega^h)\in N^{hS}_{s^{a-1}}$. In the first case, we set $g^*(h)=$

$((s^{a-1},s'),0)$ and proceed to $h+1$. The second case is impossible. Indeed, if $(p,q,\omega)\in N^{h,J_a}_{s^a}\cap N^{hS}_{s^{a-1}}$, by Condition Nest-3, $(p,q,\omega)\in E^h_{s^{a'},0}$, with $a'<a$. However, this contradicts s^a being minimal for h. Iterating up through H delivers the statement.$\blacksquare$

The next lemma concludes the argument.

Lemma 9 *For $(p,q,\omega)\in P\times\Omega^*$, if Lemma 8.(ii) holds, then $Z_{\mathbf{0}}(p,q,\omega)\neq 0$.*

Proof: Let g^* be the selection of Lemma 8.(ii), and consider the corresponding h^* and let $s^a=(s^{a-1},s)$ be such that $g^*(h^*)=((s^{a-1}s),0)$. By Condition Nest-0 and Nest-1, either $(p,q,\omega)\in N^{h^*,J_a}_{s^a}$ or $(p,q,\omega)\in E^{h^*}_{s^a,j}\backslash N^{h^*}_{s^a,j}$, for some $j>0$. We claim that the second case is impossible. Indeed, if $(p,q,\omega)\in E^{h^*}_{s^a,j}\backslash N^{h^*}_{s^a,j}$, consider the system $Z_g=[(Z^h_{g^*(h)})_{h\in H\backslash h^*},Z^{h^*}_{s^a,j}]$. Then, $D_\omega Z_g$ is onto, and $(p,q,\omega)\in N_g$, since $\omega\in\Omega^*$. However, by construction $Z^h_{g^*(h)}(p,q,\omega)=0$, for $h\neq h^*$. Thus, it must be that $Z^{h^*}_{s^a,j}(p,q,\omega)\neq 0$ or, equivalently, $(p,q,\omega)\in N^{h^*,J_a}_{s^a}$. Then, by Condition Nest-2, either there exists $(s^{a-1},s')\neq(s^{a-1},s)$ such that $(p,q,\omega)\in E^{h^*}_{(s^{a-1},s),0}\backslash N^{h^*}_{(s^{a-1},s),0}$, or $(p,q,\omega)\in N^{h^*S}_{s^{a-1}}$. By the same argument of Lemma 8, the second case is impossible. Thus, $(p,q,\omega)\in N^{h^*,J_a}_{s^a}\cap N^{h^*S}_{s^{a-1}}$. Then, by Condition Nest-3, $(p,q,\omega)\in E^{h^*}_{s^{a'},0}$, with $a'<a$, contradicting the fact that s^a is minimal at (p,ω) for h^*. $\blacksquare$

We are then left to show that Conditions Univ and Nest are indeed satisfied by our family of functions. This is done in the Appendix.

When $G=2$ (as in, say, Constantinides et al., 2002, or in Geanakoplos et al., 2004)) only $Z^h_{s^a,0}$ need to be carried around, $a\leq 2$. Proposition 7 proves that $Z_0(p,q,\omega)\neq 0$ for $p\in P^{1/n}$ and ω in a dense subset Ω^n of Ω. Proposition 7 and Lemma 6.i then imply that there exists a dense set $\Omega^n_{(s_{01},s_{02})}(s)$ where $N^n_{(s_{01},s_{02})}(s)$ is empty, which is what we were to prove.

5 Appendix

Proof of Lemma 1

We are going to construct one such equilibrium for each ω. Start from a $\xi\in E(\omega,\overline{m})$ and take two date-events $s^t,s^{t\prime}$. Assume that the vectors $\xi(s^t)$ and $\xi(s^{t\prime})$ are such that

$p(s^t)=p(s^{t\prime})$,

$\lambda^{ha}(s^t)=\lambda^{ha}(s^{t\prime})$ for all h, all $0<a<G$,

$w^{ha}(s^t) = y_{s_t} m^{h(a-1)}(s^{t-1}) = y_{s'_t} m^{h(a-1)}(s^{t-1\prime}) = w^{ha}(s^{t\prime})$ for all h, all $a > 0$,

and $s_t = s'_t = s$.

We show that we can replace the variables in $\xi_+(s^{t\prime})$ with the corresponding variables in $\xi_+(s^t)$ without altering the equilibrium, i.e., without falsifying any of the competitive equilibrium equations. This is obvious for $\xi(s^\tau)$ with either $\tau > t'$ and $s^\tau \notin \widetilde{S}_{s^{t\prime}}$, or $\tau \leq t'$ and $s^\tau \not\leq s^{t\prime}$, since we have not touched these variables, or the equations where they appear. It is also obvious for $\xi(s^\tau)$ with $\tau > t'$ and $s^\tau \in \widetilde{S}_{s^{t\prime}}$, since $\xi_+(s^t)$ was an equilibrium to start with. As for s^τ with $\tau \leq t'$ and for $s^\tau \leq s^{t'}$, the only equations where variables in $\xi_+(s^{t\prime})$ appear are: the budget constraints at $(s^{t\prime})_- = s^{t-1\prime}$, where $m^{h(a-1)}(s^{t-1\prime})$ is, for $a > 0$, all h; the no arbitrage conditions also at $s^{t-1\prime}$, where $\lambda^{ha}(s^{t-1\prime}, s)$ appears; the first-order conditions to problem (H), the budget constraints and the no arbitrage equations at $s^{t\prime}$, for all h, all a. By assumption, $\lambda^{ha}(s^{t\prime}) = \lambda^{ha}(s^t)$ for all h, all $0 < a < G$. Since $y_s w^{h(G-1)}(s^{t-1}) = y_s m^{h(G-1)}(s^{t-1\prime})$ for all h, and since $p(s^t) = p(s^{t\prime})$, the optimization problem for individuals of age $a = G$ at s^t and $s^{t\prime}$ is the same. Given strict concavity of u^{hG}, it has a unique solution given commodity prices and initial wealth, $\left(x^{hG}(s^{t\prime}), \lambda^{hG}(s^{t\prime})\right) = \left(x^{hG}(s^t), \lambda^{hG}(s^t)\right)$ and all no arbitrage equations at $s^{t-1\prime}$ are still satisfied after the substitution. Wealth equality at successor state $s^{t\prime}$ is the only constraint that $m^{h(a-1)}(s^{t-1\prime})$ must satisfy for all h, all $a < G$, but this has been assumed to hold. Finally, FOCs, budget constraints and no arbitrage equations can simply be substituted without altering anything else in the equilibrium.

Next, we construct the process $\hat{\xi} \in \times_{s^t \in \widetilde{S}} \Xi(s^t)$ from ξ recursively as follows. Use the natural order sum, $\sum_t S_t$, on $\widetilde{S}$. Let $s^t(n)$ be the node reached at step $n \geq 1$ of the construction algorithm. Let $\hat{\xi}_{n-1} \in \times_{s^t \in \widetilde{S}} \Xi(s^t)$ be given, with $\hat{\xi}_0 = \xi$. For any array $(p, (\lambda^{ha})_{h \in H, 0<a<G}, w, s)$ possible in $\hat{\xi}_{n-1}$, let

$$S_{n-1}(p, (\lambda^{ha})_{h \in H, 0<a<G}, w, s) =$$
$$\{s^t \in \widetilde{S} | p(s^t) = p, \lambda^{ha}(s^t) = \lambda^{ha}, h \in H, 0 < a < G, w(s^t) = w, s_t = s\}$$

be the corresponding equivalence class. Since under the order sum $\widetilde{S}$ is well-ordered, $\min S_{n-1}(p, (\lambda^{ha})_{h \in H, 0<a<G}, w, s) = s_{n-1}^{p,(\lambda^{ha})_{h \in H, 0<a<G}, w, s}$ is well defined. Define $\hat{\xi}_n$ by assigning to $s^t(n)$ the continuation $\hat{\xi}_{n,+}(s^t(n)) = \hat{\xi}_{n-1,+}(s_{n-1}^{p,(\lambda^{ha})_{h \in H, 0<a<G}, w, s})$, and letting $\hat{\xi}(s^t(n))$ be the $s^t(n)$-th element in the

vector $\hat{\xi}_{n+}(s^t(n))$, and then go on to step $n+1$. By construction and choice of the order on $\widetilde{S}$, each node s^t is eventually assigned a unique value $\hat{\xi}(s^t)$ in at most $n < +\infty$ many steps, with $s^t = s^t(n)$. Moreover, $\hat{\xi}$ is uniquely determined at each node by s, w, $(\lambda^{ha})_{h\in H, 0<a<G}$ and p: for each s^t, with $s_t = s$ and $\hat{\xi}(s^t)$ such that $\hat{p}(s^t) = p, \hat{\lambda}^{ha}(s^t) = \lambda^{ha}$, all $h \in H, 0 < a < G$, $w(s^t) = w$, there is a unique continuation $\hat{\xi}_+(s^t) = \hat{\xi}(p, (\lambda^{ha})_{h\in H, 0<a<G}, w, s)$. Since the process on S is Markov, so is the one on $(p, (\lambda^{ha})_{h\in H,0<a<G}, w, s)$. Since $\xi \in E(\omega, \overline{m})$, then also $\hat{\xi} \in E(\omega, \overline{m})$, i.e., $\hat{\xi} \in E_M(\omega, \overline{m})$ and we are done.■

Proof of Lemma 5

Consider a sequence $\{\omega_m\}_{m=1}^{+\infty} \subset \Omega$, with $\omega_m \to \omega \in \Omega$, and any associated sequence $\{z_m\}_{m=1}^{+\infty} \subset N_{s,\overline{s}}$. First, notice that $M(\omega_m) \to M(\omega) < +\infty$. Using $(IN.i; k - ii; k)$, for all h, a and s_k^a there is a subsequence $x_m^{ha}(s_k^a) \to x_a^{ha}(s_k^a)$ such that $M(\omega) \geq x_a^{ha}(s_k^a) \geq 0$. Since u^{ha} satisfies the boundary condition, $x^{ha}(s_k^a) \gg 0$. Using convergence of $x_m^{ha}(s_k^a)$, the price normalization, $Du^{ha} \gg 0$ and equations $(KT1, h; s_k^a)$, we have that $\lambda_m^{ha}(s_k^a) \to \lambda^{ha}(s_k^a) \gg 0$. Then, again the same equations imply that $p_m(s_k^a) \to p(s_k^a) \gg 0$. Equations $(KT2, h; s_k^a)$ now imply that $(q, \hat{q})_m \to (q, \hat{q})$, and using equations $(KT3, h; s_k^a)$ and no redundancy, we obtain that $m_m^{ha}(s_k^a) \to m^{ha}(s_k^a)$ as well.■

Proof of Theorem 4

Let $\omega \in \Omega^*$ be given as above. By construction, there is a $\xi \in E(\omega, \overline{m})$ which also has the property $\xi \in E_M^{NC}(\omega, \overline{m})$. Let $w = (..., w^{ha}, ...)_{h\in H, a>0}$, and $W = \{w \in \mathbb{R}^{HG} \mid w^{ha} = y_{s_{\tau+1}} m^{h(a-1)}(s^\tau)$ for some $s^\tau \in \widetilde{S}, s_{\tau+1} \in S\}$. Then, for all $s^t \in \widetilde{S}$ the vectors $\xi(s^t)$ can be expressed as $\xi(s^t) = \xi(s_t, w(s^t))$, i.e., they are the image of a function $\xi : S \times W \to \Xi$. For suppose not. First, it must be that $w = w(s^\tau)$ also for some $s^\tau \neq s^t$, otherwise the existence of such function is trivial. Second, suppose there exist $\xi^1, \xi^2 \in \xi(s, w)$ with $\xi^1 \neq \xi^2$ ($\xi(s, w)$ is not a singleton). Then, $\xi^1 = \xi(s^t)$ and $\xi^2 = \xi(s^\tau)$, with $\xi(s^t) \neq \xi(s^\tau)$. By construction of $\xi \in E_M(\omega, \overline{m})$, for all s^t,

$$\xi(s^t) = \Big(\big(x^{ha}(s^t), \lambda^{ha}(s^t), m^{ha}(s^t) \big)_{h\in H, a\leq G}, p(s^t), q(s^t) \Big) = \hat{\xi}(p, \lambda, w, s)$$

for $p = p(s^t)$, $\lambda = \left(\lambda^{ha}\right)(s^t)_{h\in H, 0<a<G}$, $w = w(s^t)$ and $s = s_t$, and where $\hat{\xi}(.)$ is the function from Lemma 1. So if $\xi^1, \xi^2 \in \xi(s, w)$, it must be that

$$(p^1, \lambda^1) = (p(s^t), \left(\lambda^{ha}\right)(s^t)_{h\in H, 0<a<G}) \neq (p(s^\tau), \left(\lambda^{ha}\right)(s^\tau)_{h\in H, 0<a<G}) = (p^2, \lambda^2)$$

while $w^1 = w(s^t) = w(s^\tau) = w^2$, a contradiction to $\xi \in E_M^{NC}(\omega, \overline{m})$.

The final step to time-invariance is accomplished by establishing the following law of motion on $S \times W$: given any $(s, w) \in S \times W$, let (s', w') be given by:

(1) $s' \in \{s\}_+$;

(2) $w^{h(a+1)'} = (w)_{s'}^{h(a+1)} = y_{s'} m^{ha}(s, w)$ all $a < G$.

Clearly, since $\xi(s^t)$ satisfies conditions $(H), (M)$ of a competitive equilibrium, then $\xi(s, w)$ also does, i.e., it satisfies $(H'), (M')$ in the definition of recursive equilibrium. Now, transitions T_a, all a, can be constructed by applying (1) and (2) repeatedly, and conditions (R) are easily established, so that the state space $S \times W$, the functions $\xi(s, w)$ and the transitions $(T_a)_{a \geq 0}$ form a recursive equilibrium for the economy ω, proving our assertion.■

For the proofs that follow, we need to specify the perturbations and, introducing new notation, perform preliminary computations of various derivatives of our equation system. As well known, for given (p, q), the set of $\{x^h, \lambda^h, m^h, e^h, u^h\}$ solving the system of equations $f^h(\cdot; p, q) = 0$ is a smooth manifold. We use endowments or utilities to perturb the system $f^h = 0$ with Z_σ^h appended, for some $\sigma \in \Sigma$.

Utility perturbations: Pick an age a, a state s and two consumption bundles $x_k \in \mathbb{R}_{++}^C$, $k = 1, 2$, with $x_1 \neq x_2$. We can always independently perturb the gradient of $u^{ha}(\cdot; s)$ around the two bundles x_1 and x_2 as follows. For each k, pick a pair of open balls $B_{\varepsilon_i}(x_k)$, $i = 1, 2$, centered around x_k and such that: 1) $B_{\varepsilon_1}(x_k) \subset B_{\varepsilon_2}(x_k)$; 2) $\cap_k B_{\varepsilon_2}(x_k) = \emptyset$. Then, pick smooth bump functions Φ_k such that $\Phi_k(x) = 1$, for $x \in B_{\varepsilon_1}(x_k)$ and $\Phi_k(x) = 0$, for $x \notin cl(B_{\varepsilon_2}(x_k)$. For given arbitrary vector $a_k \in \mathbb{R}^C$ and scalar η, define the utility function

$$u_\eta^{ha}(x, a_k; s) = u^{ha}(x; s) + \eta \sum_{k=1}^{2} \Phi_k(x)(a_k \otimes x)$$

where $a_k \otimes x = (..., a_{kc} x_c, ...)$. Evidently, $u_\eta^{ha}(x, a_{k,1}; s) = u^{ha}(x, s)$, at $a_{k,1} = 0$, $k = 1, 2$. Most importantly, for any given a_k, we can always pick η so close to zero that $u_\eta^{ha}(\cdot; s)$ is arbitrarily close to $u^{ha}(\cdot; s)$ in the $\mathcal{C}^2$–uniform convergence topology and it satisfies, therefore, all the maintained assumptions. For our purposes it suffices to identify a utility perturbation with the derivative with respect to η of the gradient of $u_\eta^{ha}(\cdot, a_k; s)$ computed at $\eta = 0$ and x_k, that is, $\Delta u_k^{ha} = D_\eta u_0^{ha}(x_k, a_k; s) = a_k$.

Endowment perturbations: Pick an age a, a state s and a vector $\Delta e \in \mathbb{R}^C$. Let $e^{ha}_{\eta}(s) = e^{ha}(s) + \eta\Delta e$. We identify an endowment perturbation with the derivative of $e^{ha}_{\eta}(s)$ with respect to η taken at $\eta = 0$, that is, with Δe.

Preliminary computations

Let $(\Delta x^h_{s^a}, \Delta\lambda^h_{s^a}, \Delta m^h_{s^a})$ denote the vector of changes in the consumption vector, multiplier and portfolio at s^a induced by some parameter perturbation, while $\left(\Delta X^h_{s^a}, \Delta\Lambda^h_{s^a}, \Delta M^h_{s^a}\right) \equiv (\Delta x^h_{s^a,s}, \Delta\lambda^h_{s^a,s}, \Delta m^h_{s^a,s})_{s\in\{0\}\cup S}$. Similarly, $(\Delta e_{a,s}, \Delta u_{a,s})$ denote a perturbation of endowments and utilities at the state $s^a = (s^{a-1}, s)$. It simplifies notation also to use $(\Delta e_{s^a,s}, \Delta u_{s^a,s})$ to denote $(\Delta e_{a+1,s}, \Delta u_{a+1,s})$, if $s > 0$, and $(\Delta e_{a-1,\hat{s}}, \Delta u_{a-1,\hat{s}})$, if $s = 0$ and $s^a = (s^{a-1}, \hat{s})$. This notation allows to set $(\Delta E_{s^a}, \Delta U_{s^a}) \equiv (\Delta e_{s^a,s}, \Delta u_{a-1,s};)_{s\in\{0\}\cup S}$, .

The corresponding vectors

$$\left(\Delta X^h_k, \Delta\Lambda^h_k, \Delta M^h_k, \Delta E^h, \Delta U^h_k\right) = \left(\left(\Delta X^h_{k,s^a}, \Delta\Lambda^h_{k,s^a}, \Delta M^h_{k,s^a}, \Delta E^h_{s^a}, \Delta U^h_{k,a}\right)_{s^a\in S^a}\right)_{a\in A}$$

are elements of the tangent space to the manifold $f^h_k = 0$, that is, they have to satisfy the equations

$$D_{x^h,\lambda^h,m^h,e^h,u^h} f^h \cdot \left(\Delta X^h_k, \Delta\Lambda^h_k, \Delta M^h_k, \Delta E^h, \Delta U^h_k; (p_k, q_k)\right) = 0, \;\; k = 1,2$$

or (dropping the superscript h and the subscript k):

$$\begin{aligned}
&H_{s^a}\Delta X_{s^a} - \Psi^T_{s^a}\Delta\Lambda_{s^a} - \Delta U_{s^a} = 0,\\
&R^T_{s^a}\Pi_{s^a}\Delta\Lambda_{s^a} = 0,\\
&\Psi_{s^a}\Delta X_{s^a} - R_{s^a}\Delta m_{s^a} - \Psi_{s^a}\Delta e_{s^a} + Q_{s^a}\Delta M_{s^a} - \begin{bmatrix} y_{s_a} \\ 0 \end{bmatrix}\Delta m_{s^{a-1}} = 0,
\end{aligned}$$

where $s^a = (s^{a-1}, s_a)$, H_{s^a} is the block-diagonal Hessian matrix (symmetric and negative definite) of dimension $C(S+1)\times C(S+1)$, with $S+1$ blocks of dimension $C\times C$, the s-th block given by $D^2u^{ha}(x^{ha}(s^a,s)) \equiv H(s^a,s)$, $s \in \{0\}\cup S$; Π_{s^a} is an $(S+1)\times(S+1)$ diagonal matrix of conditional probabilities, with s-th element $\pi(s|s^a)$ if $s > 0$, and 1 otherwise;

$$R_{s^a} = \begin{bmatrix} -q_{s^a} \\ Y \end{bmatrix}, \; \Psi_{s^a} = \begin{bmatrix} p(s^a) & 0 & & 0 \\ 0 & p(s^a,1) & & \\ & & \ddots & 0 \\ 0 & & 0 & p(s^a,S) \end{bmatrix}$$

is the $(S+1)\times C(S+1)$-dimensional matrix of commodity prices from state s^a onward, and

$$Q_{s^a} \equiv \begin{bmatrix} 0 & & & 0 \\ & q_{s^a,1} & & \\ & & \ddots & \\ 0 & & & q_{s^a,S} \end{bmatrix}$$

and the rest of the notation is standard.

Since $D_{X^h,\Lambda^h,M^h} f^h$ is invertible, we can solve the system for $\left(\Delta X^h, \Delta\Lambda^h, \Delta M^h\right)$ as functions of $\left(\Delta E^h, \Delta U^h\right)$. In fact, we can solve the system backwards to find Δm_0 as a function of $\left(\Delta E^h, \Delta U^h\right)$ only.

Since $H_{s^a}^{-1}$ is negative definite, the matrix $G_{s^a} \equiv \Psi_{s^a} H_{k,s^a}^{-1} \Psi_{k,s^a}^T$ is a well defined, negative-definite, diagonal matrix of dimension $(S+1)\times(S+1)$, with s-th element, $g(s^a,s) = p(s^a,s)H^{-1}(s^a,s)p(s^a,s)^T < 0$, $s \in \{0\}\cup S$. Also, set $C_{s^a} \equiv \Psi_{s^a}\Delta E_{s^a} - \Psi_{s^a} H_{s^a}^{-1}\Delta U_{s^a}$, a vector of dimension $(S+1)\times 1$, with $s-th$ element $c_{s^a,s} \equiv p(s^a,s)[\Delta e_{s^a,s} - [H(s^a,s)]^{-1}\Delta u_{s^a,s}]$, $s \in \{0\}\cup S$. Then, since $\Psi_{s^a}\Delta X_{s^a} = G_{s^a}\Delta\Lambda_{s^a} + \Psi_{s^a} H_{s^a}^{-1}\Delta u_{s^a}$, after straightforward and known computations, we get:

$$\Delta\Lambda_{s^a} = G_{s^a}^{-1}[R_{s^a}\Delta m_{s^a} + C_{s^a} + \begin{bmatrix} y_{s_a} \\ 0 \end{bmatrix}\Delta m_{s^{a-1}} - Q_{s^a}\Delta M_{s^a}].$$

Then, we use this equation and the no arbitrage condition, $0 = R_{s^a}^T \Pi_{s^a}\Delta\Lambda_{s^a}$. Let $\Gamma_{s^a} \equiv \Pi_{s^a} G_{s^a}^{-1}$, a negative definite diagonal matrix of dimension $(S+1)\times(S+1)$. Since R_{s^a} has full column rank, the matrix $R_{s^a}^T\hat{\Gamma}_{s^a}R_{s^a}$ is negative definite. Then, letting $A_{s^a} \equiv [R_{s^a}^T\Gamma_{s^a}R_{s^a}]^{-1}$, and taking into account that $R_{s^a}^T\Pi_{s^a}\Delta\Lambda_{s^a} = 0$, we can solve for Δm_{s^a} and get after straightforward computations:

$$\Delta m_{s^a} = -A_{s^a}R_{s^a}^T\Gamma_{s^a}[C_{s^a} + \begin{bmatrix} y_{s_a} \\ 0 \end{bmatrix}\Delta m_{s^{a-1}} - Q_{s^a}\Delta M_{s^a}]. \tag{2}$$

Equation (2) underscores the recursive structure of the programming problem. Since $\Delta m_{s^G} = 0$, all s^G, solving backwards, at s^{G-1} equation (2) is

$$\Delta m_{s^{G-1}} = -A_{s^{G-1}}R_{s^{G-1}}^T\Gamma_{s^{G-1}}[C_{s^{G-1}} + \begin{bmatrix} y_{s_{G-1}} \\ 0 \end{bmatrix}\Delta m_{s^{G-2}}]$$

and premultiplying by $q_{s^{G-1}}$, we have

$$q_{s^{G-1}}\Delta m_{s^{G-1}} = -q_{s^{G-1}}A_{s^{G-1}}R^T_{s^{G-1}}\Gamma_{s^{G-1}}C_{s^{G-1}}+q_{s^{G-1}}(\gamma_{s^{G-1}}A_{s^{G-1}})q^T_{s^{G-1}}y_{s_{G-1}}\Delta m_{s^{G-2}},$$

where γ_{s^a} is the first-row element of the matrix Γ_{s^a}. Now introduce the following matrices D_{s^a}: $D_{s^a} \equiv 0$ if $a \geq G-1$, and

$$D_{s^a} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 0 & d_{s^a,1} & & \\ \vdots & & \ddots & \\ 0 & & & d_{s^a,S} \end{bmatrix},$$

for $a \leq G-2$ where

$$d_{s^a,s} = \left\{ \; -q_{s^a,s}(\gamma_{s^a,s}\tilde{A}_{s^a,s})q^T_{s^a,s} < 0 \quad \text{for all } a \leq G-2, \right.$$

and where $\tilde{A}_{s^a}$ is defined for $a \leq G-1$ as

$$\tilde{A}_{s^a} = [R^T_{s^a}\Gamma_{s^a}(I + D_{s^a})R_{s^a}]^{-1}.$$

At the end of the Appendix, in Claim 10 we prove that $d_{s^a} > -1$ and $0 > -\gamma_{s^a}(1 + d_{s^a})y_{s_a}\tilde{A}_{s^a}y^T_{s_a} > -1$, for all $s^a \equiv (s^{a-1}, s) \in S^a$. Thus, $[R^T_{s^a}\Gamma_{s^a}(I + D_{s^a})R_{s^a}]$ is a negative definite matrix, and hence $\tilde{A}_{s^a}$ is well defined (and negative definite). Also define recursively the vector $\Theta_{s^a} = (0, \theta(s^a, 1), ..., \theta(s^a, S))^T$ as

$$\theta_{s^a} = \begin{cases} 0 & \text{if } a = G \\ -q_{s^a}\tilde{A}_{s^a}R^T_{s^a}\Gamma_{s^a}C_{s^a} & \text{if } a = G-1 \\ -q_{s^a}\tilde{A}_{s^a}R^T_{s^a}\Gamma_{s^a}[C_{s^a} + \Theta_{s^a}] & \text{if } a < G-1 \end{cases},$$

so that $\Theta_{s^{G-1}} = 0$, and $\Theta_{s^{G-2}} = (0, ..., -q_{s^a}\tilde{A}_{s^a}R^T_{s^a}\Gamma_{s^a}C_{s^a}, ...)^T$. Using these matrices and the expression for $q_{s^{G-1}}\Delta m_{s^{G-1}}$, we write:

$$Q_{s^{G-2}}\Delta M_{s^{G-2}} = -D_{s^{G-2}}R_{s^{G-2}}\Delta m_{s^{G-2}} + \Theta_{s^{G-2}}.$$

Plugging the latter into equation (2) for s^{G-2}, we have

$$\begin{aligned} \Delta m_{s^{G-2}} \quad = \quad & -A_{s^{G-2}}R^T_{s^{G-2}}\Gamma_{s^{G-2}}[C_{s^{G-2}} - \Theta_{s^{G-2}} + \begin{bmatrix} y_{s_{G-2}} \\ 0 \end{bmatrix} \Delta m_{s^{G-3}}] - \\ & -A_{s^{G-2}}R^T_{s^{G-2}}\Gamma_{s^{G-2}}D_{s^{G-2}}R_{s^{G-2}}\Delta m_{s^{G-2}} \end{aligned}$$

Since

$$[I + A_{s^{G-2}} R^T_{s^{G-2}} \Gamma_{s^{G-2}} D_{s^{G-2}} R_{s^{G-2}}] = A_{s^{G-2}} [R^T_{s^{G-2}} \Gamma_{s^{G-2}} (I + D_{s^{G-2}}) R_{s^{G-2}}],$$

we then write:

$$\Delta m_{s^{G-2}} = -\tilde{A}_{s^{G-2}} R^T_{s^{G-2}} \Gamma_{s^{G-2}} [C_{s^{G-2}} + \begin{bmatrix} y_{s_{G-2}} \\ 0 \end{bmatrix} \Delta m_{s^{G-3}} - \Theta_{s^{G-2}}]$$

By proceeding backward, we get:

$$\Delta m_{s^a} = -\tilde{A}_{s^a} R^T_{s^a} \Gamma_{s^a} [C_{s^a} - \Theta_{s^a} + \begin{bmatrix} y_{s_a} \\ 0 \end{bmatrix} \Delta m_{s^{a-1}}]. \tag{3}$$

and, since $g_{s^a} \Delta \lambda_{s^a} = -q_{s^a} \Delta m_{s^a} + c_{s^a} + y_{s_a} \Delta m_{s^{a-1}}$,

$$g_{s^a} \Delta \lambda_{s^a} = (1 + d_{s^a}) y_{s_a} \Delta m_{s^{a-1}} + c_{s^a} + q_{s^a} \tilde{A}_{s^a} R^T_{s^a} \Gamma_{s^a} [C_{s^a} - \Theta_{s^a}] \tag{4}$$

Using the latter for all $s^{a+1} \in \{s^a\}_+$, the definition of Θ_{s^a} and Γ_{s^a}, and again the equation $g_{s^a} \Delta \lambda_{s^a} = -q_{s^a} \Delta m_{s^a} + c_{s^a} + y_{s_a} \Delta m_{s^{a-1}}$, we get

$$\Pi_{s^a} \Delta \Lambda_{s^a} = \Gamma_{s^a} \{(I + D_{s^a}) R_{s^a} \Delta m_{s^a} + C_{s^a} - \Theta_{s^a} + \begin{bmatrix} y_{s_a} \\ 0 \end{bmatrix} \Delta m_{s^{a-1}}\}. \tag{5}$$

We are often going to compute derivatives induced by parameter perturbations at just one s^a. We call such perturbations *simple* (for short σ-) *perturbations* applied to s^a. The perturbations are defined as follows. We drop both h and k whenever confusion does not arise. Let $\hat{s}^a \equiv (\hat{s}^{a-1}, \hat{s}) \succeq \hat{s}^1 \equiv (s_0, 1)$. If we perturb endowments, we set $\Delta U = 0$ and we let Δe have $\Delta e_{a,\hat{s}}$ as the only non-zero element with $p_{\hat{s}^a} \Delta e_{a,\hat{s}} = 1$. If we perturb utilities, we set $\Delta E = 0$ and we let ΔU have $\Delta u_{a,\hat{s}}$ as the only non-zero element with $p_{\hat{s}^a} [H(\hat{s}^a)]^{-1} g_{\hat{s}_a} \Delta u_{a,\hat{s}} = -1$. It is important to notice that a σ-perturbation applied to $\hat{s}^a$ yields $\Theta_{s^{a'}} = 0$, for all $a' \geq a$, and $c_{s^{a'}} = 0$, for all $a' > a$. With Δ_{s^a} we denote the changes in variables induced by a σ-perturbation applied to s^a. We call a perturbation *symmetric* if it generates k-invariant vectors $(C_{k,s^a}, \Theta_{k,s^a})$, all s^a, all a.

Proof of Lemma 6

Let $s^1 = (s_0, 1)$. Since $\Delta m_{k,s_{-1}} \equiv 0$, at $s^0 = 0$ equation (3) is

$$\Delta m_{k,0} = -\tilde{A}_{k,0} R_{k,0}^T \Gamma_{k,0} [C_{k,0} - \Theta_{k,0}],$$

which can also be written as

$$R_{k,0}^T \Gamma_{k,0} \{[I + D_{k,0}] R_{k,0} \Delta m_{k,0} + C_{k,0} - \Theta_{k,0}\} = 0 \tag{6}$$

an expression for $\Delta m_{k,0}$ as a function only of the perturbations $\Delta E, \Delta U$ through $C_{k,0}$ and $\Theta_{k,0}$.

If $\lambda_{1,s^1}^{h1} \neq \lambda_{2,s^1}^{h1}$, then $x_{1,s^1}^{h1} \neq x_{2,s^1}^{h1}$, and we can perturb $u^{h1}(\cdot, s^1)$ around x_{1,s^1}^{h1} without disturbing it around x_{2,s^1}^{h1}. We use a σ-perturbation applied to s^1 induced by a change of endowments, if $p_{1,s^1} \neq p_{2,s^1}$, and by a change in utilities otherwise. As already argued, in both cases, we get: $\theta_{k,s^a} = c_{k,s^a} = 0$, for $(k, s^a) \neq (1, s^1)$, while $c_{1,s^1} = 1$ and $\theta_{1,s^1} = d_{1,1}$. Thus, using regularity of demand, to show the claim it suffices to prove that $y_1 \Delta_{s^1} m_{1,0} \neq 0$. From equation (6) we get

$$y_1 \Delta_{s^1} m_{1,0} = -(\gamma_{1,1}(1 + d_{1,1})) y_1 \tilde{A}_{1,0} y_1^T$$

Since, by Claim 10, $1 + d_{1,1} > 0$, and $\tilde{A}_{1,0}$ is a negative definite matrix, $y_1 \Delta_{s^1} m_{1,0} \neq 0$. Hence, $D_{s^1} Z_0 = y_1 \Delta_{s^1} m_{1,0} \neq 0$.■

Proof that Condition Univ holds

To prove that Condition Univ is satisfied by our family $\left\{ Z_{(s^a,j)\in\Sigma}^h \right\}_{h\in H}$ at any $(p, q, \omega) \in P \times \Omega$, we need preliminary results stated under the form of two auxiliary lemmas.[8]

In almost all the remaining lemmas, we need to guarantee that for any given k and history s^a there exists a simple perturbation inducing a nonzero change in financial wealth at s^a. At the end of the Appendix, Claim 11

[8]When $G = 2$, the construction is simpler. It is enough to prove that if for $s^1 = (s_0, 1)$, x_{k,s^1}^h is k-invariant, $D_{\omega^h} Z_0^h = 0$, and $D_{\omega^h} Z_{s^1,0}^h = 0$, d_{k,s^1}^h is k-invariant. Then, the k-invariance of d_{k,s^1} as well as the implicit assumption that $(p, q, \omega) \in (P \times \Omega) \backslash E_0^h$ (otherwise there's nothing to prove) immediately implies Condition Univ and then Condition Nest-3, which is all one needs in that case. This is easily verified by applying the definition of $D_{s'} Z_{s^1,0}^h$ for an asymmetric (utility) perturbation at (s^1, s'), obtained if $x_{1,s^1,s'} \neq x_{1,s^1,s'}$, i.e., $Z_{(s^1,s'),0}^h \neq 0$, the only case that matters in P; and then showing that $D_{s'} Z_{s^1,0}^h \neq 0$.

When $G > 3$, the argument becomes more involved, as perturbing states where $x_{1,s^{a-2},s_{a-1},s'} \neq x_{2,s^{a-1},s_{a-1},s'}$ for $a > 2$ does not guarantee that the perturbation is nil at other histories $(s_k^{a-2}, s'_{a-1}, s')$ for $s'_{a-1} \neq s_{a-1}$, at same or different k. That's why we have to make sure that the family $\{Z_\sigma^h\}$ includes more elements, Condition Nest becomes more complex, and both conditions are more difficult to verify.

shows that there exists a simple perturbation $\Delta e_{a',s_{a'}}$ with $a' \leq a$ such that $\mathbf{1}_{a,a'} + [y_{s_a}\Delta_{(a',s_{a'})} m_{k,s^{a-1}}] \neq 0$, where $\mathbf{1}_{a,a'}$ is a function equal to 1 if $a = a'$, and zero otherwise. We denote the successful perturbation as 0^*, and the changes due to it as Δ_{0^*}.

Auxiliary Lemma 1 *Suppose that $(p, q, \omega) \in P \times \Omega$ is such that, for some s^a, $a \geq 1$: i) x^h_{k,s^a} is k-invariant; ii) $D_{\omega^h} Z^h_{s^a,0} = 0$; and iii) $D_{\omega^h} Z^h_{(s^a,s),0} = 0$, for all $s \in S$. Then:*

a) the matrix $\Gamma_{k,s^a}(I + D_{k,s^a})$ is k-invariant, and

$$(1 + d^h_{2,s^a}) Y \tilde{A}^h_{1,s^{a-1}} q^T_{1,s^{a-1}} = (1 + d^h_{1,s^a}) Y \tilde{A}^h_{2,s^{a-1}} q^T_{2,s^{a-1}};$$

b) if, in addition, q_{k,s^a} is k-invariant, d^h_{k,s^a} and $\tilde{A}^h_{k,s^a}$ are k-invariant;

c) if, in addition, $x^h_{k,s^a,s}$ is k-invariant, $d^h_{k,s^a,s}$ is k-invariant, all $s \in S$.

Proof. a) We apply symmetric perturbations $s' = 0^*, 1, ..., S$ to derive restrictions on the endogenous variables, using ii) and iii). We drop the superscript h. Set $\Gamma^*_{k,s^a} \equiv \Gamma_{k,s^a}(I + D_{k,s^a})$. Because of assumption i), $\gamma_{1,s^a} = \gamma_{2,s^a} = \gamma_{s^a}$; and because of ii),

$$(1 + d_{1,s^a})[\mathbf{1}_{0^*} + y_{s_a}\Delta_{0^*} m_{1,s^{a-1}}] = (1 + d_{2,s^a})[\mathbf{1}_{0^*} + y_{s_a}\Delta_{0^*} m_{2,s^{a-1}}] \quad (7)$$

as well as[9]

$$(1+d_{1,s^a})[y_{s_a}\Delta . m_{1,s^{a-1}}] + q_{1,s^a}\tilde{A}_{1,s^a} Y^T \Gamma^{*\backslash}_{1,s^a} = (1+d_{2,s^a})[y_{s_a}\Delta . m_{2,s^{a-1}}] + q_{2,s^a}\tilde{A}_{2,s^a} Y^T \Gamma^{*\backslash}_{2,s^a}, \quad (8)$$

where $[y_{s_a}\Delta . m_{k,s^{a-1}}] \equiv [y_{s_a}\Delta_s m_{k,s^{a-1}}]_{s' \in S}$. Furthermore, since by equation (3)

$$y_s \Delta_{0^*} m_{k,s^a} = y_s \tilde{A}_{k,s^a} \gamma_{s^a} q^T_{k,s^a}(\mathbf{1}_{0^*} + y_{s_a}\Delta_{0^*} m_{k,s^{a-1}}),$$

$D_{0^*} Z_{(s^a,s),0} = 0$ for all $s \in S$ only if

$$\begin{aligned} & \Gamma^{*\backslash}_{1,s^a} Y \tilde{A}_{1,s^a} \gamma_{s^a} q^T_{1,s^a}(\mathbf{1}_{0^*} + y_{s_a}\Delta_{0^*} m_{1,s^{a-1}}) \\ = \; & \Gamma^{*\backslash}_{2,s^a} Y \tilde{A}_{2,s^a} \gamma_{s^a} q^T_{2,s^a}(\mathbf{1}_{0^*} + y_{s_a}\Delta_{0^*} m_{2,s^{a-1}}). \end{aligned}$$

[9] When a \ appears above a matrix, it denotes the matrix without the first row and column.

Thus, using Claim 11 to substitute $\mathbf{1}_{0^*} + y_{s_a}\Delta_{0^*} m_{1,s^{a-1}}$ from equation (7),

$$\Gamma^{*\backslash}_{1,s^a} Y \tilde{A}_{1,s^a} q^T_{1,s^a}(1 + d_{2,s^a}) = \Gamma^{*\backslash}_{2,s^a} Y \tilde{A}_{2,s^a} q^T_{2,s^a}(1 + d_{1,s^a}). \tag{9}$$

Similarly, since

$$y_s \Delta_{s'} m_{k,s^a} = y \tilde{A}_{k,s^a} \gamma_{s^a} q^T_{k,s^a}[y_{s_a}\Delta_{s'} m_{k,s^{a-1}}] - y_s \tilde{A}_{k,s^a} y^T_{s'} \gamma^*_{k,s^a,s'},$$

$D_{s'} Z^h_{(s^a,s),0} = 0$ for all $(s, s') \in S^2$ only if

$$\begin{aligned} & \Gamma^{*\backslash}_{1,s^a} Y \tilde{A}_{1,s^a}\{\gamma_{s^a} q^T_{1,s^a}[y_{s_a}\Delta . m_{1,s^{a-1}}] - Y^T \Gamma^{*\backslash}_{1,s^a}\} + \Gamma^{*\backslash}_{1,s^a} \\ = \; & \Gamma^{*\backslash}_{2,s^a} Y \tilde{A}_{2,s^a}\{\gamma_{s^a} q^T_{2,s^a}[y_{s_a}\Delta . m_{2,s^{a-1}}] - Y^T \Gamma^{*\backslash}_{2,s^a}\} + \Gamma^{*\backslash}_{2,s^a}. \end{aligned} \tag{10}$$

By equations (8), the l.h.s. of (10) can be rewritten as

$$\begin{aligned} & \Gamma^{*\backslash}_{1,s^a} Y \tilde{A}_{1,s^a}\{\gamma_{s^a} q^T_{1,s^a} \frac{(1 + d_{2,s^a})[y_{s_a}\Delta . m_{2,s^{a-1}})]}{1 + d_{1,s^a}} + \\ & \gamma_{s^a} q^T_{1,s^a} \frac{q_{2,s^a}\tilde{A}_{2,s^a} Y^T \Gamma^{*\backslash}_{2,s^a} - q_{1,s^a}\tilde{A}_{1,s^a} Y^T \Gamma^{*\backslash}_{1,s^a}}{1 + d_{1,s^a}} - Y^T \Gamma^{*\backslash}_{1,s^a}\} + \Gamma^*_{1,s^a}. \end{aligned}$$

Using equations (9), (10) and again by (8),

$$\begin{aligned} & \Gamma^{*\backslash}_{1,s^a} Y \tilde{A}_{1,s^a}\{\gamma_{s^a} q^T_{1,s^a}[\frac{q_{2,s^a}\tilde{A}_{2,s^a} Y^T \Gamma^{*\backslash}_{2,s^a} - q_{1,s^a}\tilde{A}_{1,s^a} Y^T \Gamma^{*\backslash}_{1,s^a}}{1 + d_{1,s^a}}] - Y^T \Gamma^{*\backslash}_{1,s^a}\} + \Gamma^{*\backslash}_{1,s^a} \\ = \; & \Gamma^{*\backslash}_{2,s^a} Y \tilde{A}_{2,s^a}\{-Y^T \Gamma^{*\backslash}_{2,s^a}\} + \Gamma^{*\backslash}_{2,s^a}. \end{aligned} \tag{11}$$

However, by equations (9) the following equality holds true

$$\Gamma^{*\backslash}_{1,s^a} Y \tilde{A}_{1,s^a} q^T_{1,s^a} \frac{q_{2,s^a}\tilde{A}_{2,s^a} Y^T \Gamma^{*\backslash}_{2,s^a}}{1 + d_{1,s^a}} = \Gamma^{*\backslash}_{2,s^a} Y \tilde{A}_{2,s^a} q^T_{2,s^a} \frac{q_{2,s^a}\tilde{A}_{2,s^a} Y^T \Gamma^{*\backslash}_{2,s^a}}{1 + d_{2,s^a}}.$$

Thus, equations (11) can be finally be rewritten as

$$\begin{aligned} & \Gamma^{*\backslash}_{1,s^a} Y \tilde{A}_{1,s^a}\{\frac{\gamma_{s^a} q^T_{1,s^a} q_{1,s^a}\tilde{A}_{1,s^a}}{1 + d_1} + I\} Y^T \Gamma^{*\backslash}_{1,s^a} - \Gamma^{*\backslash}_{1,s^a} \\ = \; & \Gamma^{*\backslash}_{2,s^a} Y \tilde{A}_{2,s^a}\{\frac{\gamma_{s^a} q^T_{2,s^a} q_{2,s^a}\tilde{A}_{2,s^a}}{1 + d_{2,s^a}} + I\} Y^T \Gamma^{*\backslash}_{2,s^a} - \Gamma^{*\backslash}_{2,s^a}. \end{aligned} \tag{12}$$

Drop the subscript s^a, and let $C_k \equiv \{\tilde{A}_k \frac{\gamma q_k^T q_k}{1+d_k} \tilde{A}_k + \tilde{A}_k\}$. C_k is a negative definite and, therefore, invertible matrix. Indeed, for all $\xi \neq 0$,

$$\xi C_k \xi^T = \frac{\gamma}{1+d_k} \xi \tilde{A}_k q_k^T (q_k \tilde{A}_k \xi^T) + \xi \tilde{A}_k \xi^T < 0.$$

Hence, setting $T_k = Y C_k Y^T$ and $B_k = \Gamma_k^{*\setminus}$, equations (12) can be rewritten as:

$$B_1 T_1 B_1 - B_1 = B_2 T_2 B_2 - B_2, \tag{13}$$

These are the restrictions we were looking for. After straightforward computations, (13) is

$$B_1(T_1 - T_2)B_1 = (B_1 - B_2)[I - T_2(B_1 + B_2)].$$

We claim that the latter implies that $B_1 = \phi B_2$, some $\phi > 0$. Suppose by contradiction that $B_1 \neq \phi B_2$ for all ϕ. Then, since Y is in general position, the column spans $[B_1 Y]$ and $[B_2 Y]$ do not coincide, and, hence, $\zeta B_1 Y = 0$, but $\zeta B_2 Y \neq 0$, for some $\zeta \neq 0$. Therefore, since by definition of T_1 and T_2, $\zeta B_1 T_1 = \zeta B_1 T_2 = 0$, it must be that

$$0 = \zeta(B_1 - B_2)[I - T_2(B_1 + B_2)] = \zeta[B_1 - B_2 + B_2 T_2(B_1 + B_2)].$$

However, by equation (13), the latter can be rewritten as:

$$0 = \zeta[B_1 T_1 B_1 - B_2 T_2 B_2 + B_2 T_2 B_1 + B_2 T_2 B_2] = \zeta B_2 T_2 B_1 = \zeta B_2 Y C_2 Y^T B_1.$$

Postmultiply the latter by $B_1^{-1} B_2 \zeta^T$ and get $0 = \zeta B_2 [Y C_2 Y^T] B_2 \zeta^T$. Since $\zeta B_2 Y \neq 0$, and C_2 is negative definite, $0 = \zeta B_2 Y C_2 Y^T B_2 \zeta^T < 0$, a contradiction.

Thus, $B_1 = \phi B_2$ for some $\phi \neq 0$, or

$$\phi^2 T_1 B_2 - \phi I = T_2 B_2 - I.$$

However, for each $\eta \neq 0$ orthogonal to Y, it is $\eta B_k = 0$, thus $\phi = 1$, otherwise $\eta = \phi \eta$. Then, we get $B_1 = B_2$ or

$$\Gamma_{1,s^a}^{*\setminus} = \Gamma_{2,s^a}^{*\setminus},$$

that is, since γ_{k,s^a} is k-invariant, $\Gamma^{*\backslash}_{s^a}$ is diagonal, and using (9), we have a).

b) Suppose now that $q_{1,s^a} = q_{2,s^a}$. Since $\tilde{A}^{-1}_{k,s^a} = \gamma_{s^a} q^T_{k,s^a} q_{k,s^a} + Y^T \Gamma^{*\backslash}_{k,s^a} Y$, $\tilde{A}_{k,s^a}$ is k-invariant. Since $d_{k,s^a} = \gamma_{s^a} q_{k,s^a} \tilde{A}_{k,s^a} q^T_{k,s^a}$, d_{k,s^a} is k-invariant. Now c) is trivial.■

Auxiliary Lemma 2 *Suppose that $(p, q, \omega) \in P \times \Omega$ is such that, for some s^a, $a \geq 1$: i) x^h_{k,s^a} is k-invariant; ii) $D_{\omega^h} Z^h_{s^a,0} = 0$; iii) $x^h_{1,s^a,s'} \neq x^h_{2,s^a,s'}$, for some $s' \in S$. Then, $D_{\omega^h} Z^h_{(s^a,s),0} \neq 0$ for some $s \in S$.*

Proof. Suppose otherwise. Then, by i) and ii) and Auxiliary Lemma 1, and using the notation there, $\Gamma^*_{1,s^a} = \Gamma^*_{2,s^a} \equiv \Gamma^*_{s^a}$. Consider an asymmetric perturbation $(1, s')$ affecting $u^{a+1}(\cdot, s')$ at $x_{1,s^a,s'}$ without disturbing it around $x_{2,s^a,s'}$ —this is possible thanks to iii). For any matrix M of dimension $S \times m$, we denote by $M_\backslash$ denote the matrix obtained from M by deleting the row $s' \in S$. Then, $D_{s'} Z_{(s^{a-1},s),0} = 0$, all $s \in S \backslash \{s'\}$ only if, using (4) and (3),

$$\begin{aligned} & Y_\backslash \tilde{A}_{1,s^a} \{\gamma_{s^a} q^T_{1,s^a} [y_{s_a} \Delta_{s'} m_{1,s^{a-1}})] - y^T_{s'} \gamma^*_{1,s^a,s'}\} \qquad (14) \\ = \; & Y_\backslash \tilde{A}_{2,s^a} \gamma_{s^a} q^T_{2,s^a} [y_{s_a} \Delta_{s'} m_{2,s^{a-1}})] \end{aligned}$$

By Auxiliary Lemma 1, $(1 + d_{2,s^a}) Y \tilde{A}_{1,s^a} q^T_{1,s^a} = (1 + d_{1,s^a}) Y \tilde{A}_{2,s^a} q^T_{2,s^a}$. Thus, (14) can be rewritten as

$$Y_\backslash \tilde{A}_{1,s^a} \{\beta_{s^a} q^T_{1,s^a} - y^T_{s'} \gamma^*_{1,s^a,s'}\} = 0$$

for some nonzero scalar β_{s^a}. Using the general position of Y, the latter implies that $\beta_{s^a} q^T_{1,s^a} - y^T_{s'} \gamma^*_{1,s^a,s'} = 0$. However, if this is the case, then

$$D_{s'} Z^h_{(s^a,s'),0} = \gamma^*_{1,s^a,s'} y_{s'} \tilde{A}_{1,s^a} \{\beta_{s^a} q^T_{1,s^a} - y^T_{s'} \gamma^*_{1,s^a,s'}\} + \gamma^*_{1,s^a,s'} = \gamma^*_{1,s^a,s'} \neq 0,$$

a contradiction. ■

We are now ready to prove the original claim for any $G \geq 2$. By definition of P, given $(p, q) \in P$, there exists $a \geq 1$ such that either i) $p_{1,s^a} \neq p_{2,s^a}$ (and then $a > 1$), or ii) $q_{1,s^a} \neq q_{2,s^a}$. If i), then $x_{1,s^a} \neq x_{2,s^a}$. If ii) and if $x_{1,s^a} = x_{2,s^a}$, by the optimality conditions there exists at least one $s' \in S$ such that $\lambda_{1,s^a,s'} \neq \lambda_{2,s^a,s'}$, and hence $x_{1,s^a,s'} \neq x_{2,s^a,s'}$. Thus, for $(p, q) \in P$, there exists an $a \geq 1$ such that $x_{1,s^a} \neq x_{2,s^a}$. Let $\bar{a}$ be the smallest such a. If $\bar{a} = 1$, Lemma 6.ii applies. If $a > 1$, the claim is then a consequence of Auxiliary Lemma 2. ■

Proof that Condition Nest holds

We prove that our family satisfies the several items that make up Condition Nest. We need five auxiliary lemmas to arrive at the desired conclusion.

First we look the properties of the maps $Z^h_{s^{a+2},j}(p,q,\omega)$ with $j > S^2$, i.e., for some $(s^a, \bar{s}, \sigma)$, $a \geq 1$, $Z^h_{(s^a,\bar{s},\sigma),j} = \lambda^h_{k,s^a,\bar{s},\sigma_l} - \lambda^h_{k,s^a,s_{j'},\sigma_l}$, for some $l \geq 0$, $j' > 0$, and $k = 1, 2$.

We simplify notation by fixing once for all and dropping the indexes h and k, and identify j as $j(l, j')$. Given $(\bar{s}, j')$, $j' > 0$, let

$$S_-(\bar{s}, j') = \{\sigma_l, l \in \{0, 1, ..., S-1\} | x_{s^a,\bar{s},\sigma_l} \neq x_{s^a,s_{j'}(\bar{s}),\sigma_l}\}.$$

Auxiliary Lemma 3 *Suppose that $\pi(\cdot|s)$ is one-to-one in s. Let $(s^a, \bar{s}, \sigma)$ and k be given, $a \geq 1$. For any $j' > 0$, if $S_-(\bar{s}, j') = \varnothing$, then $D_\omega Z_{(s^a,\bar{s},\sigma),j(l,j')} \neq 0$ for some $l \geq 0$.*

Proof. We use the 0^* perturbation at s^a, and the s-perturbations at s^{a+1}(denoted by $\hat{s}$) and at (s^a, s, σ'), $\sigma' \in S$, (denoted by σ'); now ω denotes one generic such perturbation.

Without loss of generality, fix $\bar{s} = 1$ and $s_{j'}(\bar{s}) = 2$. By contradiction, suppose $D_\omega Z_{(s^a,\bar{s},\sigma),j(l,j')} \neq 0$ *for all* $l \geq 0$. By equation (5), this is true only if

$$(I + D_{s^a,1})^{\backslash}[Y\Delta_\omega m_{s^a,1} + \mathbf{I}_\omega] = (I + D_{s^a,2})^{\backslash}[Y\Delta_\omega m_{s^a,2} + \mathbf{I}_\omega]$$

where $\mathbf{I}_\omega$ is a diagonal, S-dimensional matrix with diagonal elements $\mathbf{1}_{\omega\sigma}$ equal to one if $\omega = \sigma$, and zero otherwise. Equivalently, using equation (3),

$$\begin{array}{c}(I + D_{s^a,1})^{\backslash}\{Y\tilde{A}_{s^a,1}[\gamma_{s^a,1}q^T_{s^a,1}(y_1\Delta_\omega m_{s^a} + \mathbf{1}_{\omega 1}) - y^T_\omega\gamma^*_{s^a,1}(\omega)] + \mathbf{I}_\omega\} = \\ (I + D_{s^a,2})^{\backslash}\{Y\tilde{A}_{s^a,2}[\gamma_{s^a,2}q^T_{s^a,2}(y_2\Delta_\omega m_{s^a} + \mathbf{1}_{\omega 2}) - y^T_\omega\gamma^*_{s^a,2}(\omega)] + \mathbf{I}_\omega\}\end{array} \quad (15)$$

with $\mathbf{1}_{\omega s} = 1$ if $\omega = s$, and zero otherwise, and where $y^T_\omega\gamma^*_{s^a,s}(\omega) = y^T_\sigma\gamma^*_{s^a,s,\sigma}$ if $\omega = \sigma$, and zero otherwise (here $\gamma^*_{s^a,s} = (1 + d_{s^a,s})\gamma_{s^a,s}$). However, using once more equation (3),

$$y_s\Delta_{0^*}m_{s^a} = y_s\gamma_{s^a}\tilde{A}_{s^a}q^T_{s^a}(y_{s_a}\Delta_{0^*}m_{s^{a-1}} + \mathbf{1}_{0^*}),\ s = 1, 2$$

with $\mathbf{1}_{0^*} = 1$ if $0^* = s^a$, and zero otherwise; while

$$\Delta_{\hat{s}}m_{s^a} = \tilde{A}_{s^a}[\gamma_{s^a}q^T_{s^a}(y_{s_a}\Delta_{\hat{s}}m_{s^{a-1}}) - y^T_{\hat{s}}\gamma^*_{s^a,\hat{s}}],\ \text{all } \hat{s}$$

and by Claim (6), $(y_{s_a}\Delta_{0^*}m_{s^{a-1}} + \mathbf{1}_{0^*}) \neq 0$. Finally,

$$y_s\Delta_{\sigma'}m_{s^a} = y_s\tilde{A}_{s^a}\gamma_{s^a}q_{s^a}^T(y_{s_a}\Delta_{\sigma'}m_{s^{a-1}}) - y_s\tilde{A}_{s^a}Y^T\Gamma_{s^a}^{\setminus}\Theta_{s^a}^{\setminus}$$

with $\theta_{s^a,s} = q_{s^a,s}\tilde{A}_{s^a,s}y_{\sigma'}^T\gamma_{s^a,s,\sigma'}^*$. Thus, substituting for $Y\Delta_{0^*}m_{s^a}$ into (15) with $\omega = 0^*$, we have

$$M_{s^a}(1,2)\tilde{A}_{s^a}q_{s^a}^T = 0, \tag{16}$$

where

$$M_{s^a}(1,2) \equiv \{(I + D_{s^a,1})^{\setminus}Y\tilde{A}_{s^a,1}\gamma_{s^a,1}q_{s^a,1}^Ty_1 - (I + D_{s^a,2})^{\setminus}Y\tilde{A}_{s^a,2}\gamma_{s^a,2}q_{s^a,2}^Ty_2\}.$$

While substituting $\Delta_{\hat{s}}m_{s^a}$ into (15) for $\omega = \hat{s}$, rearranging and using (16) we get if $\hat{s} = 1$,

$$(I + D_{s^a,1})^{\setminus}Y\tilde{A}_{s^a,1}\gamma_{s^a,1}q_{s^a,1}^T = M_{s^a}(1,2)\tilde{A}_{s^a}y_1^T\gamma_{s^a,1}^*, \tag{17}$$

$$-(I + D_{s^a,2})^{\setminus}Y\tilde{A}_{s^a,2}\gamma_{s^a,2}q_{s^a,2}^T = M_{s^a}(1,2)\tilde{A}y_{s^a,2}^T\gamma_{s^a,2}^* \tag{18}$$

if $\hat{s} = 2$, and for $\hat{s} > 2$

$$M_{s^a}(1,2)\tilde{A}_{s^a}y_s^T = 0. \tag{19}$$

Finally, denoting with $(y_{s_a}\Delta.m_{s^{a-1}})$ the vector $(y_{s_a}\Delta_{\sigma'}m_{s^{a-1}})_{\sigma'\in S}$, and substituting for $y_s\Delta_{\sigma'}m_{s^a}$ into (15) with $\omega = \sigma'$, all σ', we get

$$\begin{aligned} & (I + D_{s^a,1})^{\setminus}Y\tilde{A}_{s^a,1}\{\gamma_{s^a,1}q_{s^a,1}^T[y_1\tilde{A}_{s^a}\gamma_{s^a}q_{s^a}^T(y_{s_a}\Delta.m_{s^{a-1}}) - y_1\tilde{A}_{s^a}Y^T\Gamma_{s^a}^{\setminus}\Theta_{s^a}^{\setminus}] \\ & -Y^T\Gamma_{s^a,1}^{*\setminus}\} + (I + D_{s^a,1})^{\setminus} \qquad (20) \\ = \; & (I + D_{s^a,2})^{\setminus}Y\tilde{A}_{s^a,2}\{\gamma_{s^a,2}q_{s^a,2}^T[y_2\tilde{A}_{s^a}\gamma_{s^a}q_{s^a}^T(y_{s_a}\Delta.m_{s^{a-1}}) - y_2\tilde{A}_{s^a}Y^T\Gamma_{s^a}^{\setminus}\Theta_{s^a}^{\setminus}] \\ & -Y^T\Gamma_{s^a,2}^{*\setminus}\} + (I + D_{s^a,2})^{\setminus} \end{aligned}$$

However, using equations (16) - (19), and by the definition of $\gamma_{s^a,s}^*$, equations (20) can be rewritten as

$$\begin{aligned} & (I + D_{s^a,1})^{\setminus}YC_{s^a,1}Y^T\Gamma_{s^a,1}^{*\setminus} - (I + D_{s^a,1})^{\setminus} \\ = \; & (I + D_{s^a,2})^{\setminus}YC_{s^a,2}Y^T\Gamma_{s^a,2}^{*\setminus} - (I + D_{s^a,2})^{\setminus}. \end{aligned}$$

where $C_{s^a,s} \equiv \tilde{A}_{s^a,s}\{\frac{\gamma_{s^a,s}q^T_{s^a,s}q_{s^a,s}}{1+d_{s^a,s}}+I\}$. Bear in mind that, since $S_-(\bar{s},j') = \varnothing$, $\lambda_{s^a,s,\sigma}$ is s-invariant, for $s = \bar{s}$, $s_{j'}(\bar{s})$, and so $x_{s,\sigma}$ is s-invariant, then $G_{s^a,s}$ is s-invariant. Then, drop the subscript s^a, and set $T_s = YC_sY^T\Pi_s$ and $B_s = G^{-1}(I+D_s)$. Observe that C_s is essentially the same matrix appearing in the proof of Auxiliary Lemma 1. Rewriting (20) as

$$B_1T_1B_1 - B_1 = B_2T_2B_2 - B_2, \tag{21}$$

obviously (21) is the same as equation (13). Therefore, we arrive at the same conclusion (here we postmultiply by $B_1^{-1}\Pi_2^{-1}B_2\zeta^T$ instead of by only $B_1^{-1}B_2\zeta^T$ to adjust for the different definition of B_s): $B_1 = B_2$. Here this means $YC_1Y^T\Pi_1 = YC_1Y^T\Pi_2$ or, transposing,

$$\Pi_1YC_1Y^T = \Pi_2YC_2Y^T$$

Given that C_s is invertible and that Y has rank J, the latter implies the equality of the column spans of the matrices Π_1Y and Π_2Y. Since Y is in general position and $S > J$, this is true if and only if $\Pi_1 = \Pi_2$, a contradiction.■

The next lemma is identical to the previous one in all assumptions except for $S_-(\bar{s},j') = \varnothing$.

Auxiliary Lemma 4 *Suppose that $\pi(\cdot|s)$ is one-to-one in s. Let $(s^a,\bar{s},\sigma)$ and k be given, $a \geq 1$. For any $j' > 0$, if $S_-(\bar{s},j') \neq \varnothing$, then* $D_\omega Z_{(s^a,\bar{s},\sigma),j(l,j')} \neq 0$ for some $\sigma_l \in S\backslash S_-(\bar{s},j')$.

Proof. Again, fix $\bar{s} = 1$ and $s_{j'}(\bar{s}) = 2$, and let $S_-(\bar{s},j') \equiv S_-$. We denote by $Y_\backslash$ the matrix with rows y_σ, $\sigma \in S\backslash S_-$ and with $(I+D_{s^a})_\backslash$ the $(S-S_-)\times(S-S_-)$ matrix obtained from rows indexed by $s \in S\backslash S_-$ in $(I+D_{s^a})^\backslash$. By performing symmetric perturbations, using the same argument of Auxiliary Lemma 3 we get that $D_\omega Z_{(s^a,\bar{s},\sigma),j(l,j')} \neq 0$ for all $\sigma_l \in S\backslash S_-(\bar{s},j')$ and for $\omega = 0^*, 1, ..., S$ only if (16) - (19) hold, with $(I+D_{s^a,s})_\backslash Y_\backslash$ replacing $(I+D_{s^a,s})^\backslash Y$ for $s = 1,2$ in $M_{s^a}(1,2)$ —now call it $M_{s^a}(1,2)_\backslash$.

Now perform asymmetric perturbations $(s^a,1,\sigma)$, $\sigma \in S_-$, that is, perturbations affecting $u^{a+1}(\cdot;\sigma)$ around $x_{s^a,1,\sigma}$, but not around $x_{s^a,s,\sigma}$, for all s with $x_{s^a,s,\sigma} \neq x_{s^a,1,\sigma}$, and get

$$y_s\Delta_{(s^a,1,\sigma)}m_{s^a} = y_s\tilde{A}_{s^a}\gamma_{s^a}q^T_{s^a}(y_{s_a}\Delta_{(s^a,1,\sigma)}m_{s^{a-1}}) - y_s\tilde{A}_{s^a}Y^T\Gamma^\backslash_{s^a}\Theta^\backslash_{s^a}(1)$$

where $\theta_{(s^a,s)}(1) = q_{s^a,s}\tilde{A}_{s^a,s}y^T_\sigma\gamma^*_{s^a,s,\sigma}$ if $x_{s^a,s,\sigma} = x_{s^a,1,\sigma}$, and zero otherwise. Let $S_{1,\sigma} = \{s|x_{s^a,s,\sigma} = x_{s^a,1,\sigma}\}$. Bear in mind that since $\sigma \in S_-$, $2 \notin S_{1,\sigma}$.

Thus, $D_{(s^a,1,\sigma)}Z_{(s^a,1,\sigma),j(l,j')} = 0$, for all $\sigma_l \in S\backslash S_-$ and for an asymmetric $(s^a,1,\sigma)$, $\sigma \in S_-$, only if

$$\begin{aligned}(I + D_{s^a,1})\backslash Y\backslash \tilde{A}_{s^a,1}\{\gamma_{s^a,1}q^T_{s^a,1}[y_1\tilde{A}_{s^a}\gamma_{s^a}q^T_{s^a}(y_{s_a}\Delta_{(s^a,1,\sigma)}m_{s^{a-1}}) \\ -y_1\tilde{A}_{s^a}Y^T\Gamma^{\backslash}_{s^a}\Theta^{\backslash}_{s^a}(1)] - y^T_\sigma\gamma^*_{s^a,s,\sigma}\} \\ = (I + D_{s^a,2})\backslash Y\backslash \tilde{A}_{s^a,2}\{\gamma_{s^a,2}q^T_{s^a,2}[y_2\tilde{A}_{s^a}\gamma_{s^a}q^T_{s^a}(y_{s_a}\Delta_{(s^a,1,\sigma)}m_{s^{a-1}}) \\ -y_2\tilde{A}_{s^a}Y^T\Gamma^{\backslash}_{s^a}\Theta^{\backslash}_{s^a}(1)]\}, \text{ for } \sigma \in S_-.\end{aligned} \tag{22}$$

However, by the modified equations (17) - (19) and since $2 \notin S_{1,\sigma}$, we get that

$$\begin{aligned} & M_{s^a}(1,2)\backslash \tilde{A}_{s^a}[\sum_{s\in S_{1,\sigma}} y^T_s\gamma_{s^a,s}q_{s^a,s}\tilde{A}_{s^a,s}y^T_\sigma\gamma^*_{s^a,s,\sigma}] \\ = \ & (I + D_{s^a,1})Y\backslash \tilde{A}_{s^a,1}\gamma_{s^a,1}q^T_{s^a,1}[\frac{q_{s^a,1}\tilde{A}_{s^a,1}y^T_\sigma\gamma^*_{s^a,1,\sigma}}{1 + d_{s^a,1}}]\end{aligned}$$

Thus, by exploiting this and equations (16) and (22), $D_{(s^a,1,\sigma)}Z_{(s^a,1,\sigma),j(l,j')} = 0$, for all $\sigma_l \in S\backslash S_-$, and for all such asymmetric perturbations if and only if (dropping the subscript s^a)

$$(I + D_1)\backslash Y\backslash C_1Y^T_-\Gamma^*_{1,-} = 0 \tag{23}$$

where, Y_- and $\Gamma^*_{1,-}$ are obtained from Y and Γ^*_1 by deleting rows and columns $s \in S\backslash S_-$, and $C_1 \equiv \{\tilde{A}_1\frac{\gamma_1q^T_1q_1}{1+d_1}\tilde{A}_1 + \tilde{A}_1\}$, an invertible matrix. Hence the matrix $C_1Y^T_-\Gamma^*_{-1}$ has rank equal to $\min\{J, S_-\}$. However, the rank of $Y_\backslash$ is equal to $\min(J, S - S_-\}$. Thus, there are at most $\max(0, J - (S - S_-))$ vectors in its orthogonal complement. If $\min\{J, S_-\} = J$, we have a contradiction. If $\min\{J, S_-\} = S_-$, but $\min(J, S - S_-\} = J$, once again we have a contradiction. Thus, it must be that $\min\{J, S_-\} = S_-$ and $\min(J, S - S_-\} = S - S_-$, again a contradiction. ■

Next, we study the properties of the maps $Z^h_{s^{a+2},j}(p, q, \omega)$ with $0 \leq j \leq S^2$, i.e., such that given $(s^a, \bar{s}, \sigma)$, $a \geq 0$, $Z^h_{(s^a,\bar{s},\sigma),j} = \lambda_{1,s^a,\bar{s},\sigma_l} - \lambda_{2,s^a,s_{j'},\sigma_l}$, for some $l, j' \geq 0$. Drop the superscript h. In the next two auxiliary lemmas we assume that:

i) the optimal choices $x_{k,s^{a'}}$ are k-invariant for all $a' \leq a+1$, and $D_\omega Z_{s^{a'},0} = 0$, for all perturbations ω and all $a' \leq a + 1$;

ii) for each k, s^a and σ, the optimal choices $x^{a+2}_{k,s^a,s,\sigma}$ are one-to-one in s; and

iii) $\pi(\cdot|s)$ is one-to-one in s.

We mimic the argument of the previous lemmas. Let

$$S_-(\bar{s},j') = \{\sigma_l | x^{a+2}_{1,s^a,\bar{s},\sigma_l} \neq x^{a+2}_{2,s^a,s_{j'},\sigma_l}\}.$$

Auxiliary Lemma 5 assumes that $S_-(\bar{s},j') = \varnothing$ and $j' > 0$, while Auxiliary Lemma 6 assumes that $S_-(\bar{s},j') \neq \varnothing$ and, most importantly, that $j' \geq 0$.

Auxiliary Lemma 5 *Let $(s^a,\bar{s},\sigma)$ and $j' > 0$ be given, $a \geq 1$. Suppose that assumptions i) - iii) hold, and that $S_-(\bar{s},j') = \varnothing$. Then, $D_\omega Z_{(s^a,\bar{s},\sigma),j(l,j')} \neq 0$ for some $l \geq 0$.*

Proof. Again, fix $\bar{s} = 1$ and $s_{j'}(\bar{s}) = 2 = j'$. Using i), we exploit Auxiliary Lemma 1 thereby dropping the subscript k from variables that are k-invariant when convenient. Given $S_-(\bar{s},j') = \varnothing$, since $x_{1,(s^a,1,\sigma)} = x_{2,(s^a,2,\sigma)}$ for all σ, and by ii) it must be that $x_{k,(s^a,k,\sigma)} \neq x_{k,(s^a,s,\sigma)}$, for all $s \neq k$, $k = 1,2$. For each σ, we consider perturbations that affect only $x_{1,(s^a,1,\sigma)} = x_{2,(s^a,2,\sigma)}$ without disturbing any $x_{k,(s^a,s,\sigma)} \neq x_{1,(s^a,1,\sigma)}$. We call σ such perturbation. Then, $\theta_{k,(s^a,s,\sigma)} = q_{k,(s^a,k)}\tilde{A}_{k,(s^a,k)}y^T_\sigma\gamma^*_{k,(s^a,k,\sigma)}$ if $s = k = 1,2$, while $\theta_{k,(s^a,s,\sigma)} = 0$ otherwise.

Since by i) $D_\sigma Z_{s^a,0} = 0$ for all $\sigma \in S$, denoting with $[y_{s_a}\Delta.m_{k,s^{a-1}}]$ the vector $(y_{s_a}\Delta_\sigma m_{k,s^{a-1}})_{\sigma\in S}$, and since $D_\sigma F_{(s^a,1),0} = 0$, we get that:

$$\begin{aligned} y_1\tilde{A}_{s^a}[q^T_{s^a}\gamma_{s^a}(y_{s_a}\Delta_\sigma m_{1,s^{a-1}}) - y^T_1\gamma_{1,s^a,1}\theta_{1,s^a,1,\sigma}] = \\ y_1\tilde{A}_{s^a}[q^T_{s^a}\gamma_{s^a}(y_{s_a}\Delta_\sigma m_{2,s^{a-1}}) - y^T_2\gamma_{2,s^a,2}\theta_{2,s^a,2,\sigma}] - \frac{\theta_{1,s^a,1,\sigma}}{1+d_{s^a,1}}. \end{aligned} \tag{24}$$

Suppose now, by contradiction, that $D_\sigma Z_{(s^a,1,\sigma),j(l,2)} = 0$, for all $l \geq 0$ and all perturbations $\sigma \in S$. This is true only if

$$\begin{aligned} & (I + D_{1,s^a,1})^\setminus Y\tilde{A}_{1,s^a,1}\{\gamma_{s^a,1}q^T_{1,s^a,1}y_1\tilde{A}_{s^a}[\gamma_{s^a}q^T_{s^a}(y_{s_a}\Delta.m_{1,s^{a-1}}) \\ & -Y^T\Gamma^\setminus_{s^a}\Theta^\setminus_{1,s^a,1}] - Y^T\Gamma^{*\setminus}_{1,s^a,1}\} + (I + D_{1,s^a,1})^\setminus \\ = \; & (I + D_{2,s^a,2})^\setminus Y\tilde{A}_{2,s^a,2}\{\gamma_{s^a,2}q^T_{2,s^a,2}y_2\tilde{A}_{s^a}[\gamma_{s^a}q^T_{s^a}(y_{s_a}\Delta.m_{2,s^{a-1}}) \\ & -Y^T\Gamma^\setminus_{s^a}\Theta^\setminus_{2,s^a,2}] - Y^T\Gamma^*_{2,s^a,2}\} + (I + D_{2,s^a,2})^\setminus. \end{aligned} \tag{25}$$

However, by equations (24) and by the definition of $\Theta_{k,s^a,k}$, the left-hand side of equation (25) is

$$(I + D_{1,s^a,1})^{\backslash} Y \tilde{A}_1 \{ \gamma_{s^a,1} q_{1,s^a,1}^T [y_1 \tilde{A}_{s^a} (\gamma_{s^a} q_{s^a}^T (y_{s_a} \Delta . m_{2,s^{a-1}}) - y_2^T \gamma_{s^a,2} q_{2,s^a,2} \tilde{A}_{2,s^a,2} Y^T \Gamma_{2,s^a,2}^{*\backslash}) \tag{26}$$

$$- \frac{q_{1,s^a,1} \tilde{A}_{1,s^a,1} Y^T \Gamma_{1,s^a,1}^{*\backslash}}{1 + d_{s^a,1}}] - Y^T \Gamma_{1,s^a,1}^{*\backslash} \} + (I + D_{1,s^a,1})^{\backslash}$$

Using equation (16) of Auxiliary Lemma 3, and by taking into account equation (26), equation (25) can be rewritten as

$$\begin{aligned} & (I + D_{1,s^a,1})^{\backslash} Y \tilde{A}_{1,s^a,1} \{ \gamma_{s^a,1} q_{1,s^a,1}^T [\frac{q_{1,s^a,1} \tilde{A}_{1,s^a,1} Y^T \Gamma_{1,s^a,1}^{*\backslash}}{1 + d_{s^a,1}} \\ & + y_1 \tilde{A}_{s^a} y_2^T \gamma_{s^a,2} q_{2,s^a,2} \tilde{A}_{2,s^a,2} Y^T \Gamma_{2,s^a,2}^{*\backslash}] + Y^T \Gamma_{1,s^a,1}^{*\backslash} \} - (I + D_{1,s^a,1})^{\backslash} \\ = \; & (I + D_{2,s^a,2})^{\backslash} Y \tilde{A}_{2,s^a,2} \{ Y^T \Gamma_{2,s^a,2}^{*\backslash} \\ & + \gamma_{2,s^a} q_{2,s^a,2}^T y_2 \tilde{A}_{s^a} \gamma_{2,s^a} y_2^T q_{2,s^a,2} \tilde{A}_{2,s^a,2} Y^T \Gamma_{2,s^a,2}^{*\backslash} \} - (I + D_{2,s^a,2})^{\backslash}. \end{aligned} \tag{27}$$

However, by equations (18),

$$\begin{aligned} & M_{s^a}^{12}(1,2) [\tilde{A}_{s^a} \gamma_{2,s^a,2}^* y_2^T] \frac{[q_{2,s^a,2} \tilde{A}_{2,s^a,2} Y^T \Gamma_{2,s^a,2}^{*\backslash}]}{1 + d_{s^a,2}} \\ = \; & -(I + D_{2,s^a,2})^{\backslash} Y \tilde{A}_{2,s^a,2} \gamma_{s^a,2} q_{2,s^a,2}^T \frac{q_{2,s^a,2} \tilde{A}_{2,s^a,2} Y^T \Gamma_{2,s^a,2}^{*\backslash}]}{1 + d_{s^a,2}}. \end{aligned}$$

where $M_{s^a}^{12}(1,2)$ is $M_{s^a}(1,2)$ of Auxiliary Lemma 3 with indexes (k, s^a, k) replacing (s^a, k), $k = 1, 2$. Thus, equation (27) is

$$(I{+}D_{1,s^a,1})^{\backslash} Y C_{1,s^a,1} Y^T \Gamma_{1,s^a,1}^{*\backslash} - (I{+}D_{1,s^a,1})^{\backslash} = (I{+}D_{2,s^a,2})^{\backslash} Y C_{2,s^a,2} Y^T \Gamma_{2,s^a,2}^{*\backslash} - (I{+}D_{2,s^a,2})^{\backslash}$$

where $C_{k,s^a,k}$ is the by now familiar matrix. Then, by the same argument of Auxiliary Lemma 3, we get a contradiction. ■

Auxiliary Lemma 6 *Let* $(s^a, \bar{s}, \sigma)$ *and* $j' \geq 0$ *be given,* $a \geq 0$. *Suppose that assumptions i) - iii) hold, and that* $S_-(\bar{s}, j') \neq \varnothing$. *Then* $D_\omega Z_{(s^a,\bar{s},\sigma),j(l,j')} \neq 0$ *for some* $\sigma_l \in S \backslash S_-(\bar{s}, j')$.

Proof. We break the argument in two steps. First we consider symmetric, and then asymmetric perturbations identical to those used in Auxiliary Lemma 5. As usual, we fix $\bar{s} = 1$, drop the superscript h and use the notation introduced in the previous lemmas.

Step 1) Start with 0^* and s. Bear in mind that by i) and Auxiliary Lemma 1, $\Delta_\omega m_{k,s^{a-1}}$ is k-invariant for all symmetric perturbations $\omega = 0^*, 1..., S$. Also, if $l \in S \backslash S_-$, and by ii), these perturbations affect only $x_{1,(s^a,1,\sigma_l)} = x_{2,(s^a,j(2),\sigma_l)}$ without disturbing any $x_{k,(s^a,s,\sigma)} \neq x_{1,(s^a,1,\sigma)}$. Then, $D_\omega Z_{(s^a,1,\sigma),j(l,j')} = 0$ for all $\sigma_l \in S \backslash S_-$ and any $\omega = 0^*, 1..., S$ only if

$$(I + D_{1,s^a,1})_\backslash Y_\backslash \tilde{A}_{1,s^a,1}\{\gamma_{s^a,1} q^T_{1,s^a,1}[y_1 \Delta_\omega m_{1,s^a} + 1_{\omega 1}]\} =$$
$$(I + D_{2,s^a,s_{j'}})_\backslash Y_\backslash \tilde{A}_{2,s^a,s_{j'}}\{\gamma_{s^a,s_{j'}} q^T_{2,s^a,s_{j'}}[y_{s_{j'}} \Delta_\omega m_{2,s^a} + 1_{\omega s_{j'}}]\}$$

where $1_{\omega s} = 1$ if $\omega = s$, and zero otherwise. Since by i) $D_\omega Z_{s^a,0} = 0$, we get that $\Delta_\omega m_{k,s^a}$ is k-invariant. Hence, we get back the equivalent of equations (16)-(19) of Auxiliary Lemma 4:

$$M^{12}_{s^a}(1, s_{j'})_\backslash \tilde{A}_{s^a} q^T_{s^a} = 0, \tag{28}$$
$$M^{12}_{s^a}(1, s_{j'})_\backslash \tilde{A}_{s^a} y^T_s = 0, \text{ for all } s > 2,$$
$$(I + D_{1,s^a,1})_\backslash Y_\backslash \tilde{A}_{1,s^a,1} \gamma_{s^a,1} q^T_{1,s^a,1} = M^{12}_{s^a}(1, s_{j'})_\backslash \gamma^*_{1,s^a,1} \tilde{A}_{s^a} y^T_1,$$
$$-(I + D_{2,s^a,s_{j'}})_\backslash Y_\backslash \tilde{A}_{2,s^a,s_{j'}} \gamma_{s^a,s_{j'}} q^T_{2,s^a,s_{j'}} = M^{12}_{s^a}(1, s_{j'})_\backslash \gamma^*_{2,s^a,s_{j'}} \tilde{A}_{s^a} y^T_{s_{j'}},$$

where $M^{12}_{s^a}(1, s_{j'})_\backslash$ follows the obvious notational conventions of the previous auxiliary lemmas.

Step 2) Consider now the same asymmetric perturbations $(s^a, 1, \sigma)$ of Auxiliary Lemma 5, $\sigma \in S_-$. First, for given s, let

$$S_2(s) = \{\sigma | x_{2,s^a,s,\sigma} = x_{1,s^a,1,\sigma},\ \sigma \in S_-\}$$

By ii), for each $\sigma \in S_-$ there is at most one $s \neq 1$ such that $x_{2,s^a,s,\sigma} = x_{1,s^a,1,\sigma}$.

Start by considering $j' > 0$ with, say, $s_{j'}(1) = 2$. Then, $D_{(s^a,1,\sigma)} Z_{(s^a,1,\sigma),j(l,j')} = 0$ for all $(\sigma, \sigma_l) \in S_- \times S \backslash S_-$ only if

$$\begin{aligned} & (I + D_{1,s^a,1})_\backslash Y_\backslash \tilde{A}_{1,s^a,1}\{\gamma_{s^a,1} q^T_{1,s^a,1}[y_1 \Delta . m_{1,s^a}] - Y^T_- \Gamma^*_{1,-}]\} \\ = \; & (I + D_{2,s^a,2})_\backslash Y_\backslash \tilde{A}_{2,s^a,2}\{\gamma_{s^a,2} q^T_{2,s^a,2}[y_2 \Delta . m_{2,s^a}]\} \end{aligned} \tag{29}$$

Since $D_{(s^a,1,\sigma)}Z_{(s^a,2),0} = 0$, $\sigma \in S_-$, by i) and the definition of S_- we have $\Delta.m_{1,s^a} = \Delta.m_{2,s^a}$. Hence, the right-hand side of equation (29) can be rewritten as

$$(I + D_{2,s^a,2})\backslash Y\backslash \tilde{A}_{2,s^a,2}\{\gamma_{s^a,2}q^T_{2,s^a,2}[y_2\Delta.m_{2,s^a}]\} =$$
$$(I + D_{2,s^a,2})\backslash Y\backslash \tilde{A}_{2,s^a,2}\{\gamma_{s^a,2}q^T_{2,s^a,2}y_2\tilde{A}_{s^a}[\gamma_{s^a}q^T_{s^a}y_{s_a}\Delta.m_{1,s^{a-1}} - \gamma_{s^a,1}y^T_1 q_{1,s^a,1}\tilde{A}_{1,s^a,1}Y^T_-\Gamma^*_{1,-}\}$$

and using the first and the third of equations (28), we finally get that (29) holds only if

$$(I + D_{1,s^a,1})\backslash Y\backslash C_{1,s^a,1}Y^T_-\Gamma^*_{1,-} = 0.$$

and $C_{1,s^a,1}$ is the familiar matrix. Comparison with (23) gives us the conclusion, as in Auxiliary Lemma 4.

When $j = 0$, and $s_j(1) = 1$, since $\gamma_{s^a,1}(I + D_{1,s^a,1})\backslash Y\backslash \tilde{A}_{1,s^a,1}\gamma_{s^a,1}q^T_{1,s^a,1} = \gamma_{s^a,2}(I + D_{2,s^a,2})\backslash Y\backslash \tilde{A}_{2,s^a,2}q^T_{2,s^a,2}$ and obviously $y_2 = y_1$, equation (29) reduces to $(I + D_{1,s^a,1})\backslash Y\backslash \tilde{A}_{1,s^a,1}Y^T_-\Gamma^*_{-1} = 0$, again a contradiction. ■

Auxiliary Lemma 7 *Let s^a be given, for some $a \geq 0$. Suppose that: i) $x_{k,s^a,s,\sigma}$ is one-to-one in (k, s), for all $\sigma \in S$; ii) $x_{k,s^{a'}}$ is k-invariant, for all $a' < a + 2$. Then, there exists s such that $D_\omega Z_{(s^a,s),0} \neq 0$ for some perturbation ω.*

Proof. Pick an $s \in S$, say $s = 1$, and consider asymmetric perturbation $(s^a, 1, \sigma)$ affecting $u^{a+2}(\cdot, \sigma)$ around $x_{1,s^a,1,\sigma}$. By i), it does not disturb u^{a+2} either around $x_{2,s^a,1,\sigma}$ or around $x_{k,s^a,s,\sigma}$, $s \neq 1$. Then, if by contradiction $D_\sigma F_{(s^a,s),0} = 0$ for all $s \neq 1$ and $\sigma \in S$, by ii) and Auxiliary Lemma 1 we have

$$(1 + d_{s^a,s})y_s(\Delta_\sigma m_{1,s^a} - \Delta_\sigma m_{2,s^a}) = 0, \text{ for } s \in S\backslash\{1\}.$$

Since $S > J$ and Y is in general position, the latter implies that $\Delta_\sigma m_{1,s^a} = \Delta_\sigma m_{2,s^a}$, for all $\sigma \in S$. Then $D_\sigma F_{(s^a,1),0} = 0$ only if

$$q_{1,s^a,1}\tilde{A}_{1,s^a,1}y^T_\sigma = 0 \text{ for all } \sigma \in S$$

a contradiction, given that Y is in general position. ■

Now we conclude the proof that Condition Nest holds by putting together the pieces. Given a (p, q, ω) and a minimal s^a, if $(p, q, \omega) \in \left(N^{J_a}_{s^a}\right)^c$, there exists a $J' \subset J_a$ such that $Z_{s^a,j} = 0$ for $j \in J'$, that is, $(p, q, \omega) \in N^{J_a\backslash J'}_{s^a}$: either $\lambda_{1,s^{a-2},s,\sigma_l} - \lambda_{2,s^{a-2},s_j,\sigma_l} = 0$ if $j \leq S^2$, or $\lambda_{k,s^{a-2},s,\sigma_l} - \lambda_{k,s^{a-2},s_{j'},\sigma_l} = 0$

otherwise. If the second case, Auxiliary Lemmas 3 or 4 deliver that $D_\omega Z_{s^a,j} \neq 0$ for some $j \in J' \subset J_a$, that is, $(p,q,\omega) \in E_{s^a}^{J'}$, and Conditions Nest-0 and Nest-1 hold true. Otherwise, if $j \leq S^2$, assumptions i)-iii) of Auxiliary Lemmas 5 or 6 apply, delivering the same conclusion. Next, suppose that $(p,q,\omega) \in N_{s^a}^{h,J_a} \cap N_{s^{a-1}}^{hS \backslash S'}$. Again by Auxiliary Lemmas 3 or 5, $(p,q,\omega) \in E_{s^{a-1}}^{h,S'}$, and Condition Nest-2 holds. Finally, if $(p,q,\omega) \in N_{s^a}^{h,J_a} \cap N_{s^{a-1}}^{hS}$, $x_{k,s^{a-2},s,\sigma}$ is one-to-one in (k,s), for all $\sigma \in S$, while s^a minimal means that $x_{k,s^{a'}}$ is k-invariant for $a' < a$. Then, Auxiliary Lemma 7 applies and there exists s, ω' such that $D_{\omega'} Z_{(s^{a-2},s),0} \neq 0$, i.e., $(p,q,\omega) \in E_{s^{a'},0}$ for $s^{a'} = (s^{a-2}, s) \prec s^a$, and Condition Nest-3 holds, concluding the argument.■

Proofs of Claims

Let X an $N \times M$ dimensional matrix and of full column rank M. Here x_m denotes the m-th row of X. Let D be a diagonal matrix of dimension $M \times M$, with negative diagonal elements. Let $A = (X^T D X)^{-1}$, a negative definite matrix of dimension $M \times M$. Consider the quadratic expression:

$$d_m x_m A x_m^T,\ m = 1, ..., M.$$

The quadratic expressions $\gamma_s q_s \tilde{A}_s q_s^T$, $\gamma_s (1 + d_s) y_s \tilde{A}_s y_s^T$ and $\gamma_{s,s'} y_{s'} \tilde{A}_s y_{s'}^T$ are of the this type. Therefore, it is enough to prove the following statement.

Claim 10 *$d_m x_m A x_m^T < 1$, for all m.*

Proof. Let

$$\chi_m = x_m A.$$

Observe that $(d_m x_m A x_m^T)^2 = (d_m)^2 \chi_m (x_m^T x_m) \chi_m^T$. Therefore, $(d_m)^2 \chi_m (x_m^T x_m) \chi_m^T < 1$ implies $d_m x_m A x_m^T < 1$, and it is enough and more convenient to prove the statement involving χ_m.

Let X_m be the matrix of dimension $(N-1) \times M$ obtained from X by deleting the m-th row. Let D_m be the diagonal matrix of dimension $(M-1) \times (M-1)$ obtained from D by deleting the m-th row and column. Finally, let $A_m = (X_m^T D_m X_m)^{-1}$. Then

$$x_m A x_m^T = \chi_m A^{-1} \chi_m^T = \chi_m X^T D X \chi_m^T \tag{30}$$

and

$$x_m A_m x_m^T = \chi_m A^{-1} A_m A^{-1} \chi_m^T. \tag{31}$$

However, since

$$A^{-1} = d_m x_m^T x_m + (A_m)^{-1},$$

we get

$$\begin{aligned} A^{-1} A_m A^{-1} &= (d_m x_m^T x_m A_m + I)(d_m x_m^T x_m + (A_m)^{-1}) \\ &= (d_m)^2 x_m^T x_m A_m x_m^T x_m + 2 d_m x_m^T x_m + (A_m)^{-1} \\ &= (d_m)^2 x_m^T x_m A_m x_m^T x_m + 2 d_m x_m^T x_m + A^{-1}. \end{aligned}$$

Thus,

$$\chi_m (A^{-1} A_m A^{-1}) \chi_m^T = \chi_m (A^{-1}) \chi_m^T + \chi_m (x_m^T x_m) \chi_m^T d_m (d_m x_m A {x_m}^T + 1).$$

Using (30) and (31), we get

$$\frac{x_m A_m x_m^T}{x_m A x_m^T} = 1 + \frac{\chi_m (x_m^T x_m) \chi_m^T d_m (d_m x_m A_m x_m^T + 1)}{x_m A x_m^T}$$

or equivalently

$$\frac{x_m A_m x_m^T}{x_m A x_m^T} (1 - \chi_m (x_m^T x_m) \chi_m^T (d_m)^2) = 1 + \frac{\chi_m (x_m^T x_m) \chi_m^T d_m}{x_m A x_m^T}. \tag{32}$$

Since $d_m < 0$ and $\chi_m = x_m A$, we have $\frac{\chi_m (x_m^T x_m) x_m^T d_m}{x_m A x_m^T} > 0$. Hence, the r.h.s. of equation (32) is positive. Since $\frac{x_m A_m x_m^T}{x_m A x_m^T} > 0$, the latter implies that $\chi_m (x_m^T x_m) \chi_m^T (d_m)^2 < 1$, and we are done.■

Claim 11 *Let k and s^a be given, $a \geq 1$. There exists a simple perturbation $\Delta e_{a',s}$, $a' \leq a$, such that $\mathbf{1}_{a',a} + [y_{s_a} \Delta_{(a',s)} m_{s^{a-1}}] \neq 0$.*

Proof. We consider endowment changes $\Delta e_{a',s_{a'}}$ along the realized history s^a, i.e., $\Delta e_{a',s} \neq 0$ only if $a' \leq a$, $s = \bar{s}_{a'}$ where $s^a = (s_0, s_1, ...\bar{s}_{a'}, ...s_a)$, and we choose $\Delta e_{a',s_{a'}}$ such that $p_{k,s^{a'}} \Delta e_{a',s} = 1$. The induced change in financial wealth at s^a is equal to $\mathbf{1}_{a,a'} + y_{s_a} \Delta_{(a',s_{a'})} m^k_{s^{a-1}}$. Observe that by perturbing $\Delta e(a, s_a)$ we get

$$y_{s_a} \Delta_{(a,s_a)} m_{s^{a-1}} = \gamma_{s^{a-1}} y_{s_a} \tilde{A}_{s^{a-1}} q^T_{s^{a-1}} (y_{s^{a-1}} \Delta m_{s^{a-2}}) - \gamma_{s^a} (1 + d_{s^a}) y_{s_a} \tilde{A}_{s^{a-1}} y^T_{s_a}$$

for $a > 1$, where $(y_{s^{a-1}} \Delta m_{s^{a-2}}) \equiv 0$ for $a = 1$. By Claim 10, $-\gamma_{s^a}(1 + d_{s^a}) y_{s_a} \tilde{A}_{s^{a-1}} y^T_{s_a} > -1$. Hence, if $a = 1$ or $y_{s_a} \tilde{A}_{s^{a-1}} q^T_{s^{a-1}} = 0$, the claim is proved. Suppose it is not, and that $\mathbf{1}_{a,a} + y_{s_a}\Delta_{(a,s_a)}m_{s^{a-1}} = 0$. Then, consider the perturbation $\Delta e(a-1, s_{a-1})$ at the unique predecessor of s^a. We have

$$y_{s_a}\Delta_{(a-1,s_{a-1})}m_{s^{a-1}} = \gamma_{s^{a-1}}y_{s_a}\tilde{A}_{s^{a-1}}q^T_{s^{a-1}}(1 + y_{s_{a-1}}\Delta m_{s^{a-2}})$$

and

$$y_{s_{a-1}}\Delta_{(a-1,s_{a-1})}m_{s^{a-2}} = \gamma_{s^{a-1}}y_{s_{a-1}}\tilde{A}_{s^{a-2}}q^T_{s^{a-2}}(y_{s_{a-2}}\Delta m_{s^{a-3}}) - \gamma_{s^{a-1}}(1+d_{s^{a-1}})y_{s_{a-1}}\tilde{A}_{s^{a-2}}y^T_{s_{a-1}}$$

Once again, by Claim 10 if $y_{s_{a-1}}\tilde{A}_{s^{a-2}}q^T_{s^{a-2}} = 0$, $y_{s_{a-1}}\Delta_{(a-1,s_{a-1})}m_{s^{a-2}} > -1$, and $\mathbf{1}_{a',a} + y_{s_a}\Delta_{(a-1,s_{a-1})}m_{s^{a-1}} = y_{s_a}\Delta_{(a-1,s_{a-1})}m_{s^{a-1}} \neq 0$, and we are done. Thus suppose that this is not the case, and that $\mathbf{1}+y_{s_{a-1}}\Delta_{(a-1,s_{a-1})}m_{s^{a-2}} = 0$. Proceed backward, and let $s^a = (s_1, ..., s_{a-1}, s)$. If for some $1 < j \leq a$, $y_{s_j}\tilde{A}_{s^{a-2}}q^T_{s_{j-1}} = 0$ we are done. If this is not the case, then $j = a = 1$, and it suffices to show that

$$y_{s_1}\Delta_{(1,s_1)}m_0 = -\gamma_{s^1}(1 + d_{s^1})y_{s^1}\tilde{A}_0 y^T_{s^1} > -1$$

which is the case. ■

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