Equilibrium Unemployment in a Generalized Search Model*

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Abstract

We present a generalization of the standard Diamond-Mortensen-Pissarides search model of unemployment in which there is both an intensive and extensive margin of employment creation. In our model, each firm can invest to gain access to a production technology with diminishing returns to labor and then post vacancies in order to recruit workers. Entry by new firms corresponds to the extensive margin of employment creation, while job creation by existing firms captures the intensive margin. As in the baseline Diamond-Mortensen-Pissarides search model and theories of the firm developed by Stole and Zwiebel (1996a,b) and Wolinsky (2000), wages are determined by continuous bargaining between the firm and its employees. We characterize the steady-state equilibrium in this class of models and discuss the implications of various different types of shocks on the equilibrium unemployment rate.

*This paper reports on ongoing work. Comments are welcome.

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1 Introduction

The search and matching model developed by, among others, Diamond (1982a,b), Mortensen (1982), Pissarides (1985), and Mortensen and Pissarides (1994) has become the leading model for modeling equilibrium unemployment. The baseline Diamond-Mortensen-Pissarides (DMP) model has three building blocks: first, a matching function, which incorporates the frictions involved in labor market search; second, a production structure where each firm can employ a single worker, and third, a protocol for bargaining between firms and workers for wage determination. These building blocks lead to a very tractable model, with a range of comparative statics, but the model does not incorporate the intensive and the extensive margins of employment creation (and destruction). In practice, new jobs are created both by existing firms and by the entry of new firms (see for example, Davis, Haltiwanger, and Schuh, 1996). In this paper, we extend the baseline DMP model by introducing a production technology with diminishing returns to labor, thus potentially distinguishing between the intensive and the extensive margins of employment.

In the model economy, a large number of firms can invest a fixed amount k to gain access to a production function with decreasing returns. Once active, firms can post vacancies in order to hire workers. Employment relationships come to an end both because of exogenous worker separations from continuing firms and because of firm shutdowns. Wages are determined by bargaining between individual workers and the firm as in the standard DMP model. Since there are more than two players in the bargaining game, we use the bargaining protocol of Shapley (1953) adapted to a dynamic setting. The Shapley value has been previously used in a static model of employment and wage determination by Stole and Zwiebel (1996a,b) and in a dynamic setting by Wolinsky (2000). It is an attractive wage determination protocol because it generalizes the Nash bargaining solution used in the standard DMP model in a natural way, because it captures the notion that players cannot enter into binding agreements, and because it has appealing microfoundations, especially in our environment, where the firm is "essential" for production.

Our main contribution is therefore to present a relatively tractable model that combines the general equilibrium structure of the baseline DMP model with the model due to Stole and Zwiebel and Wolinsky of wage and employment determination in the absence of binding contracts between firms and workers.

Our basic model, introduced in Section 2, assumes that firms can employ a countable number of workers. An equilibrium in this economy determines not only unemployment and vacancy rates, but also the size distribution of firms and an endogenous wage distribution. After providing a basic analysis of this setup, in Section 3 we take the limit of this economy as worker size becomes small (using the continuous limit bargaining solution of Aumann and Shapley (1974), also used in Stole and Zwiebel (1996a) and Wolinsky (2000)). This limit economy admits a set of differential equations describing equilibrium behavior, which can be solved explicitly. Using of the limiting model, we characterize the form of the steady-state equilibrium.

The steady-state equilibrium we derive is relatively tractable and can be used to address a variety of different questions that arise in equilibrium models of unemployment, as well as some new questions relating to the differential responses of intensive and extensive margins of employment creation. As an illustration of the potential applications of our framework, we investigate how unemployment responds to productivity shocks, and discuss whether adjustment takes place at the intensive or the extensive margin.

An analysis of the magnitude of employment creation to productivity shocks in this class of models is interesting as part of the study of whether search and matching models can account for the differences in the levels and in the dynamics of unemployment across countries and over time. A recent influential paper by Shimer (2005) has argued that the baseline DMP model cannot generate quantitatively plausible fluctuations in unemployment because wages are "excessively" responsive to changes in productivity (see also Hall, 2005). Since our model incorporates both intensive and extensive margins and also links wages not to average productivity but to a weighted average of marginal and average product of labor (with weights determined by the equilibrium distribution of firm sizes), the effects of productivity shocks on the equilibrium are different and potentially richer. Interestingly, however, we find that basic calibrations of the model lead to quantitative results similar to those of the baseline DMP model.

Our work is related to various different strands of the search literature. As noted above, we build on and generalize the baseline search-matching model of Diamond (1982b), Pissarides (1984, 1985) and Mortensen and Pissarides (1994), and we also perform quantitative analyses similar to those in Shimer (2005) and Hall (2005).[1] Hawkins (2006a,b) also considers a search economy with multi-worker firms, but assumes directed search and wage posting rather than bargaining.[2] Models of firm-worker bargaining with diminishing returns at the firm level without labor market frictions have been considered by Stole and Zwiebel

[1]See also Merz (1995), Andolfatto (1996) and Hagedorn and Manovskii (2005) for other quantitative investigations.

[2]See Montgomery (1991), Peters (1991), Moen (1997), Shimer (1996), Acemoglu and Shimer (1999a,b) and Burdett, Shi, and Wright (2001) for directed search models.

(1996a,b) and Acemoglu, Antràs, and Helpman (2005). Previous studies incorporating dynamic firm-worker bargaining with diminishing returns include Bertola and Caballero (1994), Smith (1999), Wolinsky (2000), Bertola and Garibaldi (2001), Cahuc and Wasmer (2001), and Cahuc, Marque, and Wasmer (2005). Of these, the paper most closely related to our work is the important paper by Wolinsky (2000), which also considers the dynamic bargaining problem between workers and firms in the presence of diminishing returns. However, Wolinsky's analysis is essentially partial equilibrium since the arrival of new workers to firms is assumed to be exogenous. Consequently, Wolinsky's model does not endogenize the unemployment rate and cannot be used for equilibrium analysis in the labor market. In addition, none of the other papers fully solve for wage determination and firm size distribution in general equilibrium with forward-looking bargaining.[3]

The rest of the paper is organized as follows. Section 2 presents the baseline model. Section 3 considers the limit economy in which workers size is small. This section provides an explicit-form characterization of the steady-state equilibrium and proves the existence of a state-state equilibrium. Section 4 explains why this class of models might generate different responses to productivity shocks than the standard DMP model, and presents a range of illustrative calibrations of the baseline model. Section 5 concludes.

2 Baseline Model

In this section we introduce the benchmark model, which we will use to illustrate the basic ideas. The model economy is infinite horizon in continuous time and is populated by a continuum of workers of size $\varepsilon > 0$ each. In this section, there is no loss of generality if the reader prefers to think of the case in which $\varepsilon = 1$; however, in Section 3 we consider the limiting economy where $\varepsilon \to 0$. It is both easier to prove theoretical results and more convenient to discuss comparative statics in this limiting economy, since when $\varepsilon > 0$, integer problems can lead to an economically uninteresting multiplicity of equilibria; therefore the model in this section should be considered as motivation for the continuous employment case.

[3] Bertola and Caballero (1994) solve a search model with multiple workers per firm assuming that wages are determined by Nash bargaining. Nash bargaining is difficult to justify when there is multilateral bargaining, however. Cahuc, Marque, and Wasmer (2005) use a multilateral bargaining rule similar to the Shapley value, but incorrectly impose the envelope condition, which holds for the last worker hired by the firm, throughout rather than fully solving for wages in general equilibrium.

2.1 Environment

Consider the following continuous time infinite-horizon economy. There is a continuum of mass 1 of identical workers. All workers supply labor inelastically, are risk neutral and discount the future at the rate r. In particular, the utility of the worker at time t is:

$$U(t) = \int_t^\infty \exp(-r(\tau - t))\, c(\tau)\, d\tau,$$

where $c(\tau)$ is consumption at time τ, equal to $b > 0$ if the worker is unemployed, and to his wage at time τ, $w(\tau)$, if employed.

On the other side of the market, there is a large mass of potential firms. All firms are owned by the workers and also act in in risk-neutral manner, maximizing the net present discounted value of profits. Since this has no effect on any aspect of the equilibrium, the exact distribution of shares of firms is not specified. At any instant, each potential firm can pay a fixed cost of k in order to become active (for example, k can be interpreted as the price of the necessary capital equipment). After doing so, the firm has access to the production function, $F : \varepsilon\mathbb{Z}_+ \to \mathbb{R}_+$, which specifies flow rate of output as a function of the mass of employees at that instant, denoted by n. Here $\varepsilon\mathbb{Z}_+$ is the set $\varepsilon\mathbb{Z}_+ \equiv \{0, \varepsilon, 2\varepsilon, 3\varepsilon, ...\}$, so that n takes values from this set.[4] This amounts to assuming that the "size" of each employee is equal to ε; it is useful since we will later take the limit as $\varepsilon \to 0$ and allow the firm to employ any number of workers. For the expressions in this section, there would be no loss in generality in imposing $\varepsilon = 1$, so that the number of employees of a firm takes integer values.

We assume that $F(n)$ is strictly increasing, continuously differentiable and strictly concave and satisfies $F(0) = 0$. Moreover, we assume that it satisfies a weaker version of the standard Inada condition, $\lim_{n\to\infty} F'(n) < b$, where the prime denotes differentiation.

Matching between firms and workers is frictional as in the standard DMP model. An active firm has the option of posting a single vacancy at any time; doing so has a flow cost of γ. If a firm does not post a vacancy, then it cannot match with new workers and thus cannot hire additional workers. The flow of firm-worker matches at every instant is determined by an aggregate matching function, $M(u, v)$, where u is the mass of unemployed workers, u, and v is the mass of vacancies posted by firms. The function $M(u, v)$ is assumed to the strictly increasing and continuously differentiable in its two arguments and to exhibit constant returns to scale. This implies that the flow mass of unemployed workers that meet a given vacancy

[4]Note that since workers are of fixed mass ε each, the mass of workers employed at a given firm must be an element of the set $\varepsilon\mathbb{Z}_+$ of non-negative integer multiples of ε. That is, for a given ε, the number of workers per firm can take on only countably many values. Recall, however, that there are a continuum of workers and firms.

can be expressed as

$$q(\theta) \equiv \frac{M(u,v)}{v} \equiv M\left(\theta^{-1},1\right),$$

where $\theta \equiv v/u$ is the tightness of the labor market, and $q : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable decreasing function. Since each worker is of mass ε, the flow rate at which a vacancy is contacted by a worker is $q(\theta)/\varepsilon$. This implies that it becomes more difficult for firms to match with workers in a tighter (high θ) labor market. The flow rate of a match for an unemployed worker is $\theta q(\theta)$, which is also continuously differentiable and is assumed to be increasing in θ, so that finding a firm becomes easier for workers in a tighter labor market. We denote the decision of the firm about whether to post a vacancy by $h \in [0,1]$; $h = 1$ denotes that the firm posts a vacancy, $h = 0$ denotes that it does not, and $h \in (0,1)$ denotes a mixed strategy.

Once a match between a firm and a worker is formed, it continues until either it is exogenously destroyed, or until one of the two agents chooses to end the match. Exogenous destruction can take place because of two reasons: first, the firm may shut down, which happens according to a Poisson process with arrival rate δ. If a firm is hit by such a shock, then it is destroyed and with a scrapping value of 0, and all workers previously employed at the firm become unemployed. In addition, each employed worker is also separated from his current employer, and returns to unemployment, according to a Poisson process with arrival rate s. All these stochastic processes are independent. This implies that a firm that employs a mass n of workers will experience a separation from one of its workers according to a Poisson process with arrival rate sn/ε.

Since matching between firms and workers is frictional, firms will only grow slowly and there will be a distribution of firm sizes. For $m \in \varepsilon\mathbb{Z}_+$, let $g(m,t)$ be the mass of firms employing m at time t. The distribution $g(\cdot,\cdot)$ will be the aggregate state variable of the economy, while $n(t)$, the number of workers employed by a firm at time t, will be the individual state variable.

For tractability, we will only consider steady-state equilibria and will suppress time dependence when this causes no confusion. Moreover, we assume that both firms and workers are *anonymous*, so that equilibria must have a symmetric Markovian structure, with strategies depending only on payoff-relevant state variables.[5]

More specifically, let $\mathcal{G}$ be the set of density functions $g : \varepsilon\mathbb{Z}_+ \to \mathbb{R}_+$, which, by definition,

[5] Anonymity is useful in ruling out "unreasonable" equilibria in which agents in the economy use punishment-type strategies against specific agents.

satisfy $\sum_{m=0}^{\infty} g(m) = 1$. Throughout we use "value" and "net present discounted value" interchangeably. An anonymous steady-state allocation can be defined as follows:

Definition 1. *A steady-state allocation is a tuple* $\langle \theta, V^u, g(\cdot), J(\cdot,\cdot), V(\cdot,\cdot), h(\cdot,\cdot), w(\cdot,\cdot), u \rangle$ *such that*

- $\theta \in \mathbb{R}_+$ *is the tightness of the labor market.*
- $V^u \in \mathbb{R}_+$ *is the value of an unemployed worker.*
- $g \in \mathcal{G}$ *is the distribution of firm sizes.*
- $J : \varepsilon\mathbb{Z}_+ \times \mathcal{G} \to \mathbb{R}_+$ *is the value of a firm with* $m \in \varepsilon\mathbb{Z}_+$ *employees when the distribution of firm sizes is given by* $g \in \mathcal{G}$.
- $V : \varepsilon\mathbb{Z}_+ \times \mathcal{G} \to \mathbb{R}_+$ *is the value of a worker employed by a firm of size* $m \in \varepsilon\mathbb{Z}_+$ *when the distribution of firm sizes is given by* $g \in \mathcal{G}$.
- $h : \varepsilon\mathbb{Z}_+ \times \mathcal{G} \to [0,1]$ *is the vacancy posting decision of a firm with* $m \in \varepsilon\mathbb{Z}_+$ *employees when the distribution of firm sizes is given by* $g \in \mathcal{G}$.
- $w : \varepsilon\mathbb{Z}_+ \times \mathcal{G} \to \mathbb{R}_+$ *be the wage of a worker employed by a firm of size* $m \in \varepsilon\mathbb{Z}_+$ *when the distribution of firm sizes is given by* $g \in \mathcal{G}$.
- $u \in [0,1]$ *is the unemployment rate.*

Notice that a steady-state allocation also defines and endogenous wage distribution, since g determines the distribution of firm sizes and w determines wages corresponding to different firm sizes.

2.2 Equilibrium Characterization

An anonymous steady-state equilibrium will be such that both active and inactive firms maximize their values (profits) and all workers maximize their utilities. Before we can define this equilibrium we need to specify wage determination, which in turn requires us to be more explicit about the value functions. Throughout, we will simplify notation by writing the value, policy and wage functions only as a function of the number of employees, e.g., $J(m)$ instead of $J(m, g(\cdot))$. Also, denote by Δ the first-difference operator, so that

$\Delta J(m) = [J(m) - J(m-\varepsilon)]/\varepsilon$. Standard arguments immediately imply that in steady state the value function for a firm of size $m \in \mathbb{Z}_+$ satisfies the following recursive equation:

$$\begin{aligned} rJ(m) &= F(m) - mw(m) - \delta J(m) - \frac{ms}{\varepsilon}[J(m) - J(m-\varepsilon)] \\ &\quad + h(m)\left\{-\gamma + \frac{q(\theta)}{\varepsilon}[J(m+\varepsilon) - J(m)]\right\} \end{aligned} \tag{1}$$

or rearranging:

$$(r+\delta)J(m) = F(m) - mw(m) - ms\Delta J(m) + h(m)\{-\gamma + q(\theta)\Delta J(m+\varepsilon)\}. \tag{2}$$

This equation is easy to understand. The left hand side is by definition the flow value of the asset of a firm of size m, $rJ(m)$. The right hand side gives the components of this flow value. A firm of size m produces a flow rate of output equal to $F(m)$ and pays a wage rate of $w(m)$ to each of its m workers. At the flow rate δ, the firm shuts down and loses its value $J(m)$ (and receives the scrap value of zero). At the flow rate sm/ε, one of the employees is separated from the firm, so the firm suffers a loss equal to $J(m) - J(m-\varepsilon)$.[6] Finally, as noted above, $h(m) \in \{0,1\}$ denotes whether a firm of size m opens a vacancy. If it does, $h(m) = 1$, then it incurs a cost of γ and hires a worker at the flow rate $q(\theta)/\varepsilon$, realizing a capital gain equal to $J(m+\varepsilon) - J(m)$. This expression is simplified due to two features. First, we have imposed that upon matching, the firm will hire the worker (which will always be the case in equilibrium). Second, since we are in steady state, there is no further term denoting the change in the asset value of $J(m)$ over time (i.e., there is no term $\dot{J}(m) = 0$).

By a similar reasoning, the value of the worker employed in a firm of size $m \in \mathbb{N}$ is given by

$$\begin{aligned} rV(m) &= w(m) - (\delta+s)[V(m) - V^u] - h(m)\frac{q(\theta)}{\varepsilon}[V(m) - V(m+\varepsilon)] \\ &\quad + \frac{s(m-\varepsilon)}{\varepsilon}[V(m-\varepsilon) - V(m)] \end{aligned} \tag{3}$$

or

$$(r+\delta+s)[V(m) - V^u] = w(m) - rV^u + h(m)q(\theta)\Delta V(m+\varepsilon) - (sm-\varepsilon)\Delta V(m). \tag{4}$$

The worker receives the wage rate $w(m)$, and loses his job either because of firm shutdown or separation (total flow rate of $\delta + s$), in which case he becomes unemployed. At the rate $s(m-\varepsilon)/\varepsilon$, one of the other workers is separated, in which case the worker realizes a capital

[6] The term "loss" here will be justified, since we will see below that in equilibrium $J(\cdot)$ is indeed increasing.

gain equal to $V(m-\varepsilon)-V(m)$.[7] Finally, if the firm chooses to post a vacancy, $h(m)=1$, then at the rate $q(\theta)/\varepsilon$, there will be a further hire and the worker will experience a loss of $V(m)-V(m+\varepsilon)$.

It is also straightforward to see that the value of an unemployed worker will be given by

$$rV^u = b + \theta q(\theta)\frac{\sum_{m=0}^{\infty} g(m)h(m)[V(m+\varepsilon)-V^u]}{\sum_{m=0}^{\infty} g(m)h(m)}. \tag{5}$$

Intuitively, the worker receives a flow of consumption equal to b when unemployed, finds a job at the flow rate $\theta q(\theta)$, and this job comes from a firm that already has m employees, thus giving the worker and value of $V(m+\varepsilon)$, with probability equal to the proportion of vacancies posted by such firms; this is equal to $g(m)h(m)/[\sum_{n=0}^{\infty} g(n)h(n)]$.

Clearly, V^u, as well as the strategies of workers and firms, depend on the distribution $g(\cdot)$. This is in turn given by a simple accounting equation:

$$\left(\delta + \frac{sm}{\varepsilon} + \frac{h(m)\,q(\theta)}{\varepsilon}\right) g(m) = \frac{h(m-\varepsilon)\,q(\theta)}{\varepsilon} g(m-\varepsilon) + \frac{s(m+\varepsilon)}{\varepsilon} g(m+\varepsilon), \tag{6}$$

for given $\theta \in \mathbb{R}_+$ and $g(0) \in [0,1]$. Intuitively, firms of size m leave this state if they shut down (which happens at the flow rate δ), if they lose a worker (which happens at the rate sm/ε), and if they hire a worker (which happens at the rate $q(\theta)/\varepsilon$ as long as they choose $h(m)=1$). Entry into this state either comes from firms of size $m-\varepsilon$ (at the rate $q(\theta)/\varepsilon$ if they post a vacancy, i.e., $h(m-\varepsilon)=1$) or from firms of size $m+\varepsilon$ that lose a worker (at the rate $s(m+\varepsilon)/\varepsilon$).

To complete the description of the environment, we also need to specify wage determination. As in the standard DMP search-matching models, and matched firm-worker pair creates a quasi-rent, since their value together exceeds the sum of their outside options. Wages are then assumed to be determined by some type of bargaining. Given that there is a relationship between one firm and many workers, a natural bargaining concept is the Shapley value, introduced in the seminal work of Shapley (1953). In this context, a natural bargaining protocol, described by Stole and Zwiebel (1996a) leads to the Shapley value. In particular, Stole and Zwiebel (1996a) show that if the firm bargains sequentially with each worker, until agreement is reached with all, the equilibrium division of rents will correspond to the Shapley value. Intuitively, when a firm contemplates disagreeing with a worker, it realizes that this will increase the bargaining power of the remaining workers because of diminishing marginal return in the production function $F(\cdot)$. Since there is a single essential player, the Shapley value takes a simple form. First, recall that the Shapley value specifies that in a bargaining

[7] Contrary to $J(\cdot)$, $V(\cdot)$ will be decreasing, so that this is indeed a gain for the worker.

game with a finite number of players every player's payoff is the average of her contributions to all coalitions that consist of players ordered below her in all feasible permutations. More explicitly, in a game with $m+1$ players, let $\pi = \{\pi(0), \pi(1), ..., \pi(m)\}$ be a permutation of $0, 1, 2, ..., m$, where player 0 denotes the firm and players $1, 2, ..., m$ are the suppliers, and let $z_\pi^j = \{j' \mid \pi(j) > \pi(j')\}$ be the set of players ordered below j in the permutation π. Let the set of feasible permutations be Π and the value of any coalition from this set of permutations be given by the function $v : \Pi \to \mathbb{R}$. Then the Shapley value of player j is

$$\text{Shapley value}_j = \frac{1}{(m+1)!} \sum_{\pi \in \Pi} \left[v\left(z_\pi^j \cup j\right) - v\left(z_\pi^j\right) \right].$$

In other words, the Shapley value of player j is the average of her contribution to possible coalitions ordered below her according to all possible permutations. The Shapley value equation is symmetric in the sense that all players are treated identically. Applying this to our context would imply the following simple equation in terms of value functions: $J(m) - J(m-\varepsilon) = \varepsilon [V(m) - V^u]$, which implies that the incremental value that the firm receives from employing the worker is equalized to the value that the worker obtains by being employed rather than unemployed; the factor ε on the right side of this equation arises from the fact that the worker is of size ε. This equation is use symmetry between the firm and the workers in the bargaining protocol. Instead, as in the standard DMP setup, we will use a slight generalization on this equation, which allows differential bargaining powers between workers and firms. In particular, our wage determination equation will be

$$\beta [J(m) - J(m-\varepsilon)] = (1-\beta)\varepsilon [V(m) - V^u],$$

or

$$\beta \Delta J(m) = (1-\beta) [V(m) - V^u], \tag{7}$$

for all $m \in \mathbb{N}$, where $\beta \in (0,1)$ denotes the bargaining power belonging to workers.

Now we can define *a steady-state equilibrium.*

Definition 2. *A tuple* $\langle \theta, V^u, g(\cdot), J(\cdot,\cdot), V(\cdot,\cdot), h(\cdot,\cdot), w(\cdot,\cdot) \rangle$ *is a steady-state equilibrium if*

- $J(m)$, $V(m)$ *and* $w(m)$ *satisfy (2), (4) and (7).*
- $g(m) \in \mathcal{G}$ *satisfies (6).*
- *the value of an unemployed worker,* V^u*, satisfies (5).*

- *there is optimal vacancy posting, i.e.,*

$$h(m) = \begin{cases} 1 & \text{if } -\gamma + q(\theta)\Delta J(m+\varepsilon) > 0 \\ 0 & \text{if } -\gamma + q(\theta)\Delta J(m+\varepsilon) < 0. \end{cases} \tag{8}$$

- *there is free entry, i.e.,*

$$J(0) \leq k \text{ and } \theta \geq 0, \text{ with complementary slackness.} \tag{9}$$

- $V^u \in \mathbb{R}_+$ *is the value of an unemployed worker.*

Notice that the steady-state equilibrium did not specify the unemployment rate u. This is because, as in the standard DMP model, the unemployment rate can be determined after the other endogenous variables. In particular, in steady state, a standard accounting argument implies that the u unemployed workers will be matched and thus hired at the flow rate $\theta q(\theta)$. On the other side, workers lose their job because of separations at the flow rate s and because of firm shutdowns at the flow rate δ. Consequently, the steady-state unemployment rate is given by equating flows into unemployment, $(1-u)(s+\delta)$ with flows out of unemployment, $u\theta q(\theta)$, thus

$$u = \frac{s+\delta}{s+\delta+\theta q(\theta)}. \tag{10}$$

It is straightforward to verify that, as in the standard DMP model, u is a monotonically decreasing function of θ: steady-state unemployment is lower when the labor market is tighter.

We can provide a partial characterization of a steady-state equilibrium. In particular, a particularly neat formula relates wages to the production function and the outside option of workers (that is, the flow value of being unemployed, rV^u). This is the subject of the following lemma.

Lemma 1. *In a steady-state equilibrium, wages and firm-value functions satisfy*

$$\left(m + \frac{1-\beta}{\beta}\varepsilon\right) w(m) - (m-\varepsilon)w(m-\varepsilon) = \varepsilon\Delta F(m) + \frac{1-\beta}{\beta}\varepsilon r V^u + (h(m) - h(m-\varepsilon))(\gamma - q\Delta J(m)) \tag{11}$$

for $m \geq 2\varepsilon$. In addition,

$$\frac{\varepsilon}{\beta} w(\varepsilon) = \varepsilon \Delta F(\varepsilon) + \frac{1-\beta}{\beta}\varepsilon r V^u + (h(\varepsilon) - h(0))(\gamma - q\Delta J(\varepsilon)). \tag{12}$$

Proof. Subtracting the firm's Bellman equation (2) for $m-\varepsilon$ from that for m yields

$$qh(m)\Delta J(m+\varepsilon) - [qh(m-\varepsilon) + \varepsilon(r+\delta) + sm]\Delta J(m) + s(m-\varepsilon)\Delta J(m-\varepsilon)$$
$$= mw(m) - (m-\varepsilon)w(m-\varepsilon) - F(m) + F(m-\varepsilon) - [h(m) - h(m-\varepsilon)]\gamma;$$

this equation holds for $m \geq 2\varepsilon$. In the case $m = \varepsilon$ it follows that

$$q\Delta J(2\varepsilon) - (q + \varepsilon(r+\delta) + s\varepsilon)\,\Delta J(\varepsilon) = \varepsilon w(\varepsilon) - F(\varepsilon) + F(0) - [h(\varepsilon) - h(0)]\,\gamma.$$

Also, substituting from the Shapley bargaining equation into the worker's Bellman equation for n, multiplying by $\varepsilon(1-\beta)/\beta$, and rearranging gives that

$$qh(m)\Delta J(m+\varepsilon) - (qh(m) + \varepsilon(r+\delta) + sm)\,\Delta J(m) + s(m-\varepsilon)\Delta J(m-\varepsilon) = \frac{1-\beta}{\beta}\varepsilon\,[rV^u - w(m)]\,.$$

Comparing the previous equations establishes the relationships (11) and (12). □

The wage equation derived in the lemma takes a particularly simple form in the case when there is an m^* such that $h(m) = 1$ for $m < m^*$ and $h(m) = 0$ for $m \geq m^*$. An equilibrium of this form is referred to as a *threshold equilibrium* (with hiring cutoff m^*). Naturally, only steady-state equilibria of this sort are of interest and in the rest of the paper, by a (steady-state) equilibrium, we always mean a threshold steady-state equilibrium.[8]

The proof of the following corollary is immediate on substituting the form of $h(\cdot)$ in (11).

Corollary 1. *In a steady-state equilibrium with hiring cutoff m^*, for all $m \neq \varepsilon$ and $m \neq m^*$, wages satisfy the following difference equation:*

$$\left(m + \frac{1-\beta}{\beta}\varepsilon\right) w(m) - (m-\varepsilon)w(m-\varepsilon) = \varepsilon\Delta F(m) + \frac{1-\beta}{\beta}\varepsilon r V^u.$$

In addition, for $m = \varepsilon$ and $m = m^*$, we have

$$\frac{\varepsilon}{\beta}w(\varepsilon) = \varepsilon\Delta F(\varepsilon) + \frac{1-\beta}{\beta}\varepsilon r V^u$$

$$\left(m^* + \frac{1-\beta}{\beta}\varepsilon\right) w(m^*) - (m^*-\varepsilon)w(m^*-\varepsilon) = \varepsilon\Delta F(m^*) + \frac{1-\beta}{\beta}\varepsilon r V^u + (q\Delta J(m^*) - \gamma)\,.$$

The recurrence relation can be solved forward inductively to give an expression for each $w(m)$, $m \in \varepsilon\mathbb{Z}_+$ as a weighted sum of the terms $\Delta F(\mu)$, $\mu \leq m$. For the special case in which the bargaining powers of firms and workers are symmetric, i.e., $\beta = 1/2$, this representation takes a particularly useful form, which is recorded in the following corollary:

[8]Suppose that the equilibrium does not take this form, in the sense that there exist m_1 and $m_2 > m_1$, such that $h(m) = 1$ for some $m > m_2$, but $h(m) = 0$ for $m \in [m_1, m_2]$. Let m^* be the smallest firm size such that $h(m^* + \varepsilon) = 0$. It is straightforward to verify that, when we view firm size as a Markov process, all states $m > m^*$ are transient. In particular, let $P^\tau(m, [\tilde{m}, \tilde{m}'])$ be the probability of reaching a state in the interval $[\tilde{m}, \tilde{m}']$ starting from state m in τ steps. Then from the accounting equations in (6), it is evident that $\lim_{\tau\to\infty} P^\tau(m, [\tilde{m}, \tilde{m}']) = 0$ for all $\tilde{m}' \geq \tilde{m} > m^*$ and for all m, since firms always lose workers at positive flow rates and will never grow beyond m^*. It follows that there is no loss of generality in restricting attention to threshold steady-state equilibria.

Corollary 2. *Suppose that $\beta = 1/2$. In an equilibrium with hiring cutoff m^*, then wages are given by*

$$w(m) = \begin{cases} \frac{1}{2} rV^u + \frac{1}{m}\left[F(m) - \frac{\varepsilon}{m+\varepsilon}\sum_{\mu=0}^{m/\varepsilon} F(\mu\varepsilon)\right] & m < m^* \\ \frac{1}{2} rV^u + \frac{1}{m}\left[F(m) - \frac{\varepsilon}{m+\varepsilon}\sum_{\mu=0}^{m/\varepsilon} F(\mu\varepsilon)\right] + \frac{m^*}{m(m+\varepsilon)}\left[q\Delta J(m^*) - \gamma\right] & m \geq m^*. \end{cases} \tag{13}$$

That is, wages at a firm with m workers are given by an average of the flow value of the worker's outside option, rV^u, and a term that measures the difference between the average labor productivity at the firm with m workers and a weighted average of the marginal product over the whole range of workers from the first up to the mth worker. The presence of this difference term reflects the essence of the Shapley value assumption, which recognizes the endogeneity of the outside option of the firm. In particular, if the firm were to disagree in its bargaining with the mth worker, this would increase the bargaining power of all the other workers and force the firm to pay higher wages to them. These wages would be related to these workers' marginal products after the mth worker has quit, thus, with this reasoning, the marginal product of each worker up to the mth features in the wage equation.

Equation (13) may become more intuitive when we rewrite the second term as follows:

$$\begin{aligned} \frac{1}{m}\left[F(m) - \frac{\varepsilon}{m+\varepsilon}\sum_{\mu=0}^{m/\varepsilon} F(\mu\varepsilon)\right] &= \frac{\varepsilon}{m\,(m+\varepsilon)}\sum_{\mu=0}^{m/\varepsilon}\left[F(m) - F(\mu\varepsilon)\right] \\ &= \frac{1}{m(m+\varepsilon)}\sum_{\mu=0}^{m/\varepsilon}\sum_{\nu=\mu+1}^{m/\varepsilon} \Delta F(\nu\varepsilon) \\ &= \frac{1}{m(m+\varepsilon)}\sum_{\nu=1}^{m/\varepsilon} \nu\Delta F(\nu\varepsilon). \end{aligned}$$

This expression makes it clear that wages paid by a firm employing m workers will depend most heavily on the marginal products that would apply if the firm were to employ slightly fewer than m workers, while marginal products when it employs fewer workers receive less weight in the wage formula. This is intuitive in light of the discussion above; disagreement with the mth worker will force the firm to pay higher wages to remaining workers commensurate with their marginal products and the threat points they would have in that case, which relate to the marginal products with just a few less workers.

Taking rV^u and $q(\theta)$, the optimal hiring behavior of firms and equilibrium wage bargains are straightforward to characterize as solutions to a set of linear equations or as a solution to a difference equation (see also Wolinsky, 2000). However, characterizing or even proving the existence of a general equilibrium is difficult. The reason for this can be seen in the equation for wage determination (13). Consider a cutoff equilibrium at m^*, and suppose that, given the values of rV^u and $q(\theta)$, a firm is indifferent between hiring an additional worker when it has $m^* - \varepsilon$ employees (thus $q\Delta(m^*) = \gamma$). Then for appropriately-chosen parameter values, there may also exist a cutoff equilibrium at $m^* - \varepsilon$. Now a small change in rV^u and $q(\theta)$ will induce a discontinuous change in the hiring behavior of firms. Thus generally the worker's value function $V(\cdot)$ does not vary continuously in $(q(\theta), rV^u)$, and therefore, neither does the right side of equation (5). This makes it difficult to establish the existence of a solution to the system of equations that characterize the equilibrium The same issues also lead to a large range of multiplicity of equilibrium for most parameterizations, since whether firms stop hiring at m^* or $m^* - \varepsilon$ can have a first-order effect on the vector $(q(\theta), rV^u)$, making multiple such vectors potentially part of steady-state equilibria. Both of these difficulties arise from the presence of workers with discrete size. Motivated by this reasoning, we next study the limiting economy as worker size ε becomes small.

3 Model With Continuous Employment

3.1 Basics

In this section, we consider the results of the model defined by taking the limit of the model introduced in the previous Section as $\varepsilon \to 0$. That is, we transform the number of workers employed by the firm from a discrete variable $m \in \varepsilon\mathbb{Z}_+$ to a continuous variable $n \in \mathbb{R}_+$. This has several technical advantages; in particular, the equations that define the solution are easier to work with. As an added advantage, this setting allows us to transform the difference equations that arise in the discrete-worker setting to differential equations, which have explicit form solutions.

More formally, we shall take the limit as $\varepsilon \to 0$ in equations (2), (4), (6), and (7) from the previous section. This essentially amounts to replacing the discrete first difference operator with a differential operator, while leaving the equations of the model otherwise unchanged. We therefore begin by introducing these close analogs of the basic equations of the model; the intuition is identical to that provided in the discussion in the previous section, and so discussion of the model setup will be brief here.

As in the previous section, we focus on anonymous steady-state equilibria and further limit attention to threshold equilibria. In particular, this implies that all firms will use the same equilibrium hiring strategy, which takes the form of a cutoff value n^* such that firms with fewer than n^* employees post a vacancy and those with more than n^* do not; i.e., $h(n^*) = 1$ if $n < n^*$

and $h(n^*) = 0$ if $n > n^*$. As we will see below, the appropriate limits also require firms with exactly n^* workers to post vacancies, thus in fact, $h(n^*) = 1$ if $n \leq n^*$. Finally, for natural reasons, we will focus on solutions in which the firm's value function $J(\cdot)$ is twice continuously differentiable in the number of employees the firm currently has, n (i.e., $J(\cdot) \in C^2(\mathbb{R}_+)$). The Bellman equations and the Shapley value bargaining equations (the differential analogs of (4) and (7) above) then require the wage and the worker's value function be once continuously differentiable in the number of employees at the firm (i.e., $V(\cdot) \in C(\mathbb{R}_+)$, $w(\cdot) \in C(\mathbb{R}_+)$). Below, we will prove the existence of a steady-state equilibrium with these features.

The setup is otherwise identical to that in the previous section.

Recalling equation (1) and taking the limit as $\varepsilon \to 0$ (under the assumption that differentiable solution $J(\cdot)$ exists) immediately yields

$$(r+\delta)\, J(n) = F(n) - nw(n) + \max\{0, -\gamma + q(\theta)J'(n)\} - snJ'(n). \tag{14}$$

Under the assumption that firms are using a hiring strategy with cutoff at n^*, this equation can be further simplified to

$$(r+\delta)\, J(n) = \begin{cases} F(n) - nw(n) - \gamma + (q(\theta) - sn)J'(n) & n \leq n^* \\ F(n) - nw(n) - snJ'(n) & n > n^* \end{cases} \tag{15}$$

Recall that $J(\cdot) \in C^2$. Consequently, optimality of the firm's hiring strategy at $n = n^*$ requires the following boundary condition to be satisfied:

$$-\gamma + q(\theta)J'(n^*) = 0. \tag{16}$$

This condition simply states that at n^* the firm is indeed indifferent between posting a vacancy and not. This boundary condition is not sufficient to characterize the solution to the differential equation (15). An additional boundary condition comes from a standard smooth pasting argument. Intuitively, notice that the firm is solving an optimal stopping problem—at what point to stop posting additional vacancies. Thus the necessity of a smooth pasting condition is not surprising. Intuitively, the smooth pasting condition requires that n^* is an optimal stopping point for the firm, in the sense that a small change in n^* should have no impact on the value of the firm. This is equivalent to the second-order condition

$$J''(n^*) = 0. \tag{17}$$

In addition, as in the discrete worker case of the previous section, we have a free entry condition, which takes the form

$$J(0) \leq k \text{ and } \theta \geq 0, \text{ with complementary slackness,} \tag{18}$$

which is identical to (9) in the previous section, and requires that in order for there to be positive activity in equilibrium, the value of k needs to be sufficiently low for firms to find it attractive to gain access to the production technology.

To make further progress, we need to use the Shapley bargaining formula, together with the worker's Bellman equation, to solve for the wage function $w(\cdot)$. Similarly to the argument for the firm's Bellman equation, taking the limit as $\varepsilon \to 0$ in the worker's Bellman equation (3) in the previous section yields:

$$(r+s+\delta)\,V\,(n) = \begin{cases} w\,(n) + [q(\theta) - sn]\,V'\,(n) + (s+\delta)\,V^u & n \leq n^* \\ w\,(n) - snV'\,(n) + (s+\delta)\,V^u & n > n^*. \end{cases} \tag{19}$$

By an analogous argument, the Shapley value equation is

$$(1-\beta)\,[V\,(n) - V^u] = \beta J'\,(n)\,. \tag{20}$$

To simplify the resulting differential equations and communicate the basic qualitative features of the model, let us now focus on the case where $\beta = 1/2$, so that the Shapley bargaining equation takes the simpler form

$$V\,(n) - V^u = J'\,(n)\,. \tag{21}$$

Moreover, for future use, let us differentiate this equation to observe that

$$V'\,(n) = J''\,(n)\,. \tag{22}$$

The equation for rV^u that arises from the Bellman equation for an unemployed worker closes the model. To write this equation, it is necessary to introduce additional notation for the steady-state distribution of firms as a function of the number of workers already employed by the firm; this differs slightly from the case with discrete worker size. As before, we will solve the model only in steady-state, so we do not need to index the distribution of firms by size according to time.

Denote the firm-size distribution by $G(n)$. First, suppose that there exists a steady-state for $G(\cdot)$ such that the density function $g\,(n)$ is continuous on $(0, n^*)$, together possibly with an atom of mass G^* at n^*. Such a distribution $G(\cdot)$ is a steady-state distribution if $g(\cdot)$ satisfies the steady state accounting equation equating, for each $n \in (0, n^*)$ and each $\varepsilon > 0$ sufficiently small, the flow of firms into and out of the interval $(n - \varepsilon/2, n + \varepsilon/2)$:

$$\left(\frac{q(\theta)}{\varepsilon} + \frac{sn}{\varepsilon} + \delta\right) g(n) = \frac{q(\theta)}{\varepsilon} g(n-\varepsilon) + \frac{s(n+\varepsilon)}{\varepsilon} g(n+\varepsilon) + O(\varepsilon). \tag{23}$$

Taking limits as $\varepsilon \to 0$, it follows that any differentiable solution to this equation must satisfy the following differential equation for $g(n)$:

$$\frac{g'(n)}{g(n)} = \frac{\delta - s}{sn - q}. \tag{24}$$

Integrating (24) gives that the general solution (for $q - sn \geq 0$) is given by

$$g(n) = A(q - sn)^{\frac{\delta - s}{s}}. \tag{25}$$

There are two cases to consider: the case where $n^* \geq q/s$ and the one where $n^* < q/s$. The importance of this distinction is that when $n = q/s$, then ignoring firm death, flows due to hiring and worker separation balance out; for n larger, there is a net loss of firms of size n even disregarding firm death. Thus the firm size distribution is supported on $[0, \min\{n^*, q/s\}]$. In the case of interest, in which $n^* < q/s$, then even at $n = n^*$, there is a positive "flow" of firms at $n - \varepsilon$ for any $\varepsilon > 0$; this leads to an atom at n^*. To calculate the size of this atom, again equate the flow of firms into and out of n^*, to obtain:

$$(q - sn^*)g(n^*) = \delta G^*. \tag{26}$$

Notice that (26) assumes the outflow of firms from the state n^* is given only by firm death. This is a consequence of the feature that $h(n^*) = sn^*/q$, i.e., firms at n^* still post vacancies and are hiring continuously, but only at a rate that allows them to counteract their loss of workers to the separation shock. This hiring is what maintains the atom at G^*. A more intuitive explanation for this is that at n^*, the firm is indifferent between posting a vacancy or not. If it posted a vacancy for sure when at n^*, so that $h(n^*) = 1$, it would quickly hire workers faster than it loses them to separation, since $q > sn^*$ by assumption; the firm would then move to having $n > n^*$ workers. However, as soon as this happens, it would then strictly prefer not to post a vacancy, and would therefore lose workers again until its workforce size falls to n^*. Conversely, if the firm does not post a vacancy at n^*, so that $h(n^*) = 0$, then it would quickly drop to $n < n^*$ and begin hiring again. The only possibility is that the firm must use a mixed strategy and post a vacancy with a flow probability of $h(n^*) = sn^*/q$; this ensures that the rate at which the firm attracts new workers, $h(n^*)q$, equals the rate at which it loses workers, sn^*.[9]

[9] In the discrete model with workers of size $\varepsilon > 0$ of the previous section, mixed strategies are not required, but an analogous result holds. Firms with n^* workers in this case generally strictly prefer not to post a vacancy, while firms with $n < n^*$ strictly prefer to do so. Thus a firm that has n^* workers will lose a worker with flow probability sn^*/ε, while a firm with $n^* - \varepsilon$ workers will gain an additional worker with probability q/ε and lose

Now combining (25) and (26), we obtain:

$$A(q - sn^*)^{\frac{\delta}{s}} = \delta G^*.$$

To solve for A, note that since $G(\cdot)$ is a probability distribution, we must have

$$G^* + \int_0^{n^*} g(n)\, dn = 1.$$

This implies $A = \delta q^{-\delta/s}$, so that the steady-state firm size distribution is given by

$$g(n) = \frac{\delta}{q}\left(1 - \frac{sn}{q}\right)^{\frac{\delta - s}{s}} = \frac{\delta}{q^{\frac{\delta}{s}}}\left(q - sn\right)^{\frac{\delta - s}{s}} \qquad \text{if } n \in [0, \min(n^*, \frac{q}{s}) \tag{27}$$

with an atom of mass

$$G^* \equiv \left(1 - \frac{sn^*}{q}\right)^{\frac{\delta}{s}} = \left(\frac{q - sn^*}{q}\right)^{\frac{\delta}{s}} \tag{28}$$

at n^* in the case where $n^* < \frac{q}{s}$.

Figure 1 shows this distribution for a case in which $sn^* < q$ and $\delta < s$, which is the empirically relevant case. The size G^* of the atom at n^* is not shown, although its location n^* is indicated with a vertical line.

Finally, observe that since the stochastic process for a firm's size satisfies an ergodicity condition, the steady state distribution $G(\cdot)$ is unique, so there was no loss of generality in solving only for a distribution in which $g(\cdot)$ is continuously differentiable on $(0, n^*)$.[10]

We can now write the Bellman equation for an unemployed worker. In the case where $q > sn^*$ (i.e., when there is an atom of positive mass of firms at n^*), recall that we have $h(n^*) = \frac{sn^*}{q}$; thus we have to incorporates the probability of an unemployed worker being hired by a firm of size n^*. Consequently, the Bellman equation for an unemployed worker is simply

$$rV^u = b + \theta q \left[-V^u + \frac{\int_0^{n^*} V(n) g(n)\, dn + \frac{sn^*}{q} V(n^*) G^*}{1 - \left(1 - \frac{sn^*}{q}\right) G^*} \right] \tag{29}$$

---

one with probability $sn^*/\varepsilon - 1$. Since as $\varepsilon \to 0$, $q/\varepsilon - [sn^*/\varepsilon - 1] \to \infty$, one can verify that a firm that has n^* workers will, in the future, have exactly n^* workers for a fraction $1 - sn^*/q$ of the time until it is destroyed, and exactly $n^* - \varepsilon$ workers for a fraction sn^*/q of this time. That is, the firm will post a vacancy a fraction sn^*/q of the time. Averaging over all firms with n^* workers establishes the result.

It is also useful to note the difference between the $\varepsilon > 0$ economy with workers on discrete size and the limit $\varepsilon \to 0$ economy in this respect. While in the economy with workers of discrete size, aggregate dynamics of the firm size distribution are deterministic, each individual firm's size follows a nontrivial stochastic process. In contrast, in the limit economy, the growth process of each firm is deterministic.

[10] For the ergodicity argument it is convenient to think of firm death as a shock that changes the size of a firm to 0, rather than causing entry of a new firm; in this case the uniqueness of the invariant distribution is immediate from Theorem 11.9 of Stokey and Lucas with Prescott (1989).

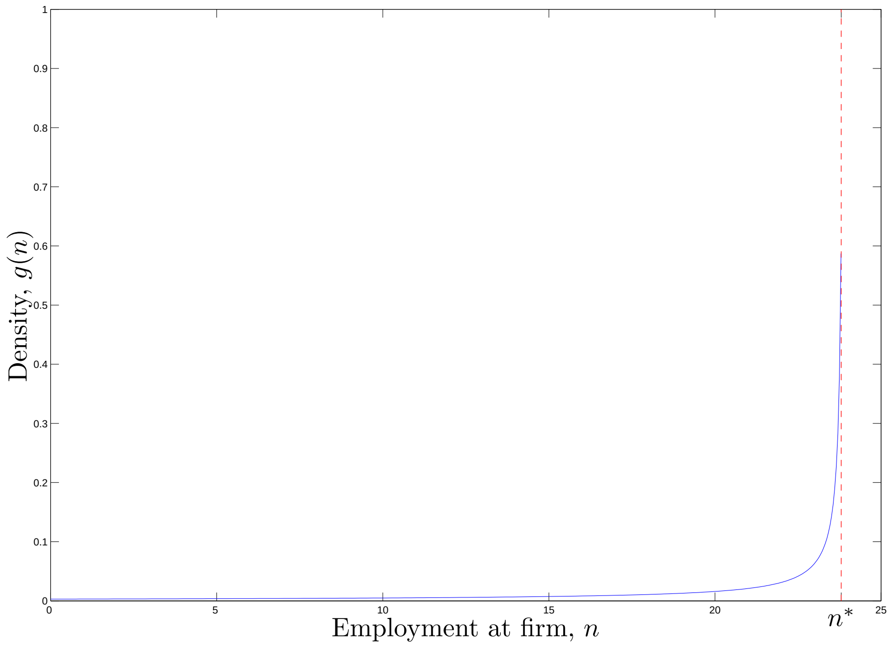

Figure 1: Steady state density function $g(n)$

The equations given above characterize almost completely any equilibrium of the model in the class we are considering (those that satisfy the differentiability assumptions and in which firms use symmetric cutoff hiring strategies). The only additional requirement is to check that firms and workers are behaving optimally.

This completes the characterization of an equilibrium, which we record as the following definition, analogous to Definition 2.

Definition 3. *A tuple* $\langle \theta, V^u, g(\cdot), J(\cdot,\cdot), V(\cdot,\cdot), h(\cdot,\cdot), w(\cdot,\cdot)\rangle$ *is an* anonymous (threshold) steady-state equilibrium *if*

- $J(m)$, $V(m)$, *and* $w(m)$ *satisfy (15), (19) and (20).*
- $g(m) \in \mathcal{G}$ *satisfies (27) and (28).*
- *the value of an unemployed worker,* V^u, *satisfies (29).*
- *there is optimal vacancy posting, i.e.,*

$$h(m) = \begin{cases} 1 & \text{if } -\gamma + q(\theta) J'(m) > 0 \\ 0 & \text{if } -\gamma + q(\theta) J'(m) < 0. \end{cases} \tag{30}$$

- *there is free entry in the sense that equation (18) holds.*

As in the previous section, the steady-state unemployment rate can be determined after the other endogenous variables, and is given by (10), making the unemployment rate, u, once again a monotonically decreasing function of the tightness of the labor market, θ.

3.2 Equilibrium Characterization

To begin analysis of such equilibria, first observe that not surprisingly, a continuous analog of equation (13) holds and gives a simple expression for the wage function $w(\cdot)$. To obtain this equation, substitute from the bargaining equations (20) and (22) into the worker's Bellman equation (19) and rearrange to observe that for $n \in (0, n^*)$,

$$(r + s + \delta)J'(n) - (q - sn)J''(n) = w - rV^u. \tag{31}$$

On the other hand, differentiating the firm's Bellman equation gives that for $n \in (0, n^*)$,

$$(r + s + \delta)J'(n) - (q - sn)J''(n) = F'(n) - w(n) - nw'(n). \tag{32}$$

Equating (31) and (32) shows that in any twice differentiable solution to the program above, it must be that for $n \in (0, n^*)$,

$$2w(n) + nw'(n) = F'(n) + rV^u. \tag{33}$$

A similar argument shows that this equation also holds for $n > n^*$, though this is less important for our analysis. Integrating (33) by parts gives:

$$w(n) = \frac{1}{2}rV^u + \frac{1}{n}\left[F(n) - \frac{1}{n}\int_0^n F(m)\,dm\right] + \frac{K}{n^2},$$

where K is a constant of integration. Next note that if $K \neq 0$, then as $n \to 0$, the firm's total wage bill satisfies $nw(n) = O(1/n)$. For finite q, this implies that the firm's value function can be written as

$$J(0) = E\int_0^\infty e^{-(r+\delta)t}\left[F(n(t)) - n(t)\,w(n(t))\right]\,dt,$$

where $n(t)$ denotes the number of employees at time t. Evidently, $J(0)$ is equal to $+\infty$ if $K < 0$ and to $-\infty$ if $K > 0$. Therefore, the constant of integration must be equal to zero, i.e., $K = 0$. This argument establishes the following lemma.

Lemma 2. *In a steady-state equilibrium, wages satisfy*

$$w(n) = \frac{1}{2}rV^u + \frac{1}{n}\left[F(n) - \frac{1}{n}\int_0^n F(m)\,dm\right] \qquad (\forall n > 0). \tag{34}$$

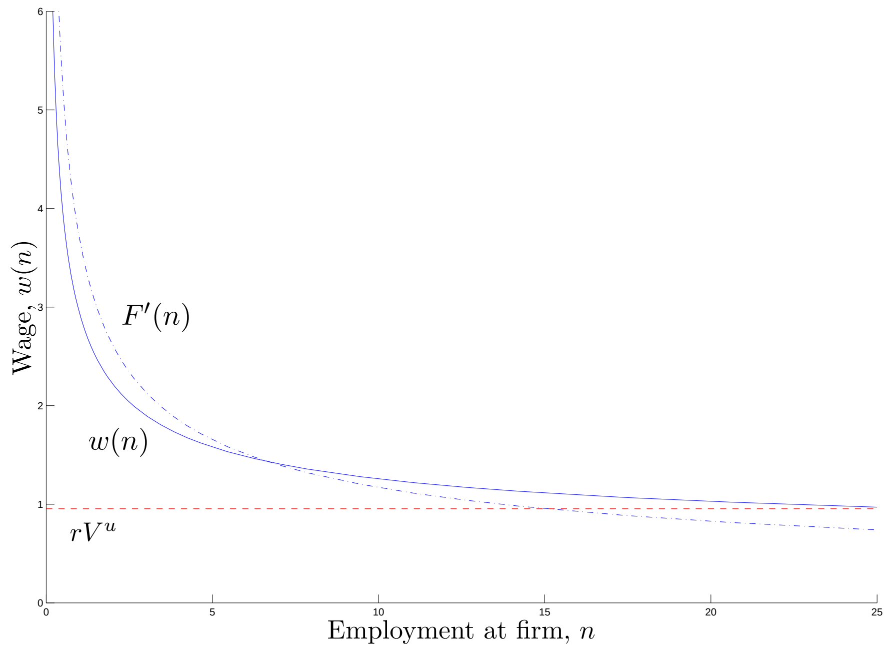

Figure 2: Wage, marginal product, and outside option

This lemma shows that despite the additional general equilibrium interactions, wages in this model take a form identical to those in Stole and Zwiebel (1996a,b) and Wolinsky (2000). This explicit form characterization of wages will be important in further characterization and proving the existence of a steady-state equilibrium. A graphical representation of the dependence of wages on the number of workers employed at the firm is indicated in Figure 2. (This figure shows the wage function arising in the calibrated example of a Cobb-Douglas production function used in Section 4.2.) Also shown are a horizontal line indicating the flow value of the unemployed, rV^u, and the marginal product function, $F'(n)$.

Our next results show that, as depicted in Figure 2, wages and flow profits satisfy convenient boundary conditions.

Lemma 3. *In a steady-state equilibrium, wages are strictly positive, strictly decreasing with firm size, and, satisfy*

$$\lim_{n\to 0^+} w(n) = +\infty \text{ and } \lim_{n\to\infty} w(n) = \frac{1}{2} rV^u.$$

Moreover, the flow profit $\pi(n) = F(n) - nw(n)$ of the firm is maximized at some $n \in [0, \infty)$ and satisfies

$$\lim_{n\to 0^+} \pi(n) = 0 \text{ and } \lim_{n\to\infty} \pi(n) = -\infty.$$

Proof. See Appendix. □

The fact that wages at very small firms become very large arises from the Inada condition on the firm's production function, since the marginal product also becomes arbitrarily large as n decreases to 0.

The closed form equation for wages also allows the value functions for firms to be derived in closed form as shown in the following lemma.

Lemma 4. *In a steady-state equilibrium with labor market tightness θ, the firm's value function $J(\cdot)$ satisfies*

$$J(n) = (q(\theta) - sn)^{-\frac{r+\delta}{s}} \left[q(\theta)^{\frac{r+\delta}{s}} k + \int_0^n (q - sn)^{\frac{r+\delta}{s} - 1} \left(\gamma + \frac{m}{2} r V^u - \frac{1}{m} \int_0^m F(\nu) \, d\nu \right) dm \right] \tag{35}$$

for all $0 < n < n^*$.

Proof. See Appendix. □

While the closed form solution for the firm's value function looks complicated, it has a relatively simple structure and enables us to further characterize the form of the equilibrium. In particular, differentiating this value function, $J(\cdot)$, with respect to n allows the boundary conditions at n^* also to be written in closed form. More specifically, equation (16) becomes

$$J(n^*) = \frac{1}{r+\delta} \left[-\frac{n^*}{2} r V^u - \frac{s n^* \gamma}{q(\theta)} + \frac{1}{n^*} \int_0^{n^*} F(m) \, dm \right], \tag{36}$$

while the smooth pasting condition (17) becomes

$$\frac{1}{n^*} \left[F(n^*) - \frac{1}{n^*} \int_0^{n^*} F(m) \, dm \right] = \frac{1}{2} r V^u + \frac{(r+\delta+s)\gamma}{q(\theta)}. \tag{37}$$

It is interesting to observe that this latter can also be expressed as

$$w(n^*) = r V^u + \frac{(r+\delta+s)\gamma}{q(\theta)}. \tag{38}$$

Equation (38) is very intuitive and states that the firm continues to hire until the wage it pays equals the outside option of the worker, rV^u, plus a term that is proportional to the severity of the labor market friction (parameterized by the flow cost of posting an application, γ, divided by the productivity of that posting, $1/q(\theta)$). This wage equation is therefore comparable to the result obtained in a static setting by Stole and Zwiebel (1996a) (see their Corollary 1 on page 396) and generalized to a dynamic setting by Wolinsky (2000). In these previous analyses, since there is no hiring margin (and no frictions), the second term is absent. Consequently, those models always imply "over-hiring" relative to a hypothetical competitive benchmark; firms will hire more than this competitive benchmark in order to reduce the marginal product of workers and thus their bargaining power according to the Shapley bargaining protocol (see Stole and Zwiebel, 1996a). Our analysis shows that this over-hiring result may or may not apply in general equilibrium; when γ is small, it will, but it could fail to do so when γ is large and $q(\theta)$ is relatively small.

Substituting the closed form expressions for $J(n^*)$ given by (36) into the formula for $J(\cdot)$ given by (35) and rearranging gives an expression which will be useful in characterizing equilibria.

$$\begin{aligned} k = J(0) = J(n^*) \left(\frac{q(\theta) - sn^*}{q(\theta)} \right)^{\frac{r+\delta}{s}} \\ - q(\theta)^{-\frac{r+\delta}{s}} \int_0^{n^*} (q(\theta) - sm)^{\frac{r+\delta}{s}-1} \left[\gamma + \frac{m}{2} rV^u - \frac{1}{m} \int_0^m F(\nu)\, d\nu \right] dm. \end{aligned} \tag{39}$$

Equation (38), together with the worker's Bellman equation (19) and the closed form equation for wages given by (34), allow the worker's Bellman equation to be expressed more simply also.

Lemma 5. *For $0 < n < n^*$, the worker's value function $V(\cdot)$ satisfies*

$$\begin{aligned} V(n) = V^u + \frac{\gamma}{q(\theta)} \left(\frac{q(\theta) - sn^*}{q(\theta) - sn} \right)^{\frac{r+\delta+s}{s}} \\ + (q(\theta) - sn)^{-\frac{r+\delta+s}{s}} \int_n^{n^*} (q(\theta) - sm)^{\frac{r+\delta}{s}} w(m)\, dm. \end{aligned} \tag{40}$$

Equivalently,

$$\begin{aligned} V(n) = V^u + \frac{\gamma}{q(\theta)} \left(\frac{q(\theta) - sn^*}{q(\theta) - sn} \right)^{\frac{r+\delta+s}{s}} + \frac{1 - \left(\frac{q(\theta)-sn^*}{q(\theta)-sn} \right)^{\frac{r+\delta+s}{s}}}{2(r+\delta+s)} rV^u \\ + (q(\theta) - sn)^{-\frac{r+\delta+s}{s}} \int_n^{n^*} (q(\theta) - sm)^{\frac{r+\delta}{s}} \psi(m)\, dm. \end{aligned} \tag{41}$$

where

$$\psi(n) \equiv \frac{1}{n} \left[F(n) - \frac{1}{n} \int_0^n F(m)\, dm \right].$$

Proof. See Appendix. ☐

These equations now enable us to represent a steady-state equilibrium as the intersection of two curves in $(q(\theta), rV^u)$-space. In particular, suppose we know that in some equilibrium, the rate at which firms meet workers and the flow value of an unemployed worker equal $q(\theta)$ and rV^u respectively. Then firms take as given the path of wages that they will have to pay, $w(\cdot)$, given by (34). This means that their decision as to when to stop hiring, n^*, is given as the solution to an optimal stopping problem; n^* is then determined as the solution to (37), and the value of $J(n^*)$ can be deduced from (36). Denote this value by $n^*(q, rV^u)$. Next, solving the differential equation (35) with initial condition $(n^*(q, rV^u), J(n^*(q, rV^u)))$ allows us to solve for $J(0)$. The resulting expression is given in closed form by the right side of (39). If the free entry condition is satisfied, then the value of $J(0)$ so derived must equal the capital cost of entry k to ensure that (39) is satisfied. This provides one condition for (q, rV^u) to be part of an equilibrium. The remaining condition for (q, rV^u) to be part of an equilibrium is $(q, rV^u, n^*(q, rV^u))$ must satisfy (29), the Bellman equation of an unemployed worker. It remains to check that firm and worker behavior is optimal; for firms this is true by construction of $n^*(q, rV^u)$, while for workers, this follows since according to Lemma 3 and equation (38), the wage at any firm that is represented in equilibrium is strictly greater than rV^u, so that it is always optimal to accept any job offer. We record this conclusion as the following Proposition.

Proposition 1. *Let $q(\theta) > 0$ and $rV^u > 0$ be given. Then there is a steady-state equilibrium allocation with queue length $q(\theta)$ and value of an unemployed worker given by rV^u if and only if* (29) *and* (39) *are satisfied with n^* defined as the unique solution to* (37) *and $J(n^*)$ given by* (36).

Proof. Most of the proof is given in the discussion preceding the statement of the proposition. The uniqueness of $n^*(q, rV^u)$ follows from the proof of Lemma 3, together with (37). □

We are now in a position to prove an existence theorem.

Theorem 1. *An steady-state equilibrium with cutoff hiring strategies exists.*

Proof. See Appendix. □

The proof of the Theorem consists of showing that there exist (q, rV^u) satisfying the hypothesis of Proposition 1. Here, we present a diagrammatic exposition, emphasizing the intuition. The proof of Theorem 1 establishes that an equilibrium with positive activity exists if

$$k < \max_{n>0} \left\{ \frac{1}{n} \int_0^n F(m)\, dm - nb \right\},$$

where the existence of the maximum on the right side follows from Lemma 3. In this case, Figure 3 shows a diagram depicting in (q, rV^u)-space the two curves described in the discussion

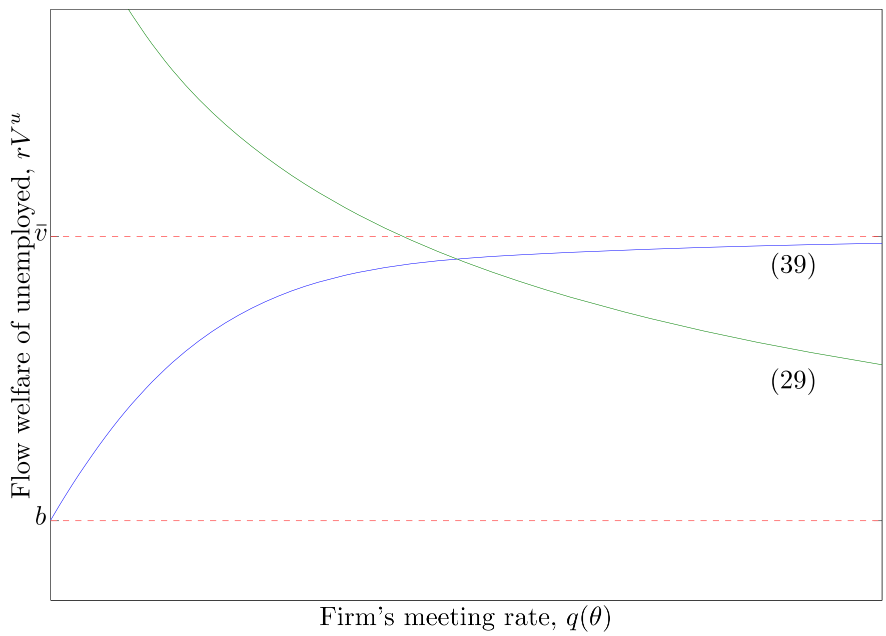

Figure 3: Equilibrium determination

preceding Proposition 1. The upward-sloping curve is the free-entry condition of firms, equation (39); it is upward-sloping since, all else equal, an increase in rV^u, must be compensated by an increase in q, which makes entry into the labor market more profitable for firms. This is because a higher rV^u translates into higher wages, so that the profit margins of firms decline. Zero profits can only be ensured by leaving vacancies unfilled for shorter durations, thus by an increase in q. The downward-sloping curve is the Bellman equation for unemployed workers. It is downward-sloping since an increase in rV^u on the right side of (29) corresponds to an increase in wages; to keep the flow value of an unemployed worker satisfying this equation, it must be that hiring is more rapid (that is, q is larger), so that when hired, the worker spends less time earning the high wage he receives when his firm is smaller.

Comparative statics of the response of the endogenous variables $q(\theta)$ and rV^u can now be obtained from the diagrammatic representation of the equilibrium. While general conclusions are difficult to draw, the general features of the comparative statics are quite clear. The movements of the free-entry condition, (39), are generally unambiguous. For example, in response to an increase in productivity, it moves upwards. This is because for a given (q, rV^u), increased productivity increases flow profits for all firms, and so increases the implied value of entry, $J(0)$; to keep the free entry condition satisfied, rV^u must increase for each q. Similarly, the free-entry condition, (39) moves downwards in response to an increase in k. Since a change in k does not affect the other curve, it has unambiguous effects on the steady-state equilibrium in the situation depicted in the diagram in which the worker's Bellman equation is downward-sloping; an increase in k reduces rV^u and increases $q(\theta)$. This also corresponds to a decline in θ and therefore, from (10), to an increase in the steady-state unemployment rate. In all calibrated examples we have investigated, the worker's Bellman equation has indeed been downward-sloping. We therefore conjecture that an increase in the cost of entry unambiguously reduces the tightness of the labor market, θ, and increases steady-state unemployment, u, but we do not at present have a proof of this assertion.

The impact of a productivity shock on equilibrium variables, on the other hand, is ambiguous because productivity shocks have a potentially ambiguous effect on the other curve. This is because the impact of a productivity increase on the optimal employment level of firms, $n^*(q, rV^u)$, is ambiguous. For a given (q, rV^u), the wages paid at a firm with any fixed number n of workers, $w(n)$, increase; however, the increase in n^* means that more workers are employed at larger firms, which, all else equal, pay lower wages. In calibrated examples, the first effect tends to dominate, so that the curve moves upwards. An example where this is the case is shown in Figure 4. The dashed lines indicate the movement of the curves after a Hicks-neutral increase in productivity. In this case, the utility of workers increases unambiguously, but the response of the equilibrium job-finding rate for firms, $q(\theta)$, is ambiguous. Nevertheless, in many calibrated examples, including the example shown in Figure 4, $q(\theta)$ decreases in response to the increase in productivity, so that workers' job-finding rate rises and steady-state unemployment falls. Another interesting feature of this example is that n^* also falls in response to the positive productivity shock. This implies that in the new steady state firms are, on average, smaller. Consequently, much of the adjustment to the new steady-state takes place *at the extensive margin*, that is, by the entry of new firms, while existing firms in fact decline in size. This is a pattern we find consistently in the calibrations, and underlines the importance of considering separating the intensive and extensive margins of employment creation.

Since the implications of an increase in productivity are potentially ambiguous, we therefore now investigate the response of unemployment, wages, and the vacancy-to-unemployment ratio to productivity shocks by using a simple calibrated version of the model.

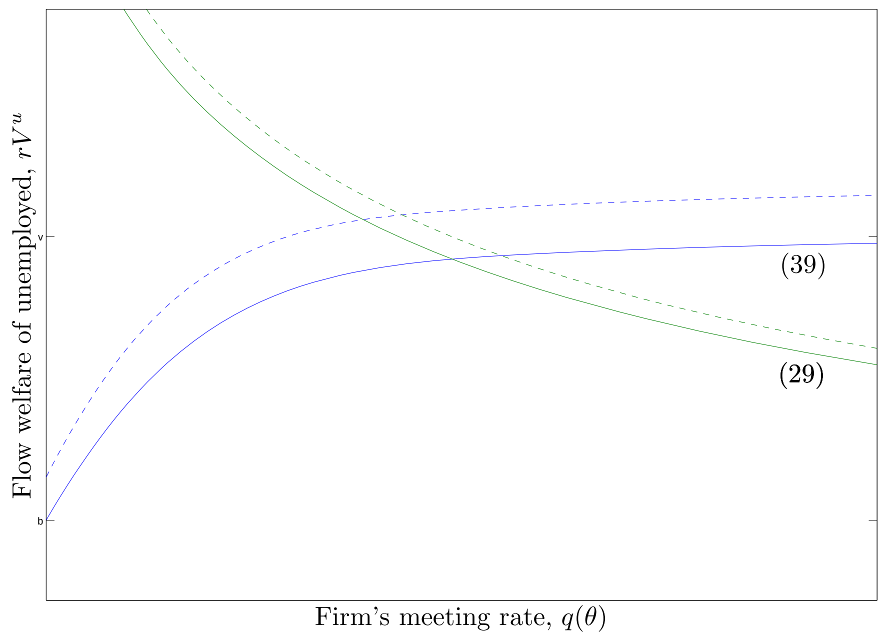

Figure 4: Response to a positive productivity shock

4 Response to Productivity Shocks: A Calibration

4.1 Motivation

In this section, we examine the effect of productivity shocks on steady-state unemployment, wage and vacancy-unemployment rates. As discussed in the previous section, while the basic trade-offs are apparent from our graphical analysis, both the direction and the magnitude of the effects are not always determinate. This motivates our more detailed analysis of a simple calibrated version of the above economy.

Another motivation for this exercise is the recent criticisms of the DMP model mentioned in the Introduction. Shimer (2005), for example, shows that a reasonably calibrated version of the DMP model also fails to generate large responses of employment (or unemployment, or the vacancy-unemployment ratio) to changes in firm productivity. Shimer (2005) and Hall (2005) document that this limited response of employment in the DMP model has its roots in the significant wage responses to changes in firm productivity. This intuition suggests that a model along the lines of the one presented in this paper might potentially generate different qualitative and quantitative patterns in response the productivity shocks. The reason for this is that, as our analysis illustrates, wages are no longer tied to average product, but are given by a weighted average of average and marginal products. In particular, recall that wages in the discrete worker version of our economy are given by

$$w(m) = \frac{1}{2} r V^u + \frac{1}{m(m+\varepsilon)} \sum_{\nu=1}^{m/\varepsilon} \nu \Delta F(\nu\varepsilon). \tag{42}$$

While the first-term is the outside option of the worker, the second term consists of a weighted average of the marginal product of different workers.

To see why a search model with wage determination given by (42) may potentially differ both qualitatively and quantitatively from the DMP model, let us first recall that in the baseline DMP model (with $\beta = 1/2$ for the sake of comparison to (42)), wages are given by

$$w = \frac{1}{2} r V^u + \frac{1}{2} y. \tag{43}$$

Here rV^u is the flow outside option of the worker (i.e., what he or she will receive per period when unemployed; see, for example, Pissarides, 2001) and y is the productivity, or equivalently, average product of the worker. An increase in output (due to any source of productivity increase) corresponds to a rise in y, which immediately translates into an increase in w. Moreover, this will typically increase the value of an unemployed worker through two channels: first, when the worker becomes employed, he or she will receive higher wages (as long as the increase in y is not transitory); and second, the rate at which workers leave unemployment will also increase. Consequently, equation (43) implies a fairly elastic response of wages to changes in productivity. For example, in Shimer's (2005) baseline simulation the wage rate increases by around 1% in response to a 1% change in labor productivity. As we have seen in our baseline model, job creation, in turn, is a function of revenue net of wages depending on the form of the matching function as captured by $q(\theta)$. In a reduced form, we can write:

$$\begin{aligned} \text{Job Creation} &= \kappa (y - w) \\ &= \kappa \left(\left(y - rJ^U \right) / 2 \right), \end{aligned}$$

where $\kappa(\cdot)$ is naturally strictly increasing. This structure implies that changes in y will have a limited impact on $y - w$, and therefore only a dampened effect on job creation.

Now let us turn to our environment and approximate the wage equation (42), with the following weighted average of average and marginal product,

$$w = \frac{1}{2} r V^u + \frac{1}{2} \left[\eta \frac{F(m \mid B)}{m} + (1-\eta) F'(m \mid B) \right], \tag{44}$$

where $F(m \mid B)$ is the production function, with B as a shift parameter, and η is an endogenously determined weight.

Now consider the widely-used constant elasticity of substitution production function for F, with B interpreted as a labor-augmenting productivity B:

$$F(m \mid B) = \left[(1-\phi) K^{\frac{\sigma-1}{\sigma}} + \phi (Bm)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

where again $\phi \in (0,1)$, K is fixed and $\sigma \in [0,\infty]$ is the elasticity of substitution between labor and K. When $\sigma = \infty$, there are no diminishing returns to labor, when $\sigma = 0$, the production function is Leontief, and when $\sigma = 1$, the production function is equivalent to the Cobb-Douglas function above. Interpreting K as capital, existing evidence suggests that an elasticity of substitution $\sigma \in (0,1)$ might be most reasonable (see, e.g., the summary of the evidence in Acemoglu, 2003). With this production function, (44) implies:

$$w(m) = \frac{1}{2} r V^U + \frac{1}{2}\left[\eta \frac{y}{m} + (1-\eta)\phi B^{\frac{\sigma-1}{\sigma}} m^{-\frac{1}{\sigma}} y^{\frac{1}{\sigma}}\right],$$

where $y = F(m \mid B)$ is the output level. Rearranging this equation in supposing that $m = 1$, we obtain:

$$w(m=1) = \frac{1}{2} r V^U + \frac{1}{2} y - \frac{1}{2}(1-\eta)(1-\phi) K^{\frac{\sigma-1}{\sigma}} y^{\frac{1}{\sigma}}. \tag{45}$$

For illustration purposes, suppose that m remains constant at 1 in response to a productivity shock. Then it can be verified that if η is sufficiently small and the marginal product changes less than average product, the change in $w(m)$ can be much less than in the baseline Nash bargaining equation (43). For example, in the extreme case where $\sigma \to 0$, the production function becomes Leontief, and if $Bm > K$, then the marginal product of the mth worker does not respond at all to an increase in productivity. Correspondingly, firm revenues net of wages can increase much more, raising job creation by more than the standard DMP model.

However, this reasoning is predicated on η being small, which is itself an endogenous variable. As we have seen above, wages depend on the distribution of firm sizes in the economy and on the exact form of the production function (since they are given by a weighted average of the marginal product of all workers up to the current level of employment m). Therefore, whether the response of wages to productivity shocks will indeed differ markedly in this model compared to the response in a model in which an equation similar to (43) occurs depends on the entire structure of the equilibrium. We next turn to investigate this question and, somewhat surprisingly, find that the response of wages to protect due to shocks in this generalized search model is quite similar to that in the baseline DMP model.

4.2 Calibration

Although our objective is to illustrate the potential wage and employment responses resulting from the generalized search model presented here, we choose parameters to approximate the labor market equilibrium in U.S. data. For comparability with the benchmark paper in this literature, we use the calibrations of Shimer (2005) where possible. Following Shimer, we first normalize the mean wage paid to workers to be 1, and set the flow utility enjoyed by the unemployed, b, to be equal to 0.4. Next, we choose the unit of time to be equal to a quarter, and therefore can again borrow Shimer's calibrations by setting the interest rate r equal to 0.012. We also follow Shimer and Petrongolo and Pissarides (2001) in assuming that the aggregate matching function $M(u,v)$ takes a Cobb-Douglas form,

$$M(u,v) = Zu^{\eta}v^{1-\eta},$$

with $\eta = 0.72$ and Z chosen so that the average job-finding rate for the unemployed, $\theta q(\theta)$, is 1.355. In addition, we take from Shimer's paper (along with Davis, Haltiwanger, and Schuh, 1996) that the gross rate of job destruction, $s + \delta$ should equal 0.10. Finally, we arbitrarily set the flow cost of posting a vacancy as $\gamma = 0.5$, half the average wage; the results are not sensitive to this choice.

We also need to match basic facts about the firm size distribution. The average employment of U.S. firms (both publicly- and privately-held) is reported by Davis, Haltiwanger, Jarmin and Miranda (2006) as 23.8, and we calibrate so that the mean firm size matches this figure. In addition, we choose to set the ratio of separations due to plant closure to those equal to 5, so that $\frac{s}{23.8\delta} = 5$; this roughly matches data for U.S. manufacturing plants from Davis, Haltiwanger, and Schuh (1996).[11]

We study four calibrations of the model, one Cobb-Douglas function $F(n) = An^{\alpha}$, and three CES functions of the form $F(n) = A\left[(1-\phi)\left(B_K K\right)^{\frac{\sigma-1}{\sigma}} + \phi(B_L L)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$, where K is the capital stock of a firm, which will be normalized to 1.

In each production function, one parameter is pinned down by the requirement that the average wage be equal to 1; the remaining parameters are chosen arbitrarily. The parameterization is recorded in Table 1, along with the levels in steady-state equilibrium of the basic endogenous variables of the model. Note that in each case, the production function is calibrated so that the average employment per firm is 23.80, and the average wage is 1.00.

The results are reported in Table 2. The numbers shown in the table are elasticities of the response of the variables shown in the first row of each column to a productivity shock. In the case of the Cobb-Douglas production function, this shock arises from a change in the

[11] Davis, Haltiwanger, and Schuh (1996) note in Figure 2.3, page 29, that if one looks at quarterly data, 11.6% of job destruction occurs in plant shutdowns; in annual data, 22.9% of job destruction occurs in plant shutdowns; the reasons for the discrepancy are that closure takes time and that transitory plant-level employment changes are more important in higher frequency data (fn. 9, p. 27). We therefore choose an intermediate value: our calculation implies that one-sixth, or 16.7%, of job destruction occurs because of firm destruction.

Production function	A	α	σ	ϕ	B_K	B_L	n^*	$\bar{F}$	$\bar{p}$	u	v/u
Cobb-Douglas	7.43	0.50	na	na	na	na	23.88	36.22	1.52	0.069	0.027
CES ($\sigma = 1.3$ $B_L = 0.06$)	31.01	na	1.30	0.50	1.00	0.06	23.88	37.18	1.56	0.069	0.027
CES ($\sigma = 0.5$ $B_L = 0.06$)	28.89	na	0.50	0.50	1.00	0.06	23.88	33.95	1.43	0.069	0.027
CES ($\sigma = 0.1$ $B_L = 0.06$)	31.31	na	0.10	0.50	1.00	0.06	23.96	33.61	1.41	0.069	0.027
CES ($\sigma = 0.5$ $B_L = 0.6$)	47.70	na	0.50	0.50	1.00	0.60	23.88	89.11	3.74	0.069	0.026
CES ($\sigma = 0.5$ $B_L = 2.0$)	96.20	na	0.50	0.50	1.00	2.00	23.87	188.38	7.92	0.069	0.026

Table 1: Parameters and equilibrium values of key endogenous variables

neutral productivity parameter A. In the case of the CES production functions, the shock can be neutral (arising from a change in A), capital-specific (arising from a change in B_K), or labor-specific (arising from a change in B_L). The two panels of the table differ with respect to how the change in productivity is calculated. In the first panel, we calculate the average output per employed worker in the economy in the steady-states before and after the change in productivity, and divide the proportional changes in the variables of interest (the unemployment rate, the vacancy-to-unemployment ratio, the average firm size, and the average wage) by the proportional change in productivity so measured. In the second panel, we calculate the change in productivity at a firm of fixed size equal to 23.8 workers and perform the analogous calculation; thus, the first panel incorporates changes in productivity arising from worker redistribution across firms between steady states, while the second panel does not.

A number of interesting features are worth noting. First, there are marked differences depending on whether we measure productivity by average product in the economy or for a firm of fixed size. In the former case, reported in the first four columns of Table 2, the quantitative impact of increase in productivity on unemployment and the vacancy unemployment ratio are very similar to the benchmark DMP model shown in the first row. In the latter case, reported in the last four columns of the table, the effects are potentially larger. This suggests that short-run and long-run responses to productivity shocks could be quite different. However, further analysis of this issue requires us to solve for non-steady-state dynamics, which we plan to do in future work. Second, when productivity is measured for fixed firm size, the exact form of technology shocks has a major impact on the response of unemployment and wages. This poses a further range of questions again related to short-run and long-run responses. Third,

		$G(\cdot)$ adjusts				Fixed firm size			
Production fn.	Shock	u	v/u	n^*	w	u	v/u	n^*	w
Benchmark DMP		-0.45	1.35	na	1.00	-0.45	1.35	na	1.00
Cobb-Douglas	Neutral	-0.38	1.08	-0.98	1.00	-0.76	2.18	-1.97	2.01
CES ($\sigma = 1.3$)	Neutral	-0.42	1.15	-1.15	0.90	-0.97	2.72	-2.69	2.09
	K-spec.	-0.44	1.28	-1.30	0.82	-1.23	3.54	-3.58	2.26
	L-spec.	-0.38	1.08	-0.99	1.00	-0.73	2.09	-1.90	1.92
CES ($\sigma = 0.5$)	Neutral	-0.29	0.83	-0.60	1.16	-0.45	1.29	-0.94	1.81
	K-spec.	-0.13	0.37	0.09	1.45	-0.12	0.35	0.09	1.37
	L-spec.	-0.38	1.09	-0.99	1.00	-0.92	2.65	-2.41	2.43
CES ($\sigma = 0.1$)	Neutral	-0.22	0.61	-0.29	1.28	-0.30	0.86	-0.40	1.77
	K-spec.	0.21	-0.59	1.57	2.01	0.08	-0.24	0.62	0.80
	L-spec.	-0.38	1.09	-0.99	1.00	-10.25	29.29	-26.64	26.85
CES	Neutral	-0.37	1.09	-0.79	1.49	-1.60	4.67	-3.36	6.36
($\sigma = 0.5,\ B_L = 0.6$)	K-spec.	-0.38	1.10	-0.74	1.64	-1.31	3.81	-2.57	5.68
	L-spec.	-0.38	1.09	-0.99	1.00	-5.88	16.80	-15.21	15.33
CES	Neutral	-0.40	1.21	-0.83	1.66	-3.55	10.78	-7.42	14.78
($\sigma = 0.5,\ B_L = 2.0$)	K-spec.	-0.41	1.22	-0.83	1.74	-3.26	9.82	-6.64	13.94
	L-spec.	-0.39	1.10	-1.00	1.00	-18.76	53.28	-48.34	48.29

Table 2: Elasticity of responses to productivity shocks

increases in productivity typically reduce n^*. This suggests that much of the adjustment to a change in productivities takes place at the *extensive margin*, by entry of new firms rather than hiring by continuing firms. This is an important implication, and may again differ between the short-run and the long-run.

5 Conclusion

This paper presented a generalization of the standard Diamond-Mortensen-Pissarides (DMP) search model of unemployment, in which there are both intensive and extensive margins of employment creation. In our model, each firm can invest to gain access to a production technology with diminishing returns to labor and then post vacancies in order to recruit workers. Entry by new firms corresponds to the extensive margin of employment creation, while job creation by existing firms captures the intensive margin. As in the baseline Diamond-Mortensen-Pissarides search model and theories of the firm developed by Stole and Zwiebel (1996a,b) and Wolinsky (2000), wages are determined by continuous bargaining between the firm and its employees.

We first characterized certain properties of the equilibrium in a baseline model where each firm can hire a countable number of workers. Although this model is a natural generalization of the standard DMP model, the fact that the choice variable of each firm is a discrete variable makes its analysis difficult. We therefore provided a more complete characterization of equilibrium and the proof of existence of a steady-state equilibrium in the limit economy, where the size of each worker becomes infinitesimally small. Another advantage of this limit economy is that the steady-state equilibrium can be characterized by a set of differential equations, which admits closed-form solutions. Consequently, despite the presence of general equilibrium interactions and forward-looking bargaining between the firm and multiple workers, the equilibrium takes a relatively tractable form.

After providing some simple comparative static results, we also undertook a simple calibration of our baseline economy to investigate whether the response of unemployment and wages is qualitatively and quantitatively different in this model than in the baseline DMP model. Since wages in our model are a weighted average of marginal and average products of different workers, such a difference is theoretically possible. Nevertheless, our calibrations indicate that the quantitative implications of the generalized search model may be quite similar to the baseline DMP model. Whether they are so or not depends on how productivity is measured (a question that arises since labor productivity differs across firms which have different sizes). When we look at productivity measured for fixed firm size, the quantitative magnitudes are larger and also more responsive to changes in the form of the production function. This suggests that short-run and long-run responses to various shocks may differ in this class of models. We intend to investigate this issue in the future work by undertaking a systematic analysis of non-steady-state behavior. In addition, our results suggest that the extensive margin may be quite important in adjusting to new steady states. Whether the extensive margin is also equally important in the short run is another question we will investigate in the future.

6 Appendix

Proof of Lemma 3. Define

$$\psi(n) = \frac{1}{n}\left[F(n) - \frac{1}{n}\int_0^n F(m)\,dm\right],$$

then $\psi(n) > 0$, $\psi'(n) < 0$, and $\lim_{n\to 0^+}\psi(n) = +\infty$ and $\lim_{n\to\infty}\psi(n) = 0$. To see these facts, observe first that $n^2\psi(n) = \int_0^n [F(n) - F(m)]\,dm > 0$ since F is strictly increasing. Next, observe that

$$\frac{n^3}{2}\psi'(n) = \int_0^n F(n)\,dn - nF(n) + \frac{n^2}{2}F'(n),$$

whereas from a third-order Taylor expansion for $n \mapsto \int_0^n F(m)\,dm$ at 0 about n, we have that for some $\nu \in (0, n)$,

$$0 = \int_0^n F(n)\,dn - nF(n) + \frac{n^2}{2}F'(n) - \frac{n^3}{6}F''(\nu);$$

since $F''(\nu) < 0$, it follows that $\psi'(n) < 0$ for all $n > 0$. Next, a similar second-order Taylor expansion implies that for all $n > 0$,

$$\psi(n) = \frac{1}{2}F'(\nu)$$

for some $\nu \in (0, n)$; it follows from the Inada condition that $\lim_{n\to 0^+}\psi(n) = +\infty$. Finally, that $\lim_{n\to\infty}\psi(n) = 0$ follows from observing that $0 < \psi(n) < \frac{1}{n}F(n) \to 0$ as $n \to \infty$, together with the squeeze principle.

The results concerning the profit function follow immediately from observing that

$$\pi(n) = \frac{1}{n}\int_0^n F(m)\,dm - \frac{n}{2}rV^u.$$

It follows that $\pi'(n) = \psi(n) - \frac{1}{2}rV^u$, from which it follows immediately that π'' is strictly concave. The remaining claims follow from substituting $n = 0$ in the formula for $\pi(\cdot)$, and from observing that

$$\lim_{n\to\infty}\pi'(n) = \lim_{n\to\infty}\psi(n) - \frac{1}{2}rV^u = -\frac{1}{2}rV^u < 0.$$

□

Proof of Lemma 4. First, substituting from (34) into (15) establishes that for $0 < n < n^*$, the firm's value function $J(\cdot)$ satisfies

$$(q(\theta) - sn)J'(n) = (r + \delta)\,J(n) + \gamma + \frac{n}{2}rV^u - \left[\frac{1}{n}\int_0^n F(m)\,dm\right]. \tag{46}$$

If $q(\theta) - sn^* > 0$, then the function $n \mapsto \frac{r+\delta}{q(\theta)-sn}$ satisfies a Lipschitz condition on $[0, n^*]$; it follows from Picard's theorem and associated results on ordinary differential equations that there is a unique solution to (46) on $(0, n^*)$ in this case. This solution is given by

$$J(n) = (q - sn)^{-\frac{r+\delta}{s}} \left[K + \int_0^n (q - sn)^{\frac{r+\delta}{s}-1} \left(\gamma + \frac{m}{2} rV^u - \frac{1}{m} \int_0^m F(\nu)\, d\nu \right) dm \right] \tag{47}$$

where K is a constant of integration. It follows from the free entry condition (18) that $K = q^{\frac{r+\delta}{s}} k$. □

Proof of Lemma 5. The proof is analogous to that of Lemma 4. Write $T(n) = V(n) - V^u$. Then the worker's Bellman equation (19) is

$$(r + s + \delta)T(n) = w(n) + (q - sn)T'(n).$$

A boundary condition is given by the fact that at n^*, the worker is paid $w(n^*)$ until the job ends (with flow probability $\delta + s$); this implies that $V(n^*)$ satisfies

$$rV(n^*) = w(n^*) + (s + \delta)[V^u - V(n^*)],$$

or, since $w(n^*) = rV^u + (r + \delta + s)\gamma/q$,

$$T(n^*) = \frac{\gamma}{q}.$$

The usual integration argument shows that the unique closed form solution for $T(\cdot)$ is as given in the statement of the Lemma. □

Proof of Theorem 1. Define two functions $\chi, \omega : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by

$$\chi(q, rV^u) = J(n^*) \left(\frac{q - sn^*}{q} \right)^{\frac{r+\delta}{s}} - \frac{1}{q^{\frac{r+\delta}{s}}} \int_0^{n^*} (q - sm)^{\frac{r+\delta}{s}-1} \left[\gamma + \frac{m}{2} rV^u - \frac{1}{m} \int_0^m F(\nu)\, d\nu \right] dm \tag{48}$$

$$\omega(q, rV^u) = rV^u - b - \theta q \left[-V^u + \frac{\int_0^{n^*} V(n)g(n)\, dn + \frac{sn^*}{q} V(n^*)G^*}{1 - \left(1 - \frac{sn^*}{q}\right) G^*} \right] \tag{49}$$

$$\begin{aligned} &= rV^u - b - \theta q \int_0^{n^*} \left\{ \frac{\gamma}{q} \left(\frac{q - sn^*}{q - sn} \right)^{\frac{r+\delta+s}{s}} + \frac{1 - \left(\frac{q-sn^*}{q-sn} \right)^{\frac{r+\delta+s}{s}}}{2(r + \delta + s)} rV^u \right. \\ &\qquad \left. + (q - sn)^{-\frac{r+\delta+s}{s}} \int_n^{n^*} (q - sm)^{\frac{r+\delta}{s}} \psi(m)\, dm \right\} d\hat{G}(n) \end{aligned} \tag{50}$$

where n^* and $J(n^*)$ are defined as in Proposition 1, according to (36) and (37), and where $\hat{G}(\cdot)$ is the probability measure with continuous density

$$\frac{g(n)}{1-\left(1-\frac{sn^*}{q}\right)G^*}$$

on $[0,n^*]$ and an atom of mass sn^*G^*/q at n^*. Note that $\chi(q,rV^u)$ represents the maximum value of an entrant firm that takes q and rV^u as given and expects to pay wages $w(\cdot)$ as given by (34); $rV^u-\omega(q,rV^u)$ is the value of an unemployed worker in an economy populated by such firms. As observed in the discussion preceding Proposition 1, (q,rV^u) is part of an equilibrium allocation iff

$$k=\chi(q,rV^u) \tag{51}$$

$$0=\omega(q,rV^u). \tag{52}$$

We will show that such an intersection exists by first observing that (51) defines a continuous 1-manifold in $\mathbb{R}^+\times\mathbb{R}$, then observing that ω restricted to this manifold defines a continuous function, showing that it takes positive and negative values, and applying the intermediate value theorem.

To see that (51) defines a continuous 1-manifold, first note that $\chi(q,rV^u)$ is nondecreasing in q and nonincreasing in rV^u, with both relationships being strict if there is positive activity (that is, if it is optimal for a firm with zero workers to hire). This follows immediately from the definition of $\chi(q,rV^u)$ as the maximal value of the problem for the firm as described. First, clearly if rV^u increases, then for any q, $\chi(q,rV^u)$ must decrease, since if the firm keeps the same hiring strategy as before, then it would increase the value of its program as $w(n)$ decreases for each n; reoptimizing the hiring strategy can only increase this effect. Second, if q increases to $q'>q$, then the firm could keep the value of its program the same by keeping the same cutoff n^* but posting a vacancy with probability $q/q'<1$ for each $n<n^*$. Moreover, for each $q>0$, it's clear that there is a unique $rV^u\in\mathbb{R}$ such that $\chi(q,rV^u)=k$. Finally, since $\chi(\cdot)$ is continuous, it follows that (51) defines a continuous curve in $\mathbb{R}^+\times\mathbb{R}$. Call this curve C. Since it's clear that $\omega|_C$ is continuous (since ω itself is continuous), an equilibrium will exist iff we can find points $(q_1,v_1),(q_2,v_2)\in C$ such that $\omega(q_1,v_1)$ and $\omega(q_2,v_2)$ differ in sign.

To do this, first define $\bar{v}$ to solve

$$k=\frac{1}{r+\delta}\max_{n>0}\left[\frac{1}{n}\int_0^n F(m)\,dm-\frac{n}{2}v\right].$$

It therefore follows that $\lim_{q\to\infty}\chi(q,\bar{v})=0$. Also, for $v>\bar{v}$, $\chi(q,v)<k$ by construction.

Now, if $q \to \infty$, then any firm will instantaneously hire n^*; thus in the limit, $J(n) = J(n^*)$ for all $n \in [0, n^*]$. From the definition of $\omega(\cdot)$, it follows that in this case, $\omega(q, \bar{v}) = \bar{v} - b$.

In the other extreme, let $\hat{q} > 0$ satisfy $\chi(\hat{q}, b) = 0$; such a $\hat{q}$ will exist provided that $\bar{v} > b$. Suppose that this is true. Then it is clear from the definition of $\bar{v}$ that $n^*(\hat{q}, b) > 0$. Also, by definition,

$$\omega(\hat{q}, b) = -\theta q \int_0^{n^*} \left[-V^u + V(n)\, dG(n)\right].$$

If q is finite, then $\theta q > 0$, and $\hat{G}(\cdot)$ is a probability measure that places positive measure on every subset of $[0, n^*]$ of positive Lebesgue measure, while $V(n) - V^u > 0$ for each $n > 0$. Thus the only possibilities are that $q = +\infty$ (which is impossible since $\bar{v} \neq b$), or that $\omega(\hat{(q)}, b) < 0$. Thus if $\bar{v} - b > 0$, then C contains points at which ω takes values of opposite signs, which completes the proof of the existence of an equilibrium via the intermediate value theorem.

If $\bar{v} - b \leq 0$ then it is trivial to prove that there is an equilibrium in which no firm ever enters. $\square$

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