Dynamic Price Competition with Network Effects

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Abstract

I consider a dynamic model of competition between two proprietary networks. Consumers die with a constant hazard rate and are replaced by new consumers. Firms compete for new consumers to join their network by offering network entry prices (which may be below cost). New consumers have a privately known preference for each network. Upon joining a network, in each period consumers enjoy a benefit which is increasing in network size during that period. Firms receive revenues from new consumers as well as from consumers already belonging to their network.

I discuss various properties of the equilibrium, including the pricing function, the system's expected motion, and the stationary distribution of market shares. I derive several results analytically. I then confirm and extend these results by numerical computation. Finally, I use the model to estimate the barrier to entry create by network effects.

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1 Introduction

Many industries exhibit some form of network effects, the situation whereby a consumer's valuation is increasing in the number of other consumers in the same network. There are several possible sources of network effects. The most obvious one is direct network effects. Take the example of operating systems. If I use the Windows OS then, when I travel, it is more likely I will find a computer that I can use (both in terms of knowing how to use it and in terms of being able to run files and programs I carry with me).[1]

A second source of network effects is the availability of complementary products. For example, it seems reasonable to assume that the variety and quality of the software available for the Palm system is greater the more users buy PDAs that run the Palm OS. A similar argument applies for complementary services. For example, the greater the number of Cannon photocopiers are sold, the more likely it is I will be able to find good post-sale service providers.

Finally, a third source of network effects is the pricing of network services.[2] Take the example of wireless telecommunications. To the extent that operators set different on-net and off-net prices, the utility from being connected to a given network is increasing in the number of other users on the same network.

In this paper, I consider a dynamic model of competition between two proprietary networks. Consumers die with a constant hazard rate and are replaced by new consumers. Firms compete for new consumers to join their network by offering network entry prices (which may be below cost). New consumers have a privately known preference for each network. Upon joining a network, in each period consumers enjoy a benefit which is increasing in network size during that period. Firms receive revenues from new consumers joining the network (possibly negative revenues) as well as from consumers already belonging to its network.

I develop a general model with the above features. I derive the firms' value functions as well as the consumer value functions from joining a given network. For the case when the discount factor is small, I prove a series of analytical results. I show that there exists a unique equilibrium. Given symmetry of the model, the equilibrium is itself symmetric. While the equilibrium is symmetric, the actual state of the system is generally asymmetric. In fact, I show that

[1] Another source of direct network effects would be file sharing. While this is frequently proposed as the main source of direct network effects, in the example at hand I think it is relatively less important.

[2] Laffont, Rey and Tirole (1998a) refer to this case as "tariff-mediated network externalities."

a larger network is always more likely to attract a new consumer. Moreover, if network effects are sufficiently strong, then the larger network tends to increase in size, unless it holds close to 100% of the market, in which case it tends to decrease in size. Finally, I provide sufficient conditions such that price is increasing or decreasing in network size.

I then numerically compute the model, which allows me to do three things. First, I confirm that the theoretical results obtained for small values of the discount factor δ extend to higher values of δ as well. Second, I derive more specific conclusions regarding the nature of the equilibrium. For example, I show that the pricing function is typically U shaped, with lowest prices around symmetric states. I also show that, if network effects are sufficiently strong, then the stationary distribution over states is bimodal. Finally, I use the model to derive a quantitative estimate of the barrier to entry created by network effects, both in terms of the value loss for an entrant and the expected time for an entrant to reach a certain market share.

■ **Related literature.** Following seminal work by Katz and Shapiro (1985), the early literature on oligopoly with network effects focused on relatively simple, static models.[3] Since then, the industrial organization literature developed in two directions. One strand attempts to empirically measure the size of network effects.[4] Another strand investigates further implications of network effects in an oligopoly context.[5] Despite important developments, most of this literature has followed a static, or finite period, approach.[6]

More recent work attempts to explicitly address the issue of dynamic competition between proprietary networks. Doganoglu (2003), Mitchell and Skrzypacz (2006), derive the Markov Perfect Equilibrium of an infinite period game where consumer's utility is an increasing function of past market shares. Markovich (2004), Markovich and Moenius (2004) develop computational models of industries with "hardware" and "software" components (very much like my paper). They assume consumers live for two periods and benefit from indirect network effects through the quality of products available. Doraszelski, Chen, and Harrington (2007) also develop a computational dynamic

[3]Other important early work includes Farrell and Saloner (1985), who focus on consumer behavior, and Arthur (1989), who presents an infinite period model but assumes non-proprietary networks (and thus excludes strategic behavior).

[4]See, for example, Gandal (1994), Goolsbee and Klenow (2002), Rysman (2004).

[5]See, for example, Laffont, Rey and Tirole (1998a,b).

[6]Farrell and Klemperer (2006) present an excellent survey of this literature. See also Economides (1996).

model. In many respects, their analysis goes beyond my paper: they consider R&D investments and compatibility decisions. However, like Doganoglu (2003), Mitchell and Skrzypacz (2006), they assume consumer benefits are an increasing function of network size at the time of purchase (that is, consumers are not forward looking). In sum, all of these papers assume relatively simple behavior on the part of consumers: either consumers are short-lived, or they are myopic, or they are backward looking.[7] By contrast, I assume that consumers live for potentially many periods (that is, die with a constant hazard rate), and make their decisions in a rational, forward looking way. In this sense, the paper that comes closest to mine is Driskill (2007), who considers a deterministic, continuous-time model where consumers are forward looking. My paper differs from his in that I consider idiosyncratic consumer preferences, which generate stochastic dynamics; and in the fact I focus on different issues of equilibrium and comparative dynamics.

Following the seminal contributions by Gilbert and Newbery (1982) and Reinganum (1983), a series of papers have addressed the issue of persistence of firm dominance. Contributions to this literature include Budd, Harris and Vickers (1993), Cabral and Riordan (1994), Athey and Schmutzler (2001), Cabral (2002). These papers provide conditions under which larger firms tend to become larger (increasing dominance). Intuitively, the reason for such dynamics corresponds to some form of the "efficiency effect" characterized by Gilbert and Newbery (1982), the fact that joint profits are greater the closer to monopoly market structure is. My framework and results are consistent with the idea of increasing dominance. Specifically, I show that, if network effects are sufficiently strong and the large firm is not too large, then increasing dominance holds.

Another related literature is that of switching costs, in particular the paper by Beggs and Klemperer (1992). They consider an infinite period model with two sellers and a stationary mass of consumers. In each period a fraction of consumers dies and a corresponding fraction enters the market. Each new consumer chooses one of the sellers and sticks with it for the rest of the consumer's life. Beggs and Klemperer derive Markov equilibria such that the firms' strategies and the consumers' value functions are linear functions of the state. They show that prices are higher the greater switching costs are. My paper differs in two important ways. First, I consider both a primary and a secondary market. As a result, I allow sellers to (partially) discriminate be-

[7] See also Kandori and Rob (1998), Auriol and Benaim (2000), who approach the problem from a stochastic evolutionary perspective.

tween old consumers and new consumers. Second, I consider the possibility of network events. In terms of results, there is also an important difference, namely price strategies are quite different from affine functions.

2 Model

I consider an infinite period model of price competition between two proprietary networks, owned by firms A and B. Since I analyze symmetric Markov equilibria, with some abuse of notation I denote each firm by the size of its network, i or j. Network size evolves over time due to consumer birth and death. In each period, a consumer dies and a new consumer is born. The new consumer chooses between one of the existing networks and stays with it until death.[8,9]

Specifically, the timing of moves in each period is summarized in Table 1. Initially, a total of $N-1$ consumers are distributed between the two firms, so $i+j=N-1$. A new consumer is born and firms simultaneously set prices $p(i)$, $p(j)$ for the consumer to join their network. If the new consumer opts for network i, then firm i receives a profit $p(i)$, whereas the consumer receives a onetime benefit from joining network i, ζ_i.[10]

After the new consumer makes his choice, there are a total of N consumers divided between the two networks. During the remainder of the period, firm i receives a payoff $\theta(i)$, whereas a consumer attached to network i enjoys a benefit $\lambda(i)$. In others words, I treat network choice as a durable good, and assume there is some non-durable good attached to the durable good "network membership." I denote the market for the non-durable good as the after market. Finally, at the end of the period one consumer dies (each with equal probability). In other words, a consumer from firm i's network dies with probability i/N.

[8] In this sense, my framework is similar to that of Beggs and Klemperer (1992). They too consider a stationary number of consumers and assume that a newborn consumer, having chosen one of the sellers, sticks with it until death.

[9] Although I work with a discrete time model, the underlying reality I have in mind is one of continuous time. Suppose that consumers die according to a Poisson process with arrival rate λ. Essentially, I consider the time between two consecutive deaths as a period in my discrete time model. By assuming risk-neutral agents, I can summarize the Poisson arrival process in a discount factor δ that reflects the average length of a discrete period: $\delta = \exp(-r/\lambda)$, where r is the continuous time discount rate.

[10] For simplicity, I assume zero cost. Alternatively, we can think of $p(i)$ as markup over marginal cost.

Table 1: Timing of model: events occurring in each period t.

Event	Value functions	State of the game
Firms set network entry prices $p(i)$	$v(i)$	$i \in \{0, \ldots, N-1\}$
Nature chooses x_i, new consumer's preference for network i		
New consumer chooses network	$u(i)$	$i \in \{0, \ldots, N\}$
Stage competition takes place: period profits $\theta(i)$, consumer surplus $\lambda(i)$		
One consumer dies (probability $\frac{1}{N}$)		$i \in \{0, \ldots, N-1\}$

■ **After market.** Since my main goal is to understand the evolution of network size over time, I take the values $\theta(i)$ and $\lambda(i)$ as given, that is, I treat them as the reduced form of the stage game played in the after market. I make very minimal assumptions regarding the values of $\theta(i)$, $\lambda(i)$:

Assumption 1 *(i) $\theta(i+1) - \theta(i)$ is (weakly) increasing in i; ; (ii) $\lambda(i)$ is (weakly) increasing in i.*

Note that Part (i) is equivalent to $\frac{\theta(i+1)-\theta(0)}{i+1} \geq \frac{\theta(i)-\theta(0)}{i}$. It thus states that firms (weakly) enjoy network benefits, in the sense that after market profit per consumer is nondecreasing in network size. Part (ii) states that consumers (weakly) enjoy network benefits, in the sense that after market benefit is non decreasing in the number of consumers in the same network.

Different industries are naturally be associated to different values of $\theta(i)$ and $\lambda(i)$. Here are some possible examples:

- After sales service (e.g., photocopiers, printers, cameras). In these examples, purchase of equipment leads to a stream of future benefits. Suppose that the value of after sales service depends on seller investment. By an appropriate change of units, suppose value equals investment. Let the cost of investment be given by $\alpha\, v^2$, where v is value. Suppose moreover that the seller captures all of the surplus it creates in the after market. Then $\lambda(i) = 0$ and $\theta(i) = \psi\, i^2$, where $\psi = \frac{1}{4\,\alpha}$.[11]

[11] The seller's optimal v maximizes $i\,v - \alpha\, v^2$. Substituting the optimum v in the payoff function yields the value of $\theta(i)$ indicated in the text.

- Handheld operating systems. Buying a Palm OS handheld gives the consumer access to a wealth of applications written for that OS. Most of these are written by third party suppliers. This implies that, as a first-order approximation, $\theta(i) = 0$. To the extent that the availability of applications software in greater the larger the network is, $\lambda(i)$ strictly increasing. In this case, all of the network effects are reflected in the consumer benefit function.

My focus is on the firms' pricing decision and the consumer's network choice. Specifically, I consider equilibria in Markov pricing and network choice decisions. The state is defined by i, the size of firm i's network at the beginning of the period, that is, when firms set prices and the newborn consumer chooses one of the networks.[12] I next derive the consumer's and the firm's decisions in a Markov equilibrium.

■ **Consumer choice.** Each consumer's utility is given by two components: ζ_i and $\lambda(i)$. The first component is the consumer's idiosyncratic preference for firm i, which I assume depends on the identity of firm i but not on the size of its network (thus the use of a subscript rather than an argument). The value of ζ_i is also the consumer's private information. The second component is network benefit from a network with size i (*including* the consumer in question), which I assume is independent of the firm's identity. I assume that consumers receive the ζ_i component the moment they join a network, whereas $\lambda(i)$ is received each period that a consumer is still alive (and thus varies according to the size of the network during each future period).[13]

I assume that the values of ζ_i are sufficiently high so that a newborn consumer always chooses one of the available networks (that is, the outside option is always dominated). This assumption allows me to concentrate on the value of $\xi_i \equiv \zeta_i - \zeta_j$, the consumer's idiosyncratic *relative* preference for firm i's network. Notice that $\xi_j = -\xi_i$. I assume that ξ_i is distributed according to $F(\xi)$, which satisfies the following properties:

Assumption 2 *(i) $F(\xi)$ is continuously differentiable; (ii) $f(\xi) = f(-\xi)$; (iii) $f(\xi) > 0$, $\forall \xi$; (iv) $F(\xi)/f(\xi)$ is strictly increasing.*

[12] Note that, when consumers enjoy network benefits, there are a total of N consumers, divided between the two networks. However, at the time that prices are set there are only $N-1$ consumers, so $i \in \{0, \ldots, N-1\}$ at that moment. The state and the firm value functions are defined at this moment (beginning of period).

[13] The assumption that ζ_i is received at birth is not important. I could have the consumer receives ζ_i each period of his or her lifetime. However, this way of accounting for consumer utility simplifies the calculations.

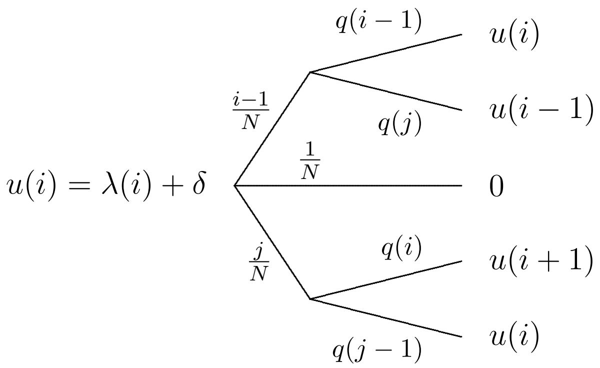

Figure 1: Consumer's value function.

Many common distributions, including the Normal, satisfy Assumption 2.

Let $u(i)$ be a consumer's aftermarket value function, that is, the discounted value of payoff streams $\lambda(i)$ received while the consumer is alive (thus excluding both ζ_i and the price paid to join the network).

Consider a new consumer's decision. At state i, the indifferent consumer will have $\xi_i = x(i)$, where the latter is given by

$$x(i) - p(i) + u(i+1) = -p(j) + u(j+1), \tag{1}$$

or simply

$$x(i) = p(i) - p(j) - u(i+1) + u(j+1). \tag{2}$$

where $p(i)$ is firm i's price. This looks very much like a Hotelling consumer decision, except for the fact that $u(i+1)$ and $u(j+1)$ and endogenous values.

Firm i's demand is the probability of attracting the new consumer to its network. It is given by

$$\begin{aligned} q(i) &= 1 - F\big(x(i)\big) \\ &= 1 - F\big(p(i) - p(j) - u(i+1) + u(j+1)\big). \end{aligned} \tag{3}$$

The consumer value functions, introduced above, are illustrated in Figure 1. The corresponding formula is given by

$$\begin{aligned} u(i) = \lambda(i) + \delta \Bigg(\frac{j}{N}\, q(i)\, u(i+1) + \left(\frac{j}{N}\, q(j-1) + \frac{i-1}{N}\, q(i-1) \right) u(i) + \\ + \frac{i-1}{N}\, q(j)\, u(i-1) \Bigg), \end{aligned} \tag{4}$$

where $q(i)$ is given by (3), $i = 1, \dots, N$, and $j = N - i$.[14] In words, a consumer's value is given by the current aftermarket benefit $\lambda(i)$. In terms of future value, there are three possibilities: with probability $1/N$, the consumer dies, in which case I assume continuation utility is zero.[15] With probability $(i-1)/N$, a consumer from the same network dies. This loss is compensated by the newborn consumer joining network i, which happens with probability $q(i-1)$, in which case next period's aftermarket state reverts back to i. With probability $1 - q(i-1)$, the new consumer opts for the rival network, in which case next period's aftermarket state drops to $i-1$. Finally, with probability j/N, a consumer from the rival network dies. This loss is compensated by the newborn consumer joining network j, which happens with probability $q(j-1)$, in which case next period's aftermarket state reverts back to i. With probability $1 - q(j-1) = q(i)$, the new consumer opts for network i, in which case next period's aftermarket state increases to $i+1$.

■ **Firm's pricing decision.** Firm i's value function is given by

$$v(i) = q(i)\left(p(i) + \theta(i+1) + \delta\,\frac{j}{N}\,v(i+1) + \delta\,\frac{i+1}{N}\,v(i)\right) + \Big(1 - q(i)\Big)\left(\theta(i) + \delta\,\frac{j+1}{N}\,v(i) + \delta\,\frac{i}{N}\,v(i-1)\right), \qquad (5)$$

where $i = 0, \dots, N-1$ and $j = N - 1 - i$.[16] This is illustrated in Figure 2. With probability $q(i)$, firm i attracts the new consumer and receives $p(i)$. This moves the aftermarket state to $i+1$, yielding a period payoff of $\theta(i+1)$; following that, with probability $(i+1)/N$ network i loses a consumer, in which case the state reverts back to i, whereas with probability j/N network j loses a consumer, in which case the state stays at $i+1$. With probability $q(j)$, the rival firm makes the current sale. Firm i get no revenues in the primary market. In the aftermarket, it gets $\theta(i)$ in the current period; following that, with probability i/N network i loses a consumer, in which case the state drops to $i-1$, whereas with probability $(j+1)/N$ network j loses a consumer, in which case the state reverts back to i.

[14] Recall that the argument of u includes the network adopter to whom the value function applies, thus i must be strictly positive in order for the value function to apply. For the extreme values $i = 1$ and $i = N$, (4) calls for values of $q(\cdot)$ and $u(\cdot)$ that are not defined. However, these values are multiplied by zero.

[15] Alternatively, I can consider a constant continuation utility upon death.

[16] Again, notice that, for the extreme case $i = 0$, (7) calls for values of $v(\cdot)$ which are not defined. However, these values are multiplied by zero.

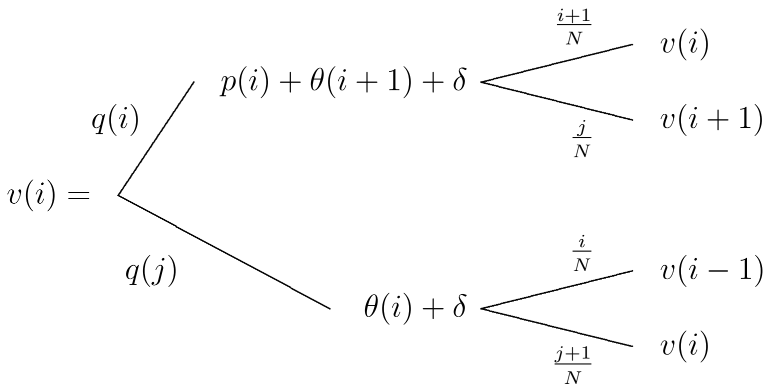

Figure 2: Firm's value function.

Equation (5) leads to the following first-order conditions for firm value maximization:

$$
\begin{aligned}
q(i) + \frac{\partial\, q(i)}{\partial\, p(i)} \Bigg(& p(i) + \theta(i+1) - \theta(i) \\
& + \delta\, \frac{j}{N}\, v(i+1) + \delta\, \frac{i+1}{N}\, v(i) \\
& - \delta\, \frac{j+1}{N}\, v(i) - \delta\, \frac{i}{N}\, v(i-1) \Bigg) = 0,
\end{aligned}
$$

or simply

$$p(i) = h(i) - w(i), \tag{6}$$

where

$$
\begin{aligned}
h(i) &\equiv -\frac{q(i)}{q'(i)} = \frac{1 - F\big(x(i)\big)}{f\big(x(i)\big)} \\
w(i) &\equiv \big(\theta(i+1) - \theta(i)\big) + \delta \left(\frac{j}{N}\, v(i+1) + \frac{i-j}{N}\, v(i) - \frac{i}{N}\, v(i-1) \right)
\end{aligned}
$$

The first-order condition (6) is fairly intuitive and plays an important role in explaining several results below. The first term on the right-hand side corresponds to the standard markup under monopoly pricing. The only difference is that consumer demand includes the endogenous value difference $u(i+1) - u(j+1)$. Recall that the indifferent consumer "address" $x(i)$ is given by $x(i) = p(i) - p(j) - u(i+1) + u(j+1)$.

The second term of the right-hand side of (6), $w(i)$, is firm i's future value of winning the current sale. By "future" I mean beginning with the current period's after market. In terms of current period's payoff, the difference comes to $\theta(i+1) - \theta(i)$. In terms of future payoffs, we have the difference in value function between states $i+1$ and i (if consumer death takes place in network j) or between states i and $i-1$ (if consumer death takes place in network i).

Finally, substituting (6) into (5) and simplifying, we get

$$v(i) = r(i) + \theta(i) + \delta \left(\frac{j+1}{N} \, v(i) + \frac{i}{N} \, v(i-1) \right),$$

where

$$r(i) \equiv \Big(1 - F\big(x(i)\big)\Big) \, h(i) = \frac{\Big(1 - F\big(x(i)\big)\Big)^2}{f\big(x(i)\big)}.$$

This is a recursive system, the solution of which is given by

$$v(i) = \left(1 - \delta \, \frac{N-i}{N}\right)^{-1} \left(r(i) + \theta(i) + \delta \, \frac{i}{N} \, v(i-1) \right), \tag{7}$$

$i = 0, \ldots, N-1$.

■ **Transition matrix and steady state distribution.** Given the equilibrium values of $q(i)$, I can compute a Markov transition matrix $M = m(i,k)$ where $m(i,k)$ is the probability of moving from state i to state k. For $0 < i < N-1$, we have

$$\begin{aligned} m(i,i-1) &= \frac{i}{N} \, \big(1 - q(i)\big) \\ m(i,i) &= \frac{i+1}{N} \, q(i) + \frac{N-i}{N} \, \big(1 - q(i)\big) \\ m(i,i+1) &= \frac{N-1-i}{N} \, q(i) \end{aligned} \tag{8}$$

Moreover, $m(i,k) = 0$ if $k < i-1$ or $k > i+1$. Finally, the boundary values are obtained as follows. For $i = 0$, I apply the general equations and add the value obtained for $m(0,-1)$ to the value of $m(0,0)$. For $i = N-1$, again I apply the general equations and add the value obtained for $m(N-1,N)$ to $m(N-1,N-1)$. As a result, I get

$$m(0,0) = 1 - \frac{N-1}{N} \, q(0)$$

$$
\begin{aligned}
m(0,1) &= \frac{N-1}{N}\, q(0) \\
m(N-1,N-2) &= \frac{N-1}{N}\, \Big(1-q(N-1)\Big) \\
m(N-1,N-1) &= 1-\frac{N-1}{N}\, \Big(1-q(N-1)\Big)
\end{aligned}
$$

Given the assumption that $F(\cdot)$ has full support (Assumption 2), $q(i) \in (0,1)\ \forall i$, that is, there are no corner solutions in the pricing stage. It follows that the Markov process is ergodic and I can compute the stationary distribution over states, that is, over market shares. This is given by the (transposed) vector d that solves $d\,M = d$.[17]

■ **Consumer welfare and social welfare.** So far I have only considered the after-market value function for an individual consumer. I now turn to the task of measuring total consumer welfare at state i. Although $u(i)$ is measured at the time the new consumer has already chosen a network, it is now simpler if I simply compute total consumer welfare at the time firms set prices $p(i)$, that is, before network choice takes place. Consumer welfare, $c(i)$, is then given by

$$
\begin{aligned}
c(i) = \sum_{\substack{k,\,\ell\,=\,i,j \\ \ell \neq k}} \quad q(k) \Bigg(& E\Big(\zeta_k \mid \xi_k > x(k)\Big) - p(k) + \\
& + (k+1)\,\lambda(k+1) + \ell\,\lambda(\ell) + \\
& + \delta\,\frac{k+1}{N}\, c(k) + \delta\,\frac{\ell}{N}\, c(k+1) \Bigg),
\end{aligned}
\tag{9}
$$

$i = 0, \ldots, N-1$. There are three components to $c(i)$, each corresponding to a row in (9). Suppose the newborn consumer joins network i. The immediate benefit for that consumer is given by the value of the idiosyncratic component ξ_i minus price $p(i)$. By joining network i, the newborn consumer increases network size to $(i+1)$, leading to a total benefit $(i+1)\,\lambda(i+1)$ to all consumers in network i. In the meantime, consumers in the rival network jointly receive a total benefit of $j\,\lambda(j)$. Finally, we need consider continuation payoffs. With probability $\frac{i+1}{N}$, a consumer from network i (now of size $i+1$) dies, bringing network size back to i and total consumer value to $c(i)$. With probability $\frac{j}{N}$, a

[17] This vector d can also be computed by repeatedly multiplying M by itself. That is, $\lim_{k\to\infty} M^k$ is a matrix with d in every row.

Table 2: Notation.

i	Firm i's network size (also j, k, ℓ).
N	Market size (number of consumers).
δ	Discount factor.
ξ_i	Consumers's idiosyncratic preference for firm i.
$F(\xi_i)$	Distribution of ξ_i.
$\theta(i)$	Firm's after market profit in state i.
$\lambda(i)$	Consumer's after market benefit in state i.
$x(i)$	Indifferent consumer's relative preference for firm i.
$p(i)$	Price in state i (for new consumer).
$q(i)$	Probability of a sale in state i (to new consumer).
$u(i)$	Individual consumer's value in state i.
$v(i)$	Firm's value in state i.
$c(i)$	Total consumer's surplus in state i.
$s(i)$	Social surplus in state i.
$m(i, j)$	Transition probability from state i to state j.
$d(i)$	Stationary probability density of state i.

consumer form network j dies, bringing the state to $i + 1$, in which case total consumer surplus is given by $c(i + 1)$.

Given the (equilibrium) values of $q(i)$, $p(i)$ and $u(i)$, (9) is a well-defined linear system, yielding a unique solution $c(i)$. Notice that, by symmetry, $c(i) = c(j)$, where $j = N - 1 - i$. Finally, social welfare, $s(i)$, is simply consumer welfare plus firm value, that is,

$$s(i) = c(i) + v(i) + v(j), \tag{10}$$

$i = 0, \ldots, N - 1$, $j = N - 1 - i$. Once again, by symmetry $s(j) = s(i)$, where $j = N - 1 - i$.

A summary of the model's notation is given in Table 2. To help the reader navigate through the extensive set of variables, I follow the rule of using Greek letters to denote exogenous values and Roman letters to denote endogenous values. The only exception is the use of $F(\cdot)$ for the distribution of the idiosyncratic preference parameter.[18]

[18] The only other apparent exception, N, stems from the fact that the symbol for capital ν is the same as capital n.

3 Equilibrium properties

The dynamic system introduced in Section 2 has no general closed-form solution. In this section I derive analytical results for the particular case when δ is small. I find a unique symmetric Markov equilibrium. I also provide a series of results that characterize equilibrium dynamics, both in terms of the pricing function and the expected transition across states (reversion to the mean vs increasing dominance). In Section 4, I proceed to compute the equilibrium numerically. I confirm the analytical results obtained in this section and derive additional results regarding equilibrium dynamics and welfare.

■ **Existence and uniqueness.** I begin my analytical characterization by establishing equilibrium existence and uniqueness.

Proposition 1 *There exists a δ' such that, if $\delta < \delta'$, then there exists a unique Markov equilibrium, which is symmetric.*

The proof of this and all other results may be found in the Appendix. Notice that symmetry is a derived property, not an assumption.

Proposition 1 is in sharp contrast with typical results from static models that feature network effects. In fact, in static models with sufficiently strong network effects there exist multiple equilibria and no stable symmetric equilibrium. The critical difference of my model with respect to these static models is that I assume an overlapping generations framework. As a result, the consumers' decisions are interdependent but not simultaneous. In other words, simultaneity of adoption decisions overstates the coordination problem.

I conjecture that, if the discount factor is sufficiently close to 1, then there may exist multiple equilibria, supported by asymmetric beliefs. Even then, I conjecture that there exists a unique symmetric equilibrium; and symmetry seems a reasonable equilibrium selection criterion.[19]

■ **Increasing dominance.** How does the system, that is, the size of each network, evolve over time? To answer this question, I first characterize the equilibrium probability of a sale by firm i. Let $i^* \equiv \frac{N-1}{2}$ denote the "symmetric" state. If N is even, there exists no symmetric state, but the result below applies nonetheless.[20]

[19] For $N = 2$, I can prove uniqueness of symmetric equilibria for *any* value of δ. The proof is available upon request.

[20] Specifically, if N is odd, then i^* is the symmetric state. If N is even, then there is no

Proposition 2 *There exists a δ' such that, if $\delta < \delta'$, then $q(i) \geq \frac{1}{2}$ if $i > i^*$.*

This result states that the leader always has a higher probability of attracting a newborn consumer. This does not imply that the state moves away from symmetry in expected value. In fact, the leader also has a higher death rate.

My next result concerns precisely the issue of expected motion across states. One of the central questions of interest in systems like the one I consider is whether it tends to revert to the mean (the symmetric state), or whether asymmetries tend to increase over time.

Proposition 3 *There exist δ', λ', θ' such that: (a) if i is close to zero or close to $N-1$, then the state moves toward i^* in expected value; (b) if i is close to i^*, and either $\lambda(i^*+1)-\lambda(i^*) > \lambda'$ or $\theta(i^*+1)+\theta(i^*-1)-2\,\theta(i^*) > \theta'$, then the state moves away from i^* in expected value.*

This result is in the same spirit as Budd, Harris and Vickers (1993). In a one-dimensional R&D race between two players, they show that, in expected terms, the system moves away from the symmetric state. In my model this is true if and only if network effects are sufficiently strong.

■ **Pricing.** I next consider the properties of the equilibrium pricing function. Which firm prices more aggressively: a firm with a large network or a firm with a small network? As I show in my next result, the answer depends critically on the shape of the aftermarket functions $\theta(i)$ and $\lambda(i)$.

Proposition 4 *There exists a δ' such that: (a) if $\delta < \delta'$, $\theta(i+1) - \theta(i)$ is constant, and $\lambda(i)$ is strictly increasing, then $p(i)$ is strictly increasing; (b) if $\delta < \delta'$, $\lambda(i)$ is constant, and $\theta(i+1) - \theta(i)$ is strictly increasing, then $p(i)$ is strictly decreasing.*

Proposition 4 highlights the two main forces impacting on the firms' pricing incentives: market power over the current newborn consumer and the quest for market power in the after market and in future periods. This can be seen from equation (6), which states that $p(i) = h(i) - w(i)$. Here the $h(i)$ term represents market power over the current newborn consumer, whereas the term $w(i)$ represents the quest for market power in the after market and in future periods. The first term is increasing in i, whereas the second term is generally

symmetric state; $[i^*]$ and $[i^*]+1$ are the two states closest to the symmetric state, where $[x]$ denotes the highest integer lower than x.

decreasing in i. Proposition 4 provides conditions under which the first or the second effects dominate.

Specifically, given my assumption that the discount factor is small, $w(i)$ is essentially given by after-market profits, $\theta(i)$. Under case (a), after market profits are an affine function of network size. This implies that the benefit from winning a new customer, in terms of after market profits, does not depend on network size. Differences in pricing are thus exclusively driven by market power considerations related to the newborn consumer. Now, consumers are willing to pay more for a firm with a bigger network. In equilibrium, this is reflected in a higher price by the firm with a larger network. Thus $p(i)$ is increasing in i.

Consider now the case (b), the case when consumers do not care about network size. Notice this does not mean there are no network externalities, rather that sellers completely capture the added consumer surplus resulting from network externalities. If it were not for after market and future profits, firms would set the same price, as their products are identical in the eyes of consumers. But to the extent that $\theta(i) - \theta(i-1)$ is increasing, the firm with a bigger network size has more to gain from making the next sale. This implies that it discounts price (with respect to the static price) to a greater extent. Thus $p(i)$ is decreasing in i.

In the next section, where I numerically compute equilibrium values, I will show that the two above effects typically lead to a U shaped pricing function: decreasing for low levels of i, increasing for high values of i.

4 Numerical computation

The dynamic system introduced in Section 2 has no closed-form solution. In this section I describe the process of numerical computation of equilibrium values.

The exogenous parameters of the model correspond to the stage game profit function, $\theta(i)$, and consumer surplus function, $\lambda(i)$; the discount factor, δ; and the distribution of ξ_i, $F(\cdot)$. The endogenous variables are the price function $p(i)$, the demand function $q(i)$, the firm value function $v(i)$, and the consumer value function $u(i)$.

In order to obtain an equilibrium solution, I follow a Gaussian method similar to that proposed by Pakes and McGuire (1994):

1. Start with "naïve" consumer and firm value functions: $u_0(i) = \lambda(i)/(1-\delta)$ and $v_0(i) = \theta(i)/(1-\delta)$.

2. At iteration $t = 1, \ldots$, use (6) to compute equilibrium prices and (3) to compute equilibrium demand given the latest estimate of u and v, that is, $u_{t-1}(i)$ and $v_{t-1}(i)$.

3. Use equilibrium prices and demands to compute the value functions at iteration t. The consumer value functions, $u_t(i)$ are obtained from (4). The firm value functions, $v_t(i)$, are obtained from (7).

4. Compute

$$\Delta \equiv \frac{\sum_{i=1}^{N} \left| u_t(i) - u_{t-1}(i) \right| + \sum_{i=0}^{N-1} \left| v_t(i) - v_{t-1}(i) \right|}{\sum_{i=1}^{N} u_t(i) + \sum_{i=0}^{N-1} v_t(i)}$$

Return to Step 2 until $\Delta < \Delta'$. I use $\Delta' = 1\text{E}{-6}$, though converge is not greatly affected by considering different values.[21]

For the purpose of numerical computation, the following is a useful theoretical result:

Lemma 1 *Given $u(i)$ and $v(i)$, there exist unique $p(i)$ and $q(i)$; given $p(i)$ and $q(i)$, there exist unique $u(i)$ and $v(i)$.*

The proof of Lemma 1 shows that (6) yields a unique solution $p(i)$ given the value functions $u(i), v(i)$. As a result, we also get a unique solution $q(i)$ given $u(i)$ and $v(i)$. Moreover, the systems determining $u(i), v(i)$ are linear, and so given $p(i), q(i)$ we get unique values of $u(i)$ and $v(i)$. Lemma 1 implies that the process described in Steps 1–4 above yields a unique solution at each iteration. Therefore, insofar as the sequence of p, q, u, v converges, we obtain a specific Nash equilibrium of the game.

A solution consists of four primary outputs: the mappings $p(i)$ and $q(i)$, which give equilibrium price and probability of a sale at each state $i = 0, \ldots, N-1$; the firm value functions $v(i)$, $i = 0, \ldots, N-1$; and the consumer value functions $u(i)$, $i = i, \ldots, N$. From these I can also derive the Markov transition matrix M, which is given by (8); the stationary distribution over states, $d(i)$, which is given by the solution to $d\,M = d$; consumer welfare at state i, $c(i)$, which is given by (9); and social welfare at state i, $s(i)$, which is given by (10).

[21] I have also experimented with computing a moving average of values of u and v from previous iterations. This alternative procedure improved the speed of convergence for some parameter values.

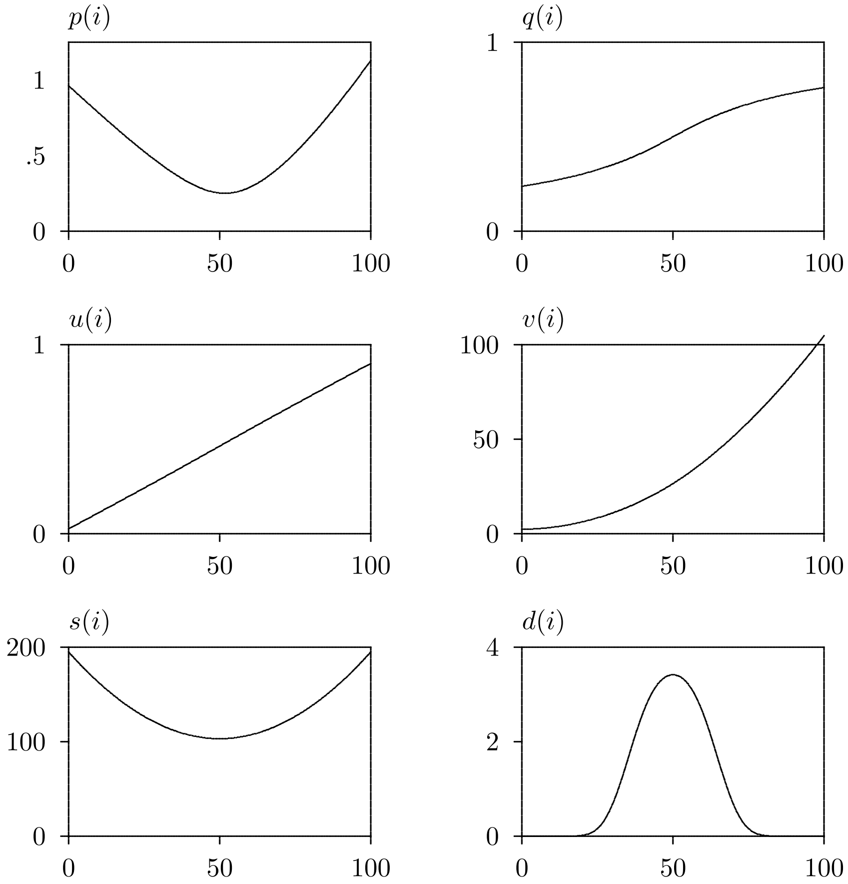

Figure 3: Equilibrium values as a function of state (horizontal axis) when F is normal with standard deviation σ and assuming $\theta(i) = \psi\, i^2/N,\ \lambda(i) = \psi\, i/N,\ \psi = .1,\ \sigma = 1,\ \delta = .9,\ N = 101.$

In computing $c(i)$, one step still needs to be taken, namely computing the value of $E\left(\zeta_i \mid \xi_i > x(i)\right)$, which appears in (9). I make the Hotelling-like assumption that $\zeta_i = \zeta' + \frac{1}{2}\,\xi_i$, where ζ' is a constant, which I normalize to zero. Moreover, I assume that $\xi_i \sim N(0, \sigma)$. This implies that

$$E\left(\zeta_i \mid \xi_i > x(i)\right) = \left(\frac{\sigma}{2}\right)\left(\frac{\phi\left(x(i)\right)}{1 - \Phi\left(x(i)\right)}\right),$$

where $\phi(x)$ and $\Phi(i)$ are the standardized normal density and distribution functions, respectively.

Figure 3 shows equilibrium values for the particular case when $\theta(i) = \psi\, i^2/N,\ \lambda(i) = \psi\, i/N,\ \psi = 1,\ \sigma = 5$, and $\delta = .9$. Recall that the state space is given by $\{0, \ldots, N-1\}$, in this case $\{0, \ldots, 100\}$. Thus, this particular choice of N has the advantage that i also denotes firm i's market share. To understand the meaning of the particular $\theta(i)$ and $\lambda(i)$ mappings I choose, suppose that the value generated in the after market is given by $2\,\psi\,i/N$ per consumer. Suppose moreover that this value is equally split between buyer and seller. Then we obtain the expressions above.

The top left corner shows the price function, which is U shaped: decreasing at lower market shares, increasing for high market shares. In Section 3, particularly in Proposition 4, I showed that there are two forces that influence equilibrium price: short run market power and long run market power. The short run effect is that the larger a firm is the higher its price is. This corresponds to firms harvesting their market power. The long-run effect is the investment effect. To the extent that the firm's value function is convex (which can be seen in the middle right panel), the greater the firm's market share the greater the benefit from making a sale, in terms of future profits; and thus the grater incentive to reduce price. We thus have two opposite effects on price. The numerical simulation suggests that they combine into a U shaped function, with lowest prices around the symmetric states.

The top right panel confirms the results of Proposition 2: the greater market share, the greater the probability of attracting a new consumer. Notice that this result does not imply that the large firm will tend to become larger. In fact, larger firms also have larger death rates. The important comparison is between birth rates and death rates. More on this below.

The two middle panels show the consumer and the firm value functions. Notice that, since $\delta = .9$, a myopic consumer, that is, a consumer who assumes that the current market share persists forever, would have $u(100) \approx \lambda(100)/(1-\delta) = 1$. However, a rational consumer knows that a large firm is bound to become smaller. Accordingly, $u(100) < \lambda(100)/(1-\delta)$. Regarding the firm's value function, notice that it is convex in market share. This property is also found in the models of Budd, Harris and Vickers (1993) and Cabral and Riordan (1994). Essentially, it corresponds to the dynamic version of Gilbert and Newbery's (1982) efficiency effect: joint profits are increasing in asymmetry, or equivalently, a large firm has more to lose from decreasing its market share than a small firm has from gaining the same increment in market share.

Network effects imply that social welfare is greater the greater the asymmetry between firms. This can be seen from the bottom left panel. Finally, the bottom right panel shows the stationary distribution of market shares. Although the distribution is unimodal and centered around 50%, it is fairly

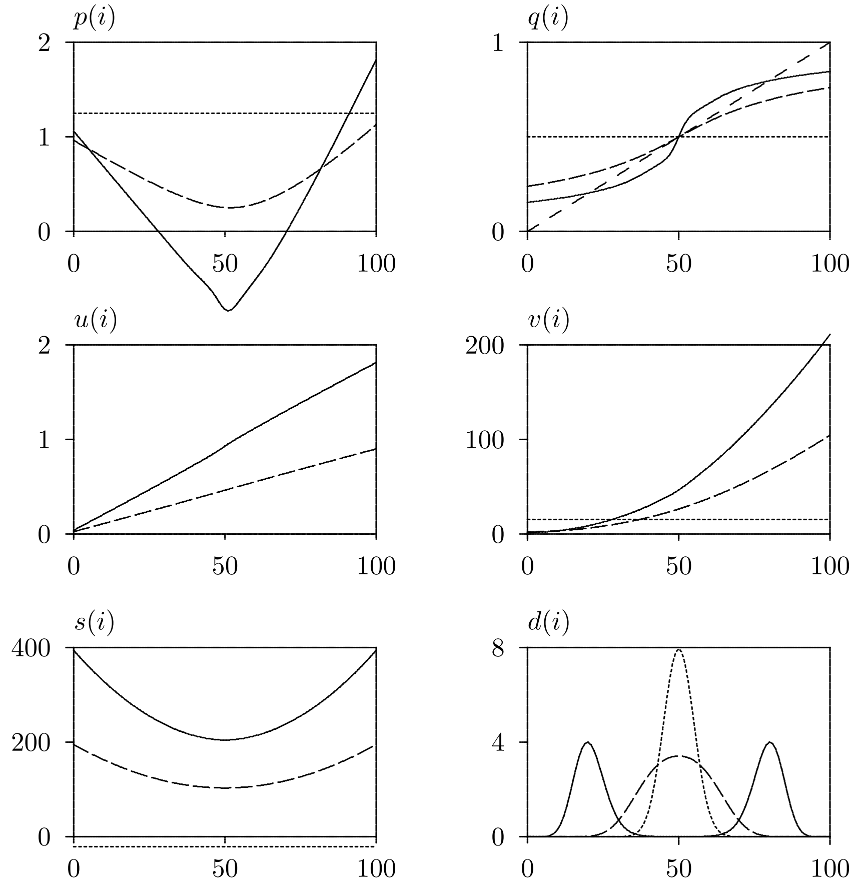

Figure 4: Equilibrium values as a function of state (horizontal axis) when F is normal with standard deviation σ and assuming $\theta(i) = \psi\, i^2/N$, $\lambda(i) = \psi\, i/N$, $\sigma = 1$, $\delta = .9$, $N = 101$. Three values of ψ are considered: 0 (dotted lines), .1 (dashed lines), .2 (solid lines).

dispersed. In other words, even though the equilibrium I consider is symmetric, most likely the actual state in each period is asymmetric. How asymmetric it is depends on the degree of network benefits, to which I turn next.

■ **Comparative dynamics: the effect of network effects.** The particular parametrization I consider, $\theta(i) = \psi\, i^2/N$ and $\lambda(i) = \psi\, i/N$ allows me to easily consider the effects of increasing network effects. Figure 4 plots equilibrium mappings for three values of ψ: 0, .1, and .2. As expected, if $\psi = 0$ then both the price function and the quantity function are flat (the latter at $\frac{1}{2}$). The same is true for the consumer and the firm's value functions (the former at zero).

As the degree of network effects increases, the firm's value function becomes increasing and convex. Notice that firm value does not increase uniformly in ψ. For low market shares a firm is worse off when ψ is higher, even though a higher ψ increases consumer willingness and firm aftermarket profits at every state. This apparently puzzling outcome, which basically results from a strategic effect that outweighs the direct effect, is referred to by Cabral and Villas-Boas (2005) as a *Bertrand supertrap*.[22]

Similarly to the value function, the price function also becomes more convex as ψ increases. This shift notwithstanding, the large firm's network becomes relatively more attractive to consumers. The price difference $p(i) - p(j)$ does not change that much as ψ increases; but the consumers utility difference $u(i+1) - u(j+1)$ does increase considerably. As a result, the demand function becomes steeper, as the top right panels shows. Ultimately, this implies a more dispersed stationary distribution. In particular, notice that as ψ goes from .1 to .2, the slope of the $q(i)$ function at the symmetric state becomes greater than 1. This implies that, for a market share greater than, but close to, the symmetric state, the birth rate is greater than the death rate. This confirms that Proposition 3 holds for higher values of the discount factor: around the symmetric state, the system tends to move away from the symmetric state, that is, increasing dominance holds. In terms of the stationary distribution, increasing dominance implies that the stationary distribution is bimodal. In fact, the modes of the stationary distribution are given by the points at which the $q(i)$ map crosses the main diagonal from above.

An alternative way to judge the impact of network effects is to generate typical time paths for the state space (market share). Figure 5 does that. Two times series are plotted, one corresponding to $\psi = 0$, one corresponding to $\psi = .2$. The two time series are constructed based on the same random series of ζ_{it}, time t's newborn consumer preference for network i. This allows for a better understanding of the relative role of randomness and network effects. (It also explains why the two time series are so positively correlated.)

Three values of market share are marked with dashed lines: 50%, the mode of the stationary distribution when $\psi = 0$; and the two modes of the asymmetric distribution corresponding to $\psi = .2$, approximately 20% and 80%. As expected, if there are no network effects ($\psi = 0$), then market share

[22]In Figure 4, $v(50)$ (that is, firm value at the symmetric state) seems to be increasing in ψ. However, there can be cases when $v(50)$ decreases with ψ. Cabral and Riordan's (1994) Theorem 3.5 corresponds to a Bertrand supertrap in a symmetric state in the context of dynamic competition with learning curves.

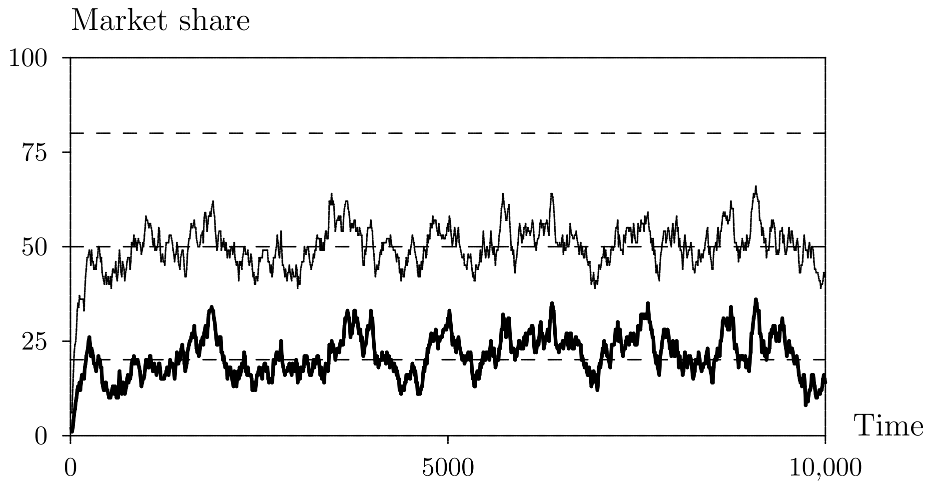

Figure 5: Time series of market share with network effects (think line) or no network effects (thin line). The dashed lines represent the modes of the stationary distribution of market shares (cf Figure 4).

is typically around 50%. Not so when network effects are significant. As can be seen, if $\psi = .2$, then, starting from $i = 0$, the system rapidly converges to one of the asymmetric states corresponding to the closest mode. Although there is significant inertia around this asymmetric state, tipping is possible (though it does not take place in the particular series plotted in Figure 5).

5 Network effects as a barrier to entry

The numerical solution of the model has two advantages over the analytical solution considered in Section 3. First, by choosing reasonable numbers for the various model parameters I can go beyond qualitative analysis and actually assign numbers to the variables of interest. Second, by performing a series of comparative dynamics exercises, I can uncover qualitative results that are difficult or impossible to obtain from the analytical solution. In this section, I present a policy experiment based on the numerical solution of the model. Specifically, I estimate the degree of barriers to entry implied by network effects.

There are several ways by which to measure barriers to entry.[23] I suggest two possible measures. The first one is the difference between the incum-

[23]See Cabral (2007) for a discussion on the concept of barriers to entry. See Llobet and Manove (2006) for a different perspective on the implications of network effects for entry.

bent's value, $v(100)$, and an entrant's value, $v(0)$. This is closest to Gilbert's (1989) notion of barriers to entry ("a rent that is derived from incumbency"). An alternative measure corresponds to comparing the entrant's value to the counterfactual value that would prevail if there were no network effects.

Not surprisingly, both measures are positive, that is, both measures indicate that network effects imply a barrier to entry.[24] At this stage, the relevant question is about size. For this purpose, it is useful to have an idea of what different parameter values mean. Accordingly, I construct a measure of "die-hard" fans of a network. Consider the following experiment: a new consumer must choose between a network of size zero and a network of size N. Suppose both networks set the same entry price. Suppose moreover that the new consumer is myopic, that is, assumes the current network size will persist into the indefinite future. A die-hard fan is one who chooses a network of size zero in these circumstances. Given the distribution of ξ_i, the measure of die-hard fans is given by

$$\Phi\left(\frac{-\psi/\sigma}{1-\frac{N-1}{N}\,\delta}\right),$$

where $\Phi(\cdot)$ is the standardized normal cumulative distribution function.[25]

An alternative measure to get a feel for the value of ψ is the mode of the stationary distribution. In fact, the greater the value of ψ, the closer the modes are to 0 and 100%. One problem with this measure is that, for an open set of value of ψ, the mode is equal to 50% even though there are significant changes in dispersion.[26]

In Table 3 I consider four different values of ψ: 0, .1, .2, .3. I present the values of the mode of the stationary distribution and the measure of die-hard fans. These help get a feel for what the different values of ψ mean. I then present the values of $v(0)$ and $v(100)$. Finally, I compute (in percent terms) the measures of barriers to entry introduced above.

Two facts are noticeable. First, there is a significant difference between the two measures of barriers to entry. Network effects lead to a value function

[24] Notice that neither the Bain nor the Stigler definitions would indicate a barrier to entry. See Gilbert (1989), Cabral (2006).

[25] Notice that, as expected, I have a degree of freedom in determining the monetary unit in which to express consumer benefit. Different units would change both ψ and σ, and so my measure is invariant to the particular unit chosen.

[26] Regarding the other parameters in the model, it can easily been shown that the churn rate at the symmetric state is given by $\frac{1}{2N}$. For the value of N I am considering, this comes to .5% per period. If the period is one quarter, this is roughly the churn rate observed in the U.S. wireless telecommunications industry.

Table 3: Network effects as a barrier to entry.

ψ	0	0.1	0.2	0.3
Mode st. dist. (%)	50	50	20	14
Die-hard fans (%)	50	17.9	3.3	.3
$v(0)$	3.13	2.37	1.69	1.43
$v(100)$	3.13	104.75	210.97	319.00
$\frac{v(100)-v(0)}{v(100)}$ (%)	0	98	99	100
$\frac{v(0)\vert_{\psi=0}-v(0)}{v(0)\vert_{\psi=0}}$ (%)	0	24	46	54

Assumptions: Normal F with $\sigma = 1$, $\theta(i) = \psi\, i^2/N$, $\lambda(i) = \psi\, i/N$, $\delta = .9$, $N = 101$.

that is strongly increasing and convex in i. As a result, the difference between $v(0)$ and $v(100)$ is quite significant, even for modest degrees of network effects. The effect of increasing network effects is greater in increasing $v(100)$ then in decreasing $v(0)$. Thus, the "counterfactual" measure of network effects, $v(0)\mid_{\psi=0} - v(0)$, is significantly smaller.

The second fact of notice is that both measures of barriers to entry are increasing in the degree of network effects. This is not very surprising. What is interesting is that, whereas $v(100) - v(0)$ is roughly proportional to ψ, $v(0)\mid_{\psi=0} - v(0)$ increases less than proportionally with respect to ψ.

An alternative way of measuring the difficulty to enter is to compute the average time it takes for an entrant to achieve a certain market share. Let $T(i, k)$ be expected time to reach k starting from $i < k$. This can be computed as follows. First notice that $T(k, k) = 0$. For $i < k$, we have

$$T(i, k) = m(i, i+1)\, T(i+1, k) + m(i, i)\, T(i, k) + m(i, i-1)\, T(i-1, k).$$

where $m(i, \ell)$ is the transition probability from i to ℓ. This forms a band linear system of k equations with k unknowns.

Solving this system I obtain the values in Figure 6. The vertical axis shows the expected time to first getting to a given state; the horizontal axis shows the specific state (market share). The qualitative conclusion from this figure is that network effects don't greatly affect the time it takes for an entrant to reach a low market share (say, 10%), but do significantly affect the time required for an entrant to achieve a 50% market share. To see this, consider the extreme cases of network effects I consider, $\psi = 0$ and $\psi = .3$. It takes an

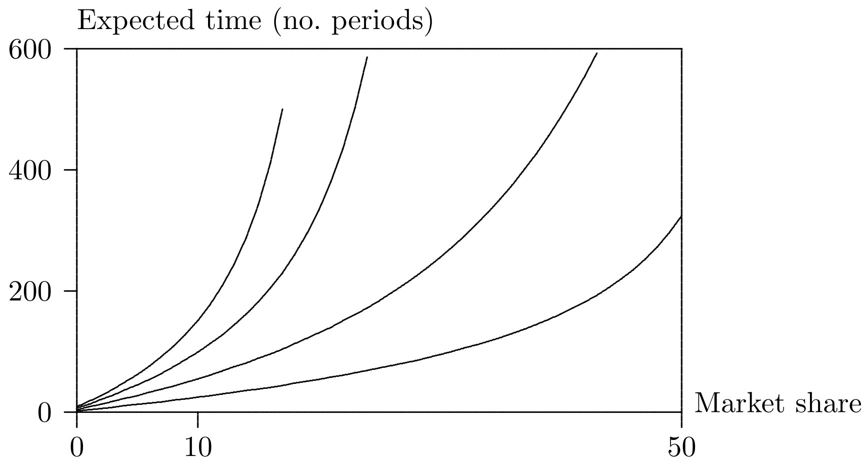

Figure 6: Expected time for an entrant to reach a given market share, assuming $\theta(i) = \psi\, i^2/N,\ \lambda(i) = \psi\, i/N,\ \sigma = 1,\ \delta = .9,\ N = 101$. Three values of ψ are considered: 0, 0.1, 0.2

average of 25 periods for an entrant to achieve a 10% market share if there are no network effects. With strong network effects, $\psi = .3$, it takes six times as long for the entrant to achieve the same market share. If however we consider how long it takes to get to a 50% market share, then the difference between $\psi = 0$ and $\psi = .3$ is a factor of 50,200,284 (the ratio 14,909,484,327/297)!

6 Conclusion

In this paper, I propose a novel framework with which to analyze the dynamics of price competition with network effects. I also use the framework to address a question of interest in industrial organization: the extent to which network effects imply a barrier to entry.

I believe that there are many other questions of interest that my framework, or extensions of it, can also address. In a followup paper, I study the impact of aftermarket power on installed base dynamics. Antitrust cases such as *Kodak* hinge on such effects; nevertheless, most of the extant analysis is based on static models. Also, in network industries such as wireless telecommunications, specific regulations in the aftermarket, such as the level of termination charges, are likely to have an impact on the evolution of market shares — an effect that, to the best of my knowledge, has not been addressed.

Another interesting question to study is the relation between private and social incentives for innovation. Suppose a new firm has a product that is better in the eyes of consumers but starts from a zero market share. Is the adoption of the new technology too slow or two fast from a social standpoint? By adopting the new technology, the new consumer imposes a negative externality on existing users; but by adopting the old technology the consumer may impose a negative externality on future users. In addition to these externalities, we must also consider the business stealing effect of entrant penetration. It is not clear what the net effect will be.

Appendix

Proof of Proposition 1: In order to consider the possibility of asymmetric equilibria, I add a firm subscript to the relevant functions. Let k and ℓ be the firm subscripts associated to the firms with networks of size i and j, respectively.

Firm k's first-order condition (6) then becomes

$$p_k(i) = h_k(i) - w_k(i), \tag{11}$$

where

$$\begin{aligned} h_k(i) &= \frac{1 - F\Big(p_k(i) - p_\ell(j) - u_k(i+1) + u_\ell(j+1)\Big)}{f\Big(p_k(i) - p_\ell(j) - u_k(i+1) + u_\ell(j+1)\Big)} \\ w_k(i) &= v_k(i+1) - v_k(i) \end{aligned}$$

Similar expressions hold for firm ℓ. Define

$$\begin{aligned} P_k(i) &\equiv p_k(i) - p_\ell(j) \\ H_k(i) &\equiv h_k(i) - h_\ell(j) \\ W_k(i) &\equiv w_k(i) - w_\ell(j) \\ U_k(i) &\equiv u_k(i+1) - u_\ell(j+1) \end{aligned}$$

where $j = N - 1 - i$. Subtracting firm j's first-order conditions from (11), I get

$$P_k(i) = H_k(i) - W_k(i) \tag{12}$$

Suppose that $\delta = 0$. Then

$$w_k(i) = \theta(i+1) - \theta(i)$$

and

$$U_k(i) = \lambda(i+1) - \lambda(j+1).$$

In other words, both $U_k(i)$ and $w_k(i)$ (and thus $W_k(i)$) are constants. It follows that (12) is an equation with one unknown, $P_k(i)$.

Specifically, the "hazard" function $H_k(i)$ is given by

$$\begin{aligned} H_k(i) &= \frac{1 - F\Big(P_k(i) - U_k(i)\Big)}{f\Big(P_k(i) - U_k(i)\Big)} - \frac{1 - F\Big(P_\ell(j) - U_\ell(j)\Big)}{f\Big(P_\ell(j) - U_\ell(j)\Big)} \\ &= \frac{1 - 2\,F\Big(P_k(i) - U_k(i)\Big)}{f\Big(P_k(i) - U_k(i)\Big)}, \end{aligned}$$

since $P_\ell(j) = -P_k(i)$, $U_\ell(j) = -U_k(i)$, and $F(-x) = 1 - F(x)$. We can therefore re-write (12) as

$$P_k(i) + \frac{2\,F\left(P_k(i) - \lambda(i+1) + \lambda(j+1)\right) - 1}{f\left(P_k(i) - \lambda(i+1) + \lambda(j+1)\right)} = \\ = -\theta(i+1) + \theta(i) + \theta(j+1) - \theta(j), \qquad (13)$$

Assumption 2 implies that the left-hand side of (13) is strictly increasing in $P_k(i)$, ranging from $-\infty$ to $+\infty$. The right-hand side, in turn, is a constant. The intermediate value theorem then implies that there exists a unique equilibrium value $P_k(i)$. Given symmetry of the $\theta(i)$ and $\lambda(i)$ functions, we conclude that the $P_\ell(j)$ function is identical to the $P_k(i) =$ function. Given $P_k(i)$, the values of $p_k(i)$ are uniquely determined by (11). Again, symmetry of the $\theta(i)$ and $\lambda(i)$ functions implies symmetry of the pricing function. Finally, the result follows by continuity around $\delta = 0$. ■

Proof of Proposition 2: Similarly to the proof of Proposition 1, I define (now without firm subscripts)

$$\begin{aligned} P(i) &= p(i) - p(j) \\ U(i) &= u(i+1) - u(j+1) \\ H(i) &= h(i) - h(j) \\ W(i) &= w(i) - w(j) \end{aligned}$$

Subtracting the first-order conditions, I have

$$P(i) = H(i) - W(i). \qquad (14)$$

By the same argument as in the proof of Proposition 1,

$$H(i) = \frac{1 - 2\,F\left(P(i) - U(i)\right)}{f\left(P(i) - U(i)\right)}$$

I can thus re-write (14) as

$$P(i) + \frac{2\,F\left(P(i) - U(i)\right) - 1}{f\left(P(i) - U(i)\right)} = -W(i) \qquad (15)$$

Suppose that $\delta = 0$. Suppose also that $U(i) = 0$. The left-hand side of (15) is strictly increasing in $P(i)$ and strictly positive if and only if $P(i)$ is strictly positive. It follows that, if $W(i) \geq 0$, then $P(i) \leq 0$, that is, $P(i) - U(i) \geq 0$ (as I assume $U(i) = 0$).

Let $H_U(i) \equiv \frac{\partial H(i)}{\partial U(i)}$ and $H_P(i) \equiv \frac{\partial H(i)}{\partial P(i)}$. By (13) and Assumption 2, $H_U(i) > 0$ and $H_P(i) = -H_U(i)$. Since $W(i)$ does not depend on $U(i)$, I can apply the Implicit Function Theorem to (14) to get

$$\frac{d\,P(i)}{d\,U(i)} = \frac{H_U(i)}{1 + H_U(i)} \in (0, 1).$$

It follows that

$$\frac{d\,\Big(P(i) - U(i)\Big)}{d\,U(i)} < 0. \tag{16}$$

Since $P(i) - U(i) \leq 0$ when $U(i) = 0$ and $P(i) - U(i)$ is strictly decreasing in $U(i)$, I conclude that $P(i) - U(i) \leq 0$ if $U(i) \geq 0$. Symmetry implies that $U(i) \geq 0$ if $i > i^*$. And $q(i) \geq \frac{1}{2}$ if and only if $U(i) \geq P(i)$. Finally, the result follows from continuity in δ. ■

Proof of Proposition 3: Suppose that $i^* < i < N - 1$. From (8), the state moves away from i^* in expected terms if and only if

$$\frac{N - 1 - i}{N}\ q(i) > \frac{i}{N}\ \Big(1 - q(i)\Big),$$

which is equivalent to

$$q(i) > \frac{i}{N - 1}. \tag{17}$$

In other words, the system moves away from i^* if and only if the leader's birth rate, $q(i)$, is greater that the leader's death rate, $\frac{i}{N-1}$.

In the proof of Proposition 2, I show that $P(i) - U(i)$ is strictly decreasing in $U(i)$. In fact, the derivative of $P(i) - U(i)$ with respect to $U(i)$ is bounded away from zero (from above). Since $q(i) = 1 - F\Big(P(i) - U(i)\Big)$ and $\frac{i}{N-1} < 1$, it follows that, if $U(i)$ is high enough, then (17) is satisfied.

In the proof of Proposition 4, I show that $P(i)$ is decreasing in $W(i)$. (I then assume that $U(i) = 0$. However, all I need is that $U(i)$ does not change with $W(i)$, which follows from $\delta = 0$.) In fact, the derivative of $P(i)$ with respect to $W(i)$ is bounded away from zero (from above). It follows that, if $W(i)$ is high enough, then (17) is satisfied. ■

Proof of Proposition 4: Suppose that $\delta = 0$. Consider first the case when $\theta(i) = \bar{\theta}$. We then get $w(i) = 0$, and thus $W(i) = 0$ as well. As shown in the proof of Proposition 2, $P(i) - U(i)$ is strictly decreasing in $U(i)$. Since $\lambda(i) > \lambda(i-1)$, it follows that $U(i)$ is strictly increasing, which in turn implies that $P(i) - U(i)$ is strictly decreasing. Finally, since $w(i) = 0$, (6) reduces to

$$p(i) = \frac{1 - F\left(P(i) - U(i)\right)}{f\left(P(i) - U(i)\right)},$$

which implies that $p(i)$ is strictly increasing in i.

Consider now the case when $\lambda(i) = \bar{\lambda}$. This implies that $U(i) = 0$. Let $H_U(i) \equiv \frac{\partial H(i)}{\partial U(i)}$ as before. Applying the Implicit Function Theorem to (14) we get

$$\frac{d\,P(i)}{d\,W(i)} = -\frac{1}{1 + H_U(i)} < 0.$$

Since $W(i) = w(i) - w(j)$, we have

$$\frac{\partial\,P(i)}{\partial\,w(i)} = \frac{\partial\,P(i)}{\partial\,W(i)}.$$

Moreover, since

$$h(i) = \frac{1 - F\left(P(i) - U(i)\right)}{f\left(P(i) - U(i)\right)}$$

and

$$H(i) = \frac{1 - 2\,F\left(P(i) - U(i)\right)}{f\left(P(i) - U(i)\right)}$$

we have

$$\frac{\partial\,h(i)}{\partial\,P(i)} = \frac{1}{2}\,\frac{\partial\,H(i)}{\partial\,P(i)} = -\frac{1}{2}\,H_U(i).$$

Differentiating the right-hand side of (6) with respect to $w(i)$, we get

$$\frac{d\,p(i)}{d\,w(i)} = \left(-\frac{1}{2}\,H_U(i)\right)\left(-\frac{1}{1 + H_U(i)}\right) - 1 < 0. \tag{18}$$

If $\delta = 0$, then $w(i) = \theta(i+1) - \theta(i)$, which is strictly increasing, by assumption. It follows by (18) that $p(i)$ is strictly decreasing in i. Finally, the results follow from continuity in δ. ∎

Proof of Lemma 1: As shown in the proof of Proposition 1, the equilibrium price differences $P(i)$ are given by the solution to

$$P(i) + \frac{2\,F\left(P(i) - U(i)\right) - 1}{f\left(P(i) - U(i)\right)} = -W(i)$$

Since $U(i)$ is only a function of various values of u and $W(i)$ is only a function of various values of v, it follows that, given $\{u(i), v(i)\}$, the above equation has only one unknown, $P(i)$. The left-hand side is an increasing function of $P(i)$, ranging from $-\infty$ to $+\infty$. The intermediate value theorem implies that there exists a unique equilibrium value $P(i)$. Given $P(i)$, the values of $p(i)$ are uniquely determined by (6).

The reverse is straightforward. In fact, for given $p(i)$ and $q(i)$, (4) defines a linear system in $u(i)$; and (7) defines a linear (recursive, in fact) system in $v(i)$. ■

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