

Incomplete Markets and the Evolution of US Consumer Debt*

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Abstract

Consumer debt has increased substantially in the US since the 1980s. We show in a incomplete-markets model with durables and occasionally binding collateral constraints that neither the higher uninsurable income risk of US consumers nor the financial deregulation explain this increase. The reason is that uninsurable risk increases the buffer-stock saving motive and that agents who are at the collateral constraint find it optimal not to hold durables. We find instead that the observed fall in the real interest rate by 2 percentage points explains 92% of the actual increase in consumer debt.

Keywords: durables, collateral constraint, income risk, household debt, incomplete markets, heterogeneous agents.

JEL: E21, D91.

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1 Introduction

US consumer debt has increased substantially during the last decades. Comparing the net-financial asset position of US consumers in the Survey of Consumer Finances 1983 and 2004,¹ shows that the average consumer debt increased from 124% to 206% of average labor income which corresponds to a 3.2% annual growth rate.² This increase in consumer debt has triggered a debate about its determinants. Iacoviello (2006) argues that the reason is the higher income risk of US consumers today. According to Campbell and Hercowitz (2006), the increase in debt is due to changes in financial regulation that have facilitated the use of durables as collateral.

Both papers study linearizations around a deterministic steady state and assume that a fraction of consumers is borrowing constrained.³ We argue that these assumptions are not innocuous for the results. In this paper we relax the assumptions and solve for the *non-linear* policy functions in an incomplete-markets model in which the steady state changes with income risk. Moreover, collateral constraints are occasionally binding in our model so that the incidence of the constraints is endogenously determined. Our incomplete-markets model with durables does not support the hypotheses of Iacoviello (2006) and Campbell and Hercowitz (2006). The reason is that uninsurable risk increases the buffer-stock saving motive as the consumption policies become more concave in total wealth and the steady-state net-financial wealth increases. Moreover, we find that consumers who are at the collateral constraint find it optimal not to hold durables so that it is irrelevant how much of the durable can be collateralized. If higher income risk or financial deregulation do not explain the increase in consumer debt in terms of our model, what else does explain it? We find that the observed fall in the real interest rate by 2 percentage points is most important, explaining 92% of the actual increase in consumer debt.

Another important contribution of this paper is that, relying on analytic results, we extend the endogenous grid-point method of Carroll (2006) to a problem with endogenous constraints and portfolio choice. This is not trivial and allows us to solve the model efficiently. Furthermore, we allow for an interest spread between the lending and the borrowing rate in financial markets to match the empirically observed fraction of consumers with zero financial assets. We show that such a spread breaks the global concavity of the non-durable and especially the durable consumption function. Carroll (2001, section 3) noticed the small effect of the spread on the non-durable consumption function in a model *without* durables. We find in a model *with* durables that the effect of the interest spread on the

¹The SCF is a triennial survey and both survey years are after a trough in the business cycle (1982 and 2001 according to the NBER definition) so that changes do reflect long-term trends rather than business-cycle variations.

²About 40% of consumers had a negative net-financial asset position in both years (40% in 1983 and 43% in 2004). Note that we normalize by labor earnings and not by disposable income so that these numbers are larger than those reported in Iacoviello (2006). Moreover, we do not look at gross debt which is reported in the flow of funds accounts of the Federal Reserve but use household data to net out the positive financial assets.

³Both papers also allow for aggregate risk and determine the interest rate endogenously. This is, however, not important for the difference of our results as long as average financial wealth can change in general equilibrium as in the Aiyagari (1994) model with capital accumulation. See Section 2 for further details.

propensity to purchase *durables* out of additional total wealth is sizeable. This is because durables are an alternative vehicle to transfer resources intertemporally, especially if depreciation rates are low. The result is interesting because it shows that it is possible to test for the importance of financial market imperfections, like interest spreads, using consumption data.

The rest of this paper is structured as follows. In Section 2 we present, solve and calibrate the model. In Section 3 we apply the model to study the determinants of the changes of consumer debt in the US since the 1980s. We discuss our results and conclude in Section 4.

2 The model

There is a continuum of risk averse consumers who have an infinite time horizon. They derive utility from a durable good d and a non-durable good c . The instantaneous utility is given by $U(c, d) = u(c) + w(d)$ where $u(\cdot)$ and $w(\cdot)$ are both strictly concave. We assume that the marginal utility $w'(d)$ is well defined at $d = 0$ so that our model is able to generate a mass of agents with no durable stock as is observed empirically. A possible functional form is $(d + \underline{d})^\nu$, with $\nu \leq 1$ and $\underline{d} > 0$. Durables can be transformed into non-durable consumption without cost so that the relative price is unity.

In specifying utility as above we have made a number of simplifying assumptions. We assume d to be a homogenous, divisible good where the service flow derived from durables is proportional to the stock, with the factor of proportionality normalized to 1. Moreover, as in much of the literature, utility is separable over time and at each point in time it is separable between durables and non-durables. Both assumptions are made for tractability⁴ (see Waldman, 2003, for a critical review of these common assumptions).

Labor income is stochastic. Markets are incomplete so that consumers cannot fully diversify this risk.⁵ As a consequence, consumers are heterogeneous ex post although they are identical ex ante: different histories of labor income shocks imply different total wealth positions. Consumers have access to two assets, a financial risk-free asset and a durable good. Durables like housing or cars generate utility but the durability of these goods also allows consumers to use them as collateral against which they can borrow. Since most household debt in the US (about 90%, Campbell and Hercowitz, 2005) is mortgage debt or other credit that is secured by collateral and cannot be defaulted upon, we assume that all credit needs to be collateralized.

⁴Non-separable utility would complicate the numerical solution of the model since the choice of non-durable consumption is no longer independent of the current durable stock. This would not allow us to formulate the value function in cash-on-hand as we do below and thus increases the computational burden. Quantitative results in Díaz and Luengo-Prado (2005) suggest that, for the Cobb-Douglas case, non-separability has little effect on the wealth distribution and portfolio composition of consumers.

⁵See the seminal papers of Deaton (1991), Carroll (1997) and the general equilibrium analysis of Aiyagari (1994) for models of incomplete markets and non-durable consumption; and Carroll and Dunn (1997), Díaz and Luengo-Prado (2005) or Gruber and Martin (2003) for models with durables.

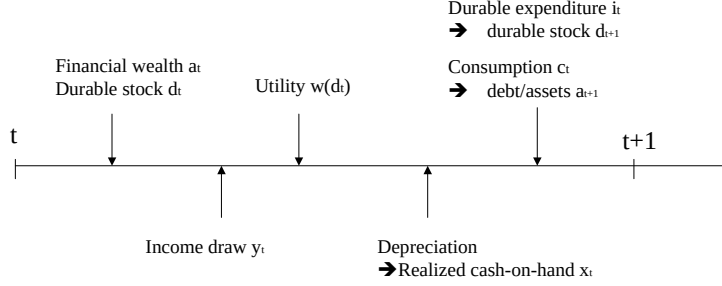


Figure 1: Timing in the model

There are transaction costs τ in the financial market so that the lending rate r^a is smaller than the borrowing rate r^b : $r^a < r^a + \tau = r^b$. This assumption implies that some consumers hold no financial assets, $a = 0$, as is the case in the data. As we will see below the spread has interesting implications for the consumption propensities and the shape of the policy functions.

Timing. We specify our model in discrete time so that we have to make assumptions about the timing. Figure 1 illustrates the time line. First uncertain exogenous labor income y_t is drawn. Then agents derive utility from the durable good d_t before the durable depreciates at rate δ . The agent then makes his choices on non-durable consumption c_t and durable investment i_t based on the available cash-on-hand

$$x_t \equiv (1 + r^j)a_t + y_t + (1 - \delta)d_t, \quad j = a, b,$$

where r^b is the interest rate on debt and r^a is the interest rate on financial assets a_t , with $r^b > r^a$. Note that financial assets a_t and the durable stock d_t are predetermined in period t and cannot be immediately adjusted.

The recursive formulation of the household problem. Rearranging the budget constraint,

$$c_t = (1 + r^j)a_t - a_{t+1} + y_t - \underbrace{(d_{t+1} - (1 - \delta)d_t)}_{\text{durable investment } i_t},$$

we can write the Bellman equation as

$$V(x_t, d_t, y_t) = \max_{a_{t+1}, d_{t+1}} \left[u(\underbrace{x_t - a_{t+1} - d_{t+1}}_{c_t}) + w(d_t) + \beta E_t V(x_{t+1}, d_{t+1}, y_{t+1}) \right] .$$

We can further simplify the problem by noting that d_t is predetermined in period t and that the additive separable term $w(d_t)$ does not affect the optimal choices of the consumer. Defining

$$\tilde{V}(x_t, y_t) \equiv V(x_t, d_t, y_t) - w(d_t)$$

the transformed maximization problem is

$$\tilde{V}(x_t, y_t) = \max_{a_{t+1}, d_{t+1}} \left[u(\underbrace{x_t - a_{t+1} - d_{t+1}}_{c_t}) + \beta w(d_{t+1}) + \beta E_t \tilde{V}(x_{t+1}, y_{t+1}) \right] \quad (1)$$

with the constraints

$$\begin{aligned} a_{t+1} &= \begin{cases} (1 + r_t^a) a_t + y_t - c_t - i_t & \text{if } a_t \geq 0 \\ (1 + r_t^b) a_t + y_t - c_t - i_t & \text{if } a_t < 0 \end{cases} \\ d_{t+1} &= (1 - \delta) d_t + i_t \\ \underbrace{\mu(1 - \delta) d_{t+1} + \underline{y}}_{\text{collateral}} &\geq -(1 + r^b) a_{t+1} \\ d_{t+1} &\geq 0 . \end{aligned}$$

The first two constraints are the accumulation equations for the financial wealth a and the durable stock d .⁶ The third constraint is the collateral constraint. This constraint ensures that the lowest attainable cash-on-hand guarantees full repayment (if income takes its smallest possible value $\underline{y}$ and agents can use a fraction $0 \leq \mu \leq 1$ of their durables as collateral⁷). The assumption here is that the lender, who lends at the risk-free rate, knows the financial position (a_t, d_t) and the minimum of the support of the income distribution $\underline{y}$. The lender does not know individual income draws.⁸ Note that whether and how much the collateral constraint binds in $t + 1$ is entirely determined by the choices in period t .

⁶We do not allow consumers to rent durables. This is not restrictive, however, since this option is weakly dominated by the choice of purchasing the durable: owned durables provide collateral in our model. This argument no longer applies with costs for adjusting the owned durable stock (see Krueger and Villaverde, 2005). Such costs would substantially increase the computational burden and, although important to match the frequency and size of investment *flows* in the data, add less insight to understand long-term trends in *stocks* which we are concerned with in this paper.

⁷The fraction μ plays no important role and is just introduced in the model to investigate empirically observed changes of the loan-to-value ratio in the calibration section.

⁸An alternative would be to allow for default and add a constraint which ensures that the consumer is at least as well off when he repays as when he defaults. We prefer the simpler constraint proposed in the paper because it simplifies the computations of the steady state as we do not have to compute the value function for the default case.

It is well known that in an environment with incomplete markets, existence of finite steady-state wealth requires consumers to be impatient, $\beta < 1/(1 + r^a)$, where β is the discount rate and r^a is the lending rate. It follows from the results by Deaton and Laroque (1992) that agents hold a finite amount of financial assets a . Because of positive depreciation δ and $\lim_{d \rightarrow \infty} w'(d) = 0$, also the durable stock d is bounded from above. The collateral constraint and $d \geq 0$ then imply a compact state space. Since problem (1) satisfies Blackwell's sufficient conditions (monotonicity and discounting) for a contraction mapping we can apply standard dynamic programming techniques to solve for the steady state (see Araujo et al., 2002, for existence proofs in a general equilibrium context). Because of stationarity, we drop time indexes and use primes “ ’ ” to denote a one-period lead (but for $u'(\cdot)$ or $w'(\cdot)$ which denote first derivatives of the instantaneous utility functions).

The steady state definition. A steady state is given by the policy functions for non-durable consumption $c(x, y)$, durable investment $i(x, y)$, the accumulation equations $a'(x, y)$ and $d'(x, y)$, and the evolution of the state variable $x'(x, y)$ so that for given prices $\{r^a, r^b\}$

- (i) the value function $\tilde{V}(x, y)$ attains its maximal value.
- (ii) the collateral constraint is not violated, i.e., $\underline{x}' \geq 0$.
- (iii) the durable stock is weakly positive, $d' \geq 0$.
- (iv) the distribution measure $\chi(X, Y)$ over the state space $X \times Y$ of agents is stationary, so that for a transition matrix $\Gamma(y'|y)$

$$\chi(X, Y) = \int_{X \times Y} I_{\{x' = x'(x, y)\}} \Gamma(y'|y) d\chi ,$$

where $I_{\{x' = x'(x, y)\}}$ is an indicator function which takes the value 1 if the statement in braces is true.⁹

Note that we assume that domestic changes in the demand for durable goods or the supply of financial assets do not affect the respective asset price. As in a small-open economy these prices are determined exogenously (on world markets). This assumption is not restrictive for our analysis since we will feed observed price changes into our model in the calibration. In order to see this, define the aggregate financial assets in the steady state

$$A_s(r, \dots) \equiv \int_{X \times Y} a(x, y) d\chi$$

and the aggregate durable stock as

$$D_d(p, \dots) \equiv \int_{X \times Y} d(x, y) d\chi,$$

where the subscripts s and d denote supply and demand, respectively, and p is the price of the durable. The dots summarize all other relevant parameters. Asset prices would be determined endogenously by

⁹See Ríos-Rull (1999) for further discussion on the restrictions of admissible income processes which satisfy monotone mixing or the American-dream / American-nightmare condition.

the two market-clearing conditions

$$A_s(r, \dots) = A_d(r, \dots)$$

and

$$D_d(p, \dots) = D_s(p, \dots) ,$$

where Walras' law implies market clearing for non-durable goods.

When calibrating our model to data in 1983 we assume that for the inverse functions $r(A_d, \dots)$ and $p(D_s, \dots)$

$$\frac{\partial r(A_d, \dots)}{\partial A_d} = 0 \text{ and } \frac{\partial p(D_s, \dots)}{\partial D_s} = 0 .$$

This assumption is by no means important for our results as we explain in a moment. Iacoviello (2006) makes the opposite extreme assumption that

$$\frac{\partial A_d(r, \dots)}{\partial r} = 0$$

so that changes in asset supply are only borne out in prices but not in quantities. By construction, income risk then does not change the aggregate amount of assets. This assumption would not hold in a closed production economy as in Aiyagari (1994) where income risk increases the aggregate capital stock.

The approach in this paper is to take the observed prices in 1983 as given when calibrating the steady state and then feed the *observed price changes* of r and p into the model when computing the model predictions for 2004. Importantly, the assumption on the elasticity of A_d with respect to r is actually *irrelevant* for the predicted change of aggregate financial assets A or durables D from 1983 to 2004. Given that we observe the change in prices r and p , the loci of $A_s(r, \dots)$ and $D_d(p, \dots)$ suffice to determine the equilibrium change of quantities and we can be agnostic about the shape of the functions $A_d(r, \dots)$ or $D_s(p, \dots)$. It is thus important to understand that taking prices as given in the calibration for 1983 is not important for our findings below. Modelling the respective other side of the market would allow, however, to learn more about what functions of $A_d(r, \dots)$ or $D_s(p, \dots)$ are consistent with the data and to compute predictions for the evolution of consumer debt in the future for which prices are unknown.

2.1 Euler equations and analytic results

For later reference, note that in the optimum

$$u'(c) = \beta(1 + r^a)E_y u'(c') ,$$

if the agent holds positive financial assets a' and durables d' , and

$$u'(c) = \beta(1 + r^b)(E_y u'(c') + \kappa)$$

if the agent holds debt and choices imply that the collateral constraint binds, $\kappa > 0$, so that the next period is entered with a portfolio at the collateral constraint. Because of the interest spread $r^b > r^a$, both Euler equations can be slack. In this case the intertemporal rate of substitution of non-durable consumption is in-between the lending and the borrowing rate:

$$1 + r^a < \frac{u'(c)}{\beta E_y u'(c')} < 1 + r^b ,$$

where $\kappa = 0$ since financial assets $a' = 0$.

In the optimum, durable investment is chosen so that it satisfies the condition

$$u'(c) = \beta ((1 - \delta) E_y u'(c') + w'(d') + \mu(1 - \delta)\kappa + \gamma) ,$$

where $\gamma \geq 0$ is the multiplier associated with the constraint $d' \geq 0$. As is intuitive, the agent aligns the marginal utility of foregone non-durable consumption today (resulting from durable investment) with the discounted marginal utility derived from the durable tomorrow and the additional marginal utility of non-durable consumption that is afforded by re-selling the durable good (taking into account its depreciation at rate δ).

Note that if the collateral constraint binds, $\kappa > 0$, present consumption is valued less and more resources are transferred to the future period. Defining “permissible income processes” as those processes which ensure that non-durable consumption and the durable stock remain in the domain over which $u(\cdot)$ and $w(\cdot)$ are defined (as in Carroll and Kimball, 1996), we can show the following

Remark 1: *If utility is separable in the durable d and non-durable consumption c , the instantaneous utility functions $u(\cdot)$ and $w(\cdot)$ are strictly concave, of the HARA family, and satisfy prudence so that $u'''(\cdot) \geq 0$ and $w'''(\cdot) \geq 0$, we can show:*

- (i) *If the constraints are not binding, $c(x, y)$, $d'(x, y)$ are globally concave, $a'(x, y)$ is globally convex in the first argument and $\partial c(x, y)/\partial x > 0$, $\partial d'(x, y)/\partial x > 0$. Moreover, $\partial a'(x, y)/\partial x \geq 0$ if $\delta = 1$, and under additional restrictions on concavity also for $0 \leq \delta < 1$.*
- (ii) *If the collateral constraint binds, $\partial a'(x, y)/\partial x$ falls and can become negative.*
- (iii) *If the Euler equations for the financial asset are slack, $c(x, y)$, $d'(x, y)$ are no longer globally concave and $a'(x, y)$ is no longer globally convex in x .*

Proof: see the Appendix I.

Remark 1(i) is an application of Theorem 1 in Carroll and Kimball (1996) to our model with durable and non-durable consumption. The global concavity of the non-durable and durable consumption functions in models of incomplete markets is very intuitive. Precautionary motives imply that the consumption propensity falls as agents have more cash-on-hand x . Moreover, if $\delta = 1$, financial assets

increase in x . For $\delta < 1$, durables become an alternative way of transferring resources to the future. Since the return of the durable depends on the marginal utility $w'(\cdot)$, the concavity of the durable utility function becomes important.

The intuition for Remark 1(ii) is that the possibility of a binding collateral constraint increases the amount of financial wealth a' for small values of x so that the slope $\partial a'(x, y)/\partial x$ is flatter. The optimality condition of borrowing agents

$$u'(c) = \beta(1 + r^b)(E_y u'(c') + \kappa)$$

illustrates that as κ falls with cash-on-hand x (the collateral constraint is less binding), $u'(c)$ decreases, ceteris paribus. The same holds for durable investment. The slope $\partial a'/\partial x$ can be negative if the propensity of non-durable and durable consumption out of additional total wealth is larger than 1 and the collateral constraint is relaxed as the durable stock increases.

The intuition for Remark 1(iii) is that the propensity to consume out of cash-on-hand has to increase if the Euler equation for the financial asset is slack since $a' = 0$ and $\partial a'/\partial x = 0$. Hence, the consumption propensities increase so that $\partial c/\partial x + \partial d'/\partial x = 1$. The consumption functions are no longer globally concave. Moreover, it follows from the optimality conditions (without multipliers for the constraints)

$$1 + r^a \leq 1 - \delta + \frac{w'(d')}{E_y u'(c')} \leq 1 + r^b$$

that the propensity for durables increases more. The inequalities show that the expected intra-temporal rate of substitution between durable and non-durable consumption tomorrow, $w'(d')/E_y u'(c')$, has to equal $(r^a + \delta)$ if the agent lends and $(r^b + \delta)$ if the agent borrows. If $a' = 0$, more cash-on-hand decreases the intratemporal rate of substitution until it equals $(r^a + \delta)$ where the consumer starts to accumulate positive financial wealth. Durables are an alternative, albeit imperfect, way to transfer resources intertemporally as long as the depreciation rate is not too high.¹⁰

2.2 Calibration and numerical results

In this section we calibrate our model to data in 1983 before we study the model's predictions for changes in consumer debt until 2004. We pick these two dates because they span the time period in which detailed comparable data on consumers' assets is recorded in the triennial survey of consumer finances (SCF). Both years, 1983 and 2004, are after a trough in the US business cycle (1982 and 2001 according to the NBER definition) so that changes reflect long-term trends rather than cyclical variation. Before we discuss the data and calibration in more detail, we briefly present the numerical algorithm to solve the problem of the previous subsection.

¹⁰That limited access to financial funds increases the propensity for durable and non-durable consumption is supported by empirical evidence of Alessie et al. (1997) for the period of financial deregulation in the UK in the 1980s. Bertola et al. (2005) provide an alternative microfoundation to explain the higher propensity for durable purchases if there is an interest spread $r^b > r^a$ and agents are liquidity constrained ($a' = 0$). In their model, a monopolist dealer has an incentive to lower the credit price of a durable good to attract liquidity constrained customers.

Numerical algorithm. It is well known that problems like ours do not have a closed-form solution for the optimal policies. Therefore, we pursue a numerical approach which relies on value function iteration and combine this with a substantial extension of the endogenous grid-point method (EGM) which has been proposed by Carroll (2006) for a much simpler problem. Using this hybrid algorithm speeds up the computations a lot. As explained further in the numerical appendix, the EGM requires that we rewrite the value function as $v(\tilde{x}, y)$ where total wealth $\tilde{x}$ is defined as¹¹

$$\tilde{x} = (1 + r^j)a + (1 - \delta)d$$

and the realized cash-on-hand is

$$x = \tilde{x} + y .$$

The endogenous collateral constraint, which is occasionally binding, forces us to use value function iteration for levels of total wealth at which the constraint is binding. In the numerical program we check for the critical level of total wealth at which the collateral constraint stops to bind. Exploiting the positive monotonicity of $\tilde{x}'(\tilde{x}, y)$ which follows from Remark 1, we can then adapt the EGM to our problem for total wealth above that critical level which increases the speed of the computations substantially. The application of the EGM is not straightforward because we need to distinguish various different first-order conditions as the Euler equation for financial assets might be slack. We provide further details in the Appendix II.

For the numerical solution we conveniently rely on the contraction properties of the Bellman operator where one of the main challenges for this technique is to get around the curse of dimensionality. This is where the formulation of the problem that reduces the number of state variables to the minimum pays off - by subsuming the portfolio positions in the single variable total wealth.

The endogenous state variable is total wealth and the exogenous state variable is the realization of uncertain labor income, which is modeled as a 3-state Markov chain. The range of total wealth, $\tilde{x}$, is restricted to an interval $[-\underline{y}, \bar{\tilde{x}}]$. The lower bound is given by the collateral constraint. Our choice of $\bar{\tilde{x}}$ guarantees that, for every $\tilde{x}$ and for every realization of income, the equilibrium policy will imply a value for $\tilde{x}$ tomorrow that remains within that interval.¹² We solve for the value function on a grid of 1,000 points over that interval. The grid is finer at the origin where the value function has more curvature. We use linear interpolation of the value function between these grid points.

A feature of our algorithm that greatly enhances the accuracy of our solutions, for the part of the grid on which the collateral constraint binds and we use standard value-function iteration, is that the maximizing choices for the policy (at each state and each iteration) are not selected from a discretized

¹¹This does not affect the results of Remark 1 or the first-order conditions above since y is an exogenous stochastic variable.

¹²In our algorithm, we choose the grid for total wealth so that for an upper bound of total wealth $\bar{\tilde{x}}$, the optimal policies imply that for the highest realization of income $\bar{y}$, $\tilde{x}'(\bar{\tilde{x}}, \bar{y}) < \bar{\tilde{x}}$. Using $\underline{\tilde{x}} = -\underline{y}$ as a lower bound gives us a compact state space (this bound is implied by the collateral constraint).

set of choices but rather by solving these maximization problems continuously over portfolio choices. We rely on a numerical optimization routine¹³ which can also handle the collateral constraint and sign restrictions, to perform this task and to obtain the implicit multipliers on the constraints. The policy functions over the range $\left[-\underline{y}, \bar{x}\right]$ are obtained from the optimal policy choices on the grid by linear interpolation.

As has become standard in the literature (see, e.g., Judd, 1992, and Aruoba et al., 2006), we evaluate the accuracy of our solutions by the normalized Euler equation errors implied by the policy functions. These are on average smaller than 10^{-3} over the entire range where the Euler equations apply with equality.

Calibration. We use data from the survey of consumer finances (SCF) to compute most of our calibration targets. The SCF has been widely used as it provides the most accurate information on consumer finances in the US. The data collectors of the Federal Reserve System pay special attention in their sampling procedures to accurately capture the right tail of the very right-skewed wealth distribution (see Kennickell, 2003, and the references therein). The data allow us to compute precise statistics for the consumer portfolio, in particular the fraction of consumers without durables or financial assets. The definitions for the constructed variables, using the SCF, are given in the Appendix III.

We normalize average labor income y to 1, and parametrize the instantaneous utility functions as

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \text{ and } w(d) = \phi \frac{(d + \underline{d})^{1-\sigma} - 1}{1-\sigma},$$

where, as mentioned above, $\underline{d} > 0$ allows the consumer to hold no durable stock. Table 1 displays how we calibrate the parameters of the model:

1. We fix risk aversion, the lending rate and the loan-to-value ratio exogenously. We set risk aversion for the non-durable and durable good $\sigma = 2$, which is well within the range of commonly used values (see, for example, Díaz and Luengo-Prado, 2005, or Chatterjee et al., 2005). We choose the lending rate $r^a = 0.04$ which approximates the long-run value in the 1980s (see, for example, Caporale and Grier, 2000). Finally we assume that 84% of the durable stock can be used as collateral, in line with the documented equity requirements in Campbell and Hercowitz (2006) for the period before the financial market deregulation in the US in the 1980s.

2. We calibrate a 3-state Markov-chain of labor income so that we match the first-order autocorrelation and Gini coefficient of labor income in the US. We normalize mean income to 1 and target a first-order autocorrelation of 0.95, consistent with results by Storesletten et al. (2004) based on estimates for the Panel Study of Income Dynamics (PSID). The matrix of transition probabilities is chosen as in Díaz and Luengo-Prado (2005), Table 1.

The target for the Gini coefficient of labor income is 0.533, based on the SCF 1983. In order to match this target we need to assume substantial income differences between states as is common in the

¹³We use the Matlab routine `fmincon()`.

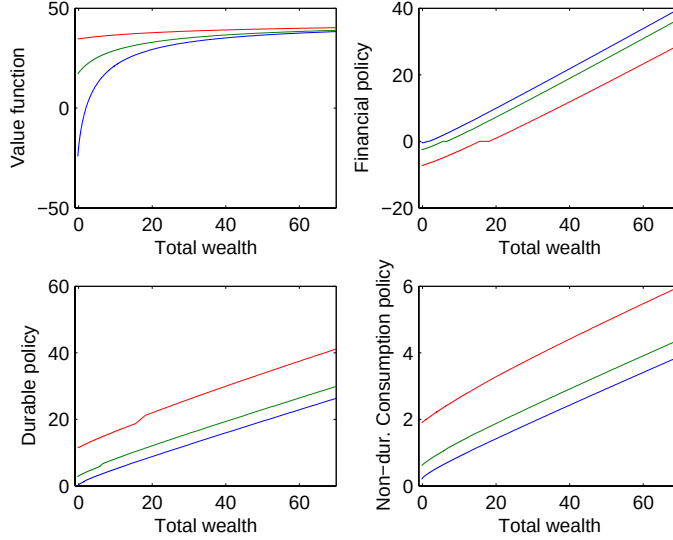


Figure 2: Value and policy functions in the three income states

literature (see Chatterjee et al., 2005, or Díaz and Luengo-Prado, 2005).

3. We jointly calibrate the remaining parameters τ , β , ϕ , $\underline{d}$ and δ . Although the parameters are jointly calibrated, in Table 1 we list the most related target statistic next to each parameter. We calibrate these parameters to match three aggregate statistics and two statistics from the distribution of financial wealth and durables in the SCF 1983. The aggregate statistics are the ratio of total wealth to average labor earnings (6.11), the ratio of durable wealth to average labor earnings (4.96) and the ratio of non-durable consumption to durable investment (6.25). The first two target statistics are computed using the SCF 1983 whereas the third target is constructed using the value of 6.14 reported in Díaz and Luengo-Prado (2005) and the NIPA Tables of the Bureau of Economic Analysis to adjust for the changes in the ratio since 1983. The other two distributional target statistics are that 4.9% of consumers hold zero net financial wealth and 8.5% of consumers hold no durable wealth (SCF 1983).

4. We check how well the calibration can match three other distributional statistics from the SCF 1983 which we did not target: a Gini of total wealth of 0.792, a Gini of financial assets of 0.956 and a Gini of durable wealth of 0.756.¹⁴

Value function and policy functions. Figure 2 displays the solution for the value function and the policy functions in the three income states. The value function is smooth and more concave in the bad income states. Not surprisingly, the function shifts down for lower levels of labor income. The policy

¹⁴Since financial assets and total wealth can be negative, the Gini index is corrected as proposed by Chen et al. (1982) so that its values lie between 0 and 1.

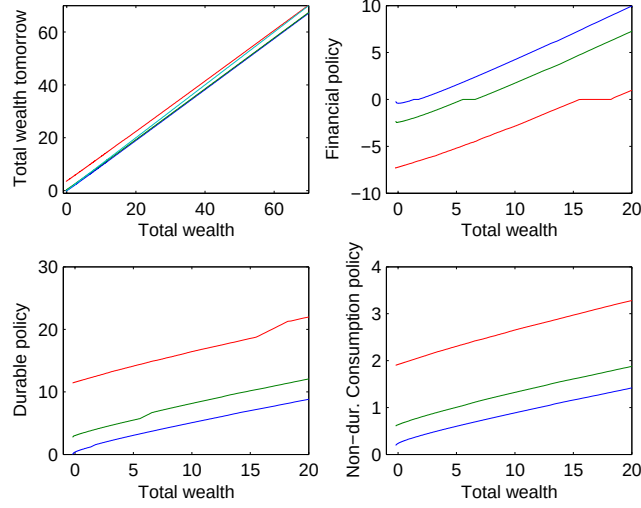


Figure 3: Total wealth accumulation and the local convexity of consumption policies

functions have a slightly non-standard shape consistent with the results of Remark 1. As is illustrated in more detail in Figure 3, financial assets $a = 0$ for an interval of total wealth because of the interest spread $r^b > r^a$. The strict concavity of the financial policy over a certain range of total wealth breaks the global concavity of the policy functions for non-durable consumption and the durable stock. This is hard to notice in the figures for non-durable consumption and is much more pronounced for the durable policy. The local convexity in the durable policy depends on whether the depreciation rate is low enough so that durables are a reasonably attractive vehicle to transfer resources intertemporally.¹⁵ The local convexity is more pronounced for the high-income state in the figure: strict concavity of the instantaneous utility function implies that more of the durable needs to be accumulated so that $w'(d')/E_y u'(c')$ falls enough until it is optimal to hold positive financial assets (see the discussion in Section 2.1).

Note that the constraint $d' = 0$ in the low income state if the collateral constraint is binding, too. This is the case if total wealth equals $-\underline{y}$. As we will see in the simulations below, agents decumulate durables in the bad income state before they hit the collateral constraint.

Although it is hard to see in the upper right panel of Figure 3, note that the policy for financial assets a falls in total wealth if the collateral constraint binds in the worst income state (inspect the upper graph within that panel). This is because the accumulation of durables relaxes the collateral constraint and allows the agent to borrow more.

Finally, the upper left panel in Figure 3 shows that total wealth is always below 65: this is where

¹⁵Since the utility from the durable is derived before depreciation, only the intertemporal transfer of utility for *non-durable* consumption is directly affected by depreciation.

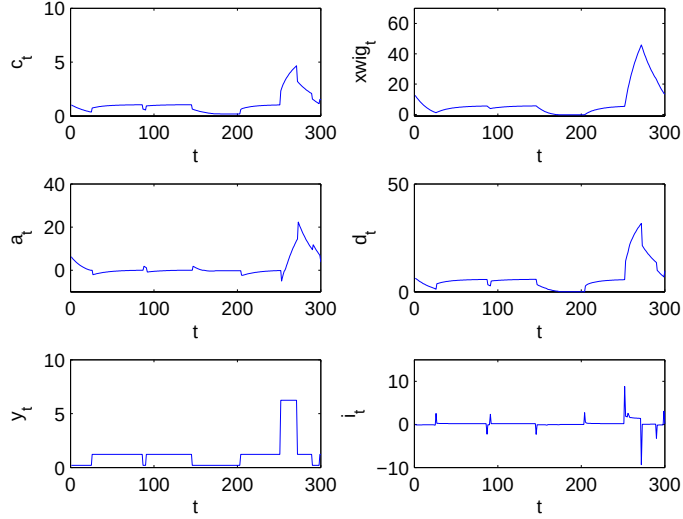


Figure 4: Time-series simulation of the economy

agents start to decumulate wealth even if they are in the best income state $\bar{y}$.¹⁶ In the figure the policy $\tilde{x}'(\tilde{x}, \bar{y})$ crosses the 45-degree line. We now simulate our economy to find out more about the means and distributions of the policy variables in the steady state.

Simulations. We simulate our economy for 10,000 periods. Figure 4 displays the results for an arbitrarily chosen subsample of 300 periods. As can be seen in the figure, the exogenous income shocks are quite persistent. Especially in the best income state, the agent accumulates substantial wealth both in terms of financial and durable assets. If the agent is in the worst income state long enough, he decumulates the durable stock and is collateral constrained. This result is crucial to understand why changes in the fraction of the durable that can be collateralized have no effect in our model, as we will see below. If consumers find it optimal to decumulate the durable before they are at the collateral constraint, then it is irrelevant for behavior how much of the durable can be collateralized once the consumers hit the constraint.

Consumers are collateral constrained 4.4% of the time which is below Jappelli's (1990) estimate of 20% for the US based on consumers' survey answers. Of course, the possibility of being constrained affects behavior of the consumer in our model more often than at the occasion when they are at the constraint. The value function incorporates the expectation that the collateral constraint can bind in the future with some probability, especially for low values of $\tilde{x}$.

Note that in our model income uncertainty prevents impatient consumers to be permanently at the

¹⁶This value of $\tilde{x}$ is well below the chosen maximum of the grid $\tilde{x}_{\max}$ in the numerical algorithm.

constraint. Income risk induces a buffer-stock saving motive so that, if income is persistently good, consumers accumulate a positive amount of financial assets $a > 0$. Concerning consumer debt, Figure 4 shows that consumers borrow against the lower bound of their labor income $\underline{y}$ if they are at the collateral constraint but also to accumulate durables if they receive a good income shock.

Steady-state statistics. Given that the income shocks are purely idiosyncratic, the law of large numbers implies that all idiosyncratic risk disappears upon aggregation (see Uhlig, 1996) and the time-series distribution can be used as an approximation of the cross-sectional distribution in the steady state. Table 2 displays the statistics in the steady-state equilibrium for the main variables of interest. In column (1) we display the empirical values of the statistics in 1983 which we compute using household data from the Survey of Consumer Finances. Column (2) shows the statistics of our model. All values are expressed in average labor-income equivalents if the statistics are not unitless. The calibration targets are matched quite closely but for the fraction of agents with no durables which is 12.3% in our model and 8.5% in the data. This is a limitation of having only 3 Markov states and persistent income shocks. In order to generate some agents with no durables we need to calibrate the model so that agents hold no durables in the worst income state as they run down their assets. With persistent shocks this occurs quite frequently *conditional* on being in the bad state.¹⁷

Concerning the statistics which we did not match explicitly, the Gini coefficient of net financial wealth (0.966) is very close to its empirical counterpart (0.956). The Gini coefficient of total wealth in the model (0.721) is smaller than in the data (0.792) because durables are more equally distributed in the model (0.607) than in the data (0.756). This is partly due to counting owned non-financial business assets as durables in the data: the Gini of durables without these assets would be lower at 0.68. Besides these measurement difficulties, our model reproduces the same ranking of the Ginis as in the data: financial wealth is most unequally distributed, followed by total wealth, then durables and finally labor earnings. This is intuitive since durable goods are also consumption goods in our model and thus should be more equally distributed than financial assets. The inequality in durable wealth is quite similar to the inequality in labor income because of the persistence of labor income which implies that current income is highly correlated with permanent income.

These results are in line with Díaz and Luengo-Prado (2005), although durables can be adjusted without cost in our model. One further difference in our paper is that we specify consumers' utility so that $d = 0$ is feasible. This difference implies that the inequality of durable wealth is slightly higher than in Díaz and Luengo-Prado (2005) since the wealth-poorest consumers hold no durables, as is the case empirically, medium-wealth consumers hold mostly durables and the wealth-richest consumers hold more financial than durable assets.¹⁸ The result that the wealth-poor consumers hold no durable

¹⁷With a more mobile income process it is easier to match the fraction of agents without durable wealth. In this case it becomes more difficult, however, to match the fraction of agents with no financial assets for plausible values of the interest spread. Results are available on request.

¹⁸This change of consumer portfolios with total wealth is similar to the findings of Krueger and Villaverde (2005) in a

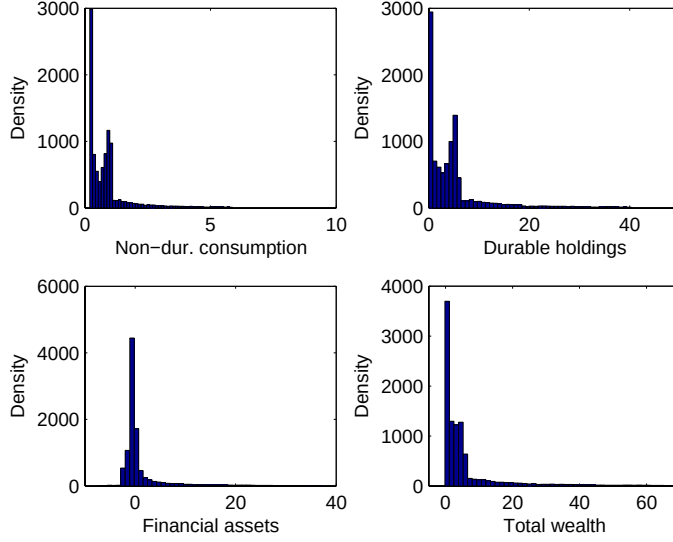


Figure 5: The steady-state distributions

wealth, if they have been in the worst income state for long enough, is important for understanding the small effect of changes in the fraction of durables that can be collateralized (the loan-to-value ratio) on the steady state in the next section.

Figure 5 displays the steady-state distributions of non-durable consumption c , durable holdings d , financial assets a , and total wealth $\tilde{x}$. All distributions are skewed to the right because the highest income state is attained by only 6% of the agents in the steady state. Moreover, total wealth is truncated at the left when the collateral constraint binds so that there is more probability mass at this truncation point. Thus, also the density of a has more mass at the lower bound of the support than would be the case without the constraint. Moreover, financial assets have a mass point at $a = 0$ when the financial-asset Euler equation is slack which occurs with a frequency of 4.9%, the fraction of US consumers who hold zero net-financial assets in the SCF 1983. The bimodality of the distribution of non-durable consumption and durables in the figure results mostly from the small number of persistently different income states.¹⁹ The bottom and middle income state occur with higher probability so that consumption and accumulation choices in the highest income state have less probability mass.

life-cycle model.

¹⁹The higher propensity to consume in the range where $a = 0$ also induces some of that bimodality.

3 On the evolution of consumer debt

We now apply our model to study the evolution of consumer debt in the US in the time period 1983 to 2004. We compare the steady-state distributions of the model for different parameters in 1983 and 2004. This steady-state comparison is not restrictive since the steady-state statistics are virtually identical if we change the model parameters to their values in 2004 and compute the transition for 21 periods starting from the steady state in 1983. Since there are no costs which hamper adjustment to the new steady state, we find that the statistics are already close to the steady state two periods after the unexpected parameter changes for the consumers.

We consider the following observed changes in the debt determinants as candidates which are summarized in Table 3: (i) an increase in labor income risk captured by a higher Gini coefficient of labor income. The Gini coefficients are 0.533 and 0.595 in the SCF 1983 and 2004, respectively (see also Gottschalk and Danziger (2005) and their references); (ii) a better use of the durable stock as collateral due to financial deregulation, as documented by the increase of the (maximum allowed) loan-to-value ratio from 84% to 89% (Campbell and Hercowitz, 2006);²⁰ (iii) a change in prices. We consider a fall of the real interest rate by 2 percentage points (see Caporale and Grier, 2000, and Caballero et al., 2006). We also reduce the interest spread from 1 to 1/2 percentage point. This change in the spread is an upper bound since the data reveal only a small reduction in the spread over time, especially for secured debt (see Davis et al., 2006, and Iacoviello and Pavan, 2006). Finally, we take the appreciation of durables into account. Since the largest part of durable assets is housing, we model the average 1.4% annual growth of real house prices since the 1980s (see Himmelberg et al., 2005) by lowering the depreciation rate of durables in our model from 2.9% to 1.5%.

Starting from the calibrated steady state for 1983, we compute the predicted effect of changes in the debt determinants on consumer debt for 2004. The results are summarized in Tables 4 and 5. Table 4 shows the results considering all changes together whereas Table 5 decomposes the overall effect considering each single determinant separately. Consumer debt in these tables is defined as the mean of the negative net-financial asset positions, that is $E(a|a < 0)$. This statistic has some advantages compared with the statistics on gross debt from the flow-of-funds accounts of the Federal Reserve, used by Iacoviello (2006). Many consumers hold debt and positive financial assets at the same time. Since our model has little to say on the size of the gross positions, we use the household-level data in the SCF to compute the net-financial asset position of consumers.

Columns (1) and (2) in Table 4 show that our model can explain 39% of the consumer debt position in 1983. The average net-financial asset position instead is matched precisely as it is one of the calibration

²⁰Campbell and Hercowitz discuss the reforms in financial markets that followed the Monetary Control Act of 1980 and the Garn-St.Germain Act of 1982. They distinguish between the initial equity share and the subsequent amortization rate. We refer to the documented changes of the required initial equity shares in their paper, since we do not model amortization rates in our paper.

targets. The actual and predicted changes are displayed in columns (3) and (4). In the data, the ratio of net financial assets over average labor income increased at an annualized rate of 2.8% between 1983 and 2004. The model instead would have predicted that the average net-financial asset position should have *decreased* at an annual rate of 8.5%. As we will see in the decomposition below this is mainly because of the substantial fall of the real interest rate which makes buffer-stock saving in terms of financial assets more expensive for impatient consumers. More interesting for our purposes is that the model predicts 9.4% annual growth in consumer debt in the period 1983-2004 which is nearly three times as high as the 3.2% growth rate of consumer debt observed in the data. This implies that our buffer-stock model explains a larger fraction of 70% of the total consumer debt holdings in 2004 (compare the debt reported in columns (3) and (4)).

Although our model predicts a substantial increase in consumer debt, the simultaneous drop in the net-financial asset position, which is opposite to what is observed in the data, points to important limitations of the model in explaining the evolution of the whole consumer wealth portfolio. We will discuss this further below but first decompose the predicted changes of the model into the effects of each debt determinant to understand the overall findings better.

Higher income risk. In Table 5, column (1), we isolate the effect of higher income risk. Higher income risk increases net-financial assets and reduces consumer debt because of the buffer-stock saving motive. This is not surprising since, in our model with incomplete markets, the motive to hold wealth is precisely to insure against income risk. If this risk is larger, the consumer is more willing to build up a costly stock of financial assets instead of consuming resources.

A larger loan-to-value ratio. Table 5, column (2), displays the effect of a larger loan-to-value ratio of 89%. This change has no quantitative effect on the steady state. The reason is that in our calibration consumers decumulate their durable stock in the worst income state when they start to borrow. When they hit the collateral constraint, they do not hold durables so that it is irrelevant for the debt holdings how much of the durable can be collateralized. Here the specification of the utility function is important that allows for $d = 0$. If $d > 0$ for the wealth-poorest consumers for whom the collateral constraint is most relevant, a higher loan-to-value ratio could have a stronger effect. This explains the different findings compared with Campbell and Hercowitz (2006).

We have shown so far that the first two candidate explanations that have been proposed in the literature do not explain the increase consumer debt which we observe in the data. We now turn to the effects of price changes on consumer debt.

A lower real interest rate. Table 5, column (3) shows the increase of consumer debt after a fall of the real interest rate from 4% to 2%. Comparing Table 5, column (3), and Table 4, column (4), one can see that most of the quantitative change in the financial asset position and consumer debt is due

to this change of the real interest rate.

A smaller interest spread. In Table 5, column (4), we reduce the interest spread to 0.5%. This implies very little change in the financial asset positions between 1983 and 2004. Quantitatively, the effect on debt is smaller also because not all agents hold financial debt in the steady state.

Durable value appreciation. In Table 5, column (5), we look at the effect of an appreciation of durables. This makes durables more attractive compared with financial assets so that the stock of net financial assets falls and consumer debt increases. The magnitude of this increase is smaller than for the fall of the real interest rate but also contributes to explaining the increase in consumer debt.

The bottomline of our results is that the changes in prices observed in the data are the most promising explanation for the increase of consumer debt in the US since the 1980s. Of course, this finding is qualitatively not very surprising (a fall in the price increases demand) but the interest here is on the quantitative explanatory power of the model.²¹ In terms of the absolute change in consumer debt observed in the data (0.82, the difference of columns (1) and (3) in Table 4 expressed in average labor-income equivalents), the fall of the real interest rate by 2 percentage points explains 92% of that increase (compare the difference in consumer debt in Table 4, column (2), and Table 5, column (3)). The hypotheses of Iacoviello (2006) or Campbell and Hercowitz (2006) instead are not supported by our incomplete-markets model.²² Neither higher income risk nor financial deregulation explain the rise in consumer debt observed in the data. The reason is that (i) income risk changes the steady-state financial asset position due to buffer-stock savings and (ii) financial deregulation (that allows for easier use of durables as collateral) is irrelevant if consumers find it optimal to decumulate the durable before hitting the collateral constraint. This result shows the importance of determining the incidence of the collateral constraint *endogenously* within the model if one analyzes the effect of changes in the financial market structure on household debt.

One should be careful, however, before declaring victory for the incomplete-markets model. Comparing the predictions of the model with the data in Table 2, columns (3) and (4), one can see that the model fails to match the evolution of consumer wealth in some important respects. The model underpredicts changes in total wealth because it fails to match the increase in financial wealth. The fall of financial wealth predicted by the model also reduces the Gini coefficient of financial wealth which is

²¹The results are in line with Barnes and Young (2003) who argue that the decline of real interest rates in the 1990s can explain part of the increase in household debt in the US. The focus of that paper, however, is on demographic changes which are analyzed in an overlapping generations model.

²²Iacoviello (2006), Appendix A, also considers a variant of his model without aggregate shocks in which agents borrow both for consumption smoothing and precautionary motives. In this case the effect of income risk depends on the strength of the two motives which imply opposite effects on savings or debt.

not borne out in the data. This suggests that buffer-stock saving alone is not enough to explain the evolution of aggregate financial wealth holdings so that extending the model to other saving motives seems promising. This does not imply that buffer-stock saving is not a reasonable description for consumers that borrow and indeed our model is doing quite well in explaining changes in aggregate consumer debt. The model also captures the increase of durable wealth but overpredicts the increase by 1/2 percentage point of annual growth.

The model does not explain the observed decline of the ratio of non-durable consumption over durable investment. The ratio increases in the model because we compare steady states and implicitly assume that the appreciation of durables is a long-run phenomenon that will continue in the future. Hence, consumers invest less into durables, as the lower depreciation rate outweighs the positive effect of a higher steady-state durable stock, and the ratio of non-durable consumption to durable investment increases. In the data instead, the appreciation of durables, especially in the period 2001-2004, has been accompanied by more housing investment. If we consider the transition between the two steady states in our model, we can capture this effect since the consumption-investment ratio falls right after an unexpected appreciation of the durable.

Finally, higher income risk in the model implies that consumers hit the collateral constraint more often in 2004 than in 1983. Since consumers decumulate durables before they are at the constraint we find a larger fraction of consumers with zero durable wealth which is not observed in the data. This shortcoming of the model is due to the parsimonious shock structure with 3 Markov states.

4 Discussion and conclusion

We have studied and solved a heterogeneous-agent model in which income risk cannot be insured and consumers derive utility from non-durable and durable consumption. Consumers only have access to secured debt that is collateralized by durables. We have characterized the shape of the policy functions analytically and have extended the endogenous grid-point method by Carroll (2006) to develop a fast numerical algorithm to solve our model with occasionally binding collateral constraints and interest spreads. We then have applied the model to investigate possible determinants for the changes in consumer debt in the US since the 1980s. We have found that higher income risk or financial deregulation do not explain the increase of consumer debt in our model. Instead the observed fall of the real interest rate by 2 percentage points can explain 92% of the absolute increase in consumer debt. These results are in contrast with findings of Campbell and Hercowitz (2006) and Iacoviello (2006). The different results of our model can be traced back to the precautionary-saving motive and occasionally binding collateral constraints where consumers find it optimal hold to decumulate durable wealth before hitting the constraint.

Our results do not imply that income risk may not explain the increase in consumer debt in different modeling environments in which the access to borrowing funds and idiosyncratic risk interact endoge-

nously. For example, in Krueger and Perri (2005), limited enforcement of credit contracts implies that financial market development interacts with income volatility. If more volatile income makes the exclusion from credit markets more costly in case of default, this might foster financial market development. In this case, more volatile income will induce a higher buffer-stock of financial assets but with respect to a laxer borrowing limit. Whether this implies more or less debt depends on which effect dominates quantitatively and is *a priori* unclear. In current research we extend our model in this direction and allow for limited commitment in credit contracts and costly default. This is particularly interesting because unsecured debt has increased substantially during the 1990s in the US so that a joint analysis of secured and unsecured debt and their determinants is warranted.

Appendix

I. Proof of Remark 1

The proof is based on results of Carroll and Kimball (1996). In order to simplify notation we drop income y_t as an argument of the functions. We also drop time indexes and primes that denote the next period in the proofs to simplify notation. Whenever we use d or a , this corresponds to a' and d' in Remark 1.

Claim (i): If the constraints are not binding, $c(x)$, $d(x)$ are concave and $a(x)$ is convex and $\partial c(x)/\partial x > 0$, $\partial d(x)/\partial x > 0$, $\partial a(x)/\partial x \geq 0$.

Proof: We want to show that if $u(\cdot)$ and $w(\cdot)$ are HARA utility functions and $u'(\cdot) > 0$, $u''(\cdot) < 0$, $u'''(\cdot) \geq 0$, and $w'(\cdot) > 0$, $w''(\cdot) < 0$, $w'''(\cdot) \geq 0$, then $c(x)$, $d(x)$ are concave and $a(x)$ is convex and $\partial c(x)/\partial x > 0$, $\partial d(x)/\partial x > 0$, $\partial a(x)/\partial x \geq 0$.

Our problem is

$$\tilde{V}_t(x_t) = \max_{a_{t+1}, d_{t+1}} \left[u(\underbrace{x_t - a_{t+1} - d_{t+1}}_{c_t}) + \beta w(d_{t+1}) + \beta E_t \tilde{V}_{t+1}(x_{t+1}) \right]$$

where $x_t \equiv (1 + r^j)a_t + y_t + (1 - \delta)d_t$ so that the budget constraint

$$c_t = x_t - a_{t+1} - d_{t+1}.$$

To start we also assume a finite horizon so that we have the terminal condition

$$c_T = x_T.$$

We then proceed analogously as in Carroll and Kimball and prove Lemmas 1-3. For this we define as $\xi_t((1 + r^j)a_{t+1}(x_t) + (1 - \delta)d_{t+1}(x_t)) \equiv \beta E_t \tilde{V}_{t+1}(x_{t+1})$, where

$$x_{t+1} \equiv (1 + r^j)a_{t+1} + y_{t+1} + (1 - \delta)d_{t+1}.$$

Note that $\xi_t(\cdot)$ is written as a function of choice variables.

The first lemma shows that the property of prudence is conserved when aggregating across states of nature.

Lemma 1: If $\tilde{V}_{t+1}''' \tilde{V}_{t+1}' / [\tilde{V}_{t+1}'']^2 \geq k$, then $\xi_t''' \xi_t' / [\xi_t'']^2 \geq k$.

Proof: see Carroll and Kimball, p. 985.

The second lemma shows that the property of prudence is conserved when aggregating intertemporally.

Lemma 2: If $\xi_t''' \xi_t' / [\xi_t'']^2 \geq k$ and $u''' u' / [u'']^2 \geq k$, $w''' w' / [w'']^2 = k$, then $\tilde{V}_t''' \tilde{V}_t' / [\tilde{V}_t'']^2 \geq k$.

Proof: Following Carroll and Kimball, p. 985/986, we denote the marginal utility of non-durable consumption at the optimal consumption level with $z_t = u'(c_t^*(x_t))$. Neglecting the collateral constraint and interest spread, we know that in our problem the following equations hold in the optimum:

$$\begin{aligned} z_t &= u'(c_t^*(x_t)) , \\ u'(c_t^*(x_t)) &= \tilde{V}'_t(x_t) , \\ u'(c_t^*(x_t)) &= \beta(1+r^j)E_t\tilde{V}'_{t+1}(x_{t+1}) = (1+r^j)\xi'_t , \\ u'(c_t^*(x_t)) &= \beta w'(d_{t+1}) + (1-\delta)\xi'_t , \end{aligned}$$

where $\xi_t((1+r^j)a_{t+1}(x_t) + (1-\delta)d_{t+1}(x_t))$. We then define the functions $f_t(z_t)$, $g_t(z_t)$, $h_t(z_t)$, $l_t(z_t)$ as

$$\begin{aligned} f_t(z_t) &= u'^{-1}(z_t) = c_t , \\ h_t(z_t) &= \tilde{V}'^{-1}_t(z_t) = x_t , \\ l_t(z_t) &= w'^{-1}\left(\frac{z_t - (1-\delta)\xi'_t(\cdot)}{\beta}\right) = d_{t+1} , \\ g_t(z_t) &= \xi'^{-1}_t\left(\frac{z_t}{1+r^j}\right) - (1-\delta)l_t(z_t) = (1+r^j)a_{t+1} . \end{aligned}$$

Noting from the last equation that

$$(1+r^j)a_{t+1} + (1-\delta)d_{t+1} = \xi'^{-1}_t\left(\frac{z_t}{1+r^j}\right) ,$$

we use this expression as the argument of $\xi'_t(\cdot)$ in the second equation which then simplifies to

$$l_t(z_t) = w'^{-1}\left(\frac{r^j + \delta}{\beta(1+r^j)}z_t\right) = d_{t+1}$$

Dropping time indexes for functions f , g , l , h , we have

$$\begin{aligned} f'(z) &= \frac{1}{u''(c(z))} , \\ f'' &= -\frac{u'''(c)}{[u''(c)]^2} \underbrace{f'}_{\partial c / \partial z} = -\frac{u'''}{[u'']^3} , \end{aligned}$$

so that

$$-\frac{zf''}{f'} = \frac{u'''u'}{[u'']^2} \geq k .$$

Similarly,

$$-\frac{zh''}{h'} = \frac{\tilde{V}'''_t\tilde{V}'_t}{[\tilde{V}''_t]^2} .$$

Furthermore,

$$\begin{aligned} l' &= \frac{r^j + \delta}{\beta(1+r^j)w''} , \\ l'' &= -\frac{(r^j + \delta)w'''}{\beta(1+r^j)[w'']^2}l' , \end{aligned}$$

so that

$$-\frac{zl''}{l'} = \frac{w'''w'}{[w'']^2} \geq k ,$$

where we use that

$$\frac{r^j + \delta}{\beta(1+r^j)} z_t = w'(d_{t+1}) .$$

Finally,

$$\begin{aligned} g' &= \frac{1}{(1+r^j)\xi'' \left(\xi_t'^{-1} \left(\frac{z_t}{1+r^j} \right) \right)} - (1-\delta) \frac{r^j + \delta}{\beta(1+r^j) w''} , \\ g'' &= -\frac{\xi'''}{(1+r^j)^2 [\xi'']^3} + (1-\delta) \frac{(r^j + \delta) w'''}{\beta(1+r^j) [w'']^2} l' . \end{aligned}$$

Thus,

$$-\frac{zg''}{g'} = \frac{\frac{\xi'''\xi'}{(1+r^j)[\xi'']^3} - (1-\delta) \frac{w'''w'}{[w'']^2} l'}{\frac{1}{(1+r^j)\xi''} - (1-\delta)l'} .$$

For $\delta = 1$, this simplifies to

$$-\frac{zg''}{g'} = \frac{\xi'''\xi'}{[\xi'']^2} \geq k ,$$

For $0 < \delta < 1$,

$$-\frac{zg''}{g'} = \frac{g'}{g' - (1-\delta)l'} \frac{\xi'''\xi'}{[\xi'']^2} - \frac{(1-\delta)l'}{g' - (1-\delta)l'} \frac{w'''w'}{[w'']^2} .$$

If we assume HARA utility so that $w'''w'/[w'']^2 = k$, then $\xi_t'''\xi_t'/[\xi_t'']^2 \geq k$ implies that

$$-\frac{zg''}{g'} \geq \frac{g'}{g' - (1-\delta)l'} k - \frac{(1-\delta)l'}{g' - (1-\delta)l'} k = k .$$

Now note that since

$$c_t = x_t - a_{t+1} - d_{t+1}$$

and

$$a_{t+1} = \frac{g}{(1+r^j)} - (1-\delta)l ,$$

we have

$$\begin{aligned} h &= f + \frac{g}{(1+r^j)} - (1-\delta)l + l \\ &= f + \frac{g}{(1+r^j)} + \delta l . \end{aligned}$$

That is, h is an additive function of f , g and l , so that

$$h' = f' + \frac{g'}{(1+r^j)} + \delta l'$$

and

$$h'' = f'' + \frac{g''}{(1+r^j)} + \delta l'' .$$

This implies that

$$\begin{aligned}
-\frac{zh''}{h'} &= -z \frac{f'' + \frac{g''}{(1+r^j)} + \delta l''}{f' + \frac{g'}{(1+r^j)} + \delta l'} \\
&= \underbrace{\frac{f'}{f' + \frac{g'}{(1+r^j)} + \delta l'}}_{>0} \underbrace{\left(-\frac{zf''}{f'}\right)}_{\geq k} + \underbrace{\frac{\frac{g'}{(1+r^j)}}{f' + \frac{g'}{(1+r^j)} + \delta l'}}_{>0} \underbrace{\left(-\frac{zg''}{g'}\right)}_{\geq k} + \underbrace{\frac{\delta l'}{f' + \frac{g'}{(1+r^j)} + \delta l'}}_{>0} \underbrace{\left(-\frac{zl''}{l'}\right)}_{\geq k} \\
&\geq k,
\end{aligned}$$

since this is a weighted average of expressions that are larger or equal than k .

As in Carroll and Kimball we move on to show Lemma 3, where we exploit again that HARA utility implies $w'''w'/[w'']^2 = k$ and $u'''u'/[u'']^2 = k$ with equality.

Lemma 3: If $\tilde{V}_t''' \tilde{V}_t' / [\tilde{V}_t'']^2 \geq k$, $w'''w'/[w'']^2 = k$ and $u'''u'/[u'']^2 = k$, then the optimal consumption policy rules $c(x)$ and $d(x)$ are concave and liquid assets $a(x)$ are convex.

Proof: Note that

$$c_t(x) = f_t(h_t^{-1}(x)).$$

Thus,

$$\frac{\partial c}{\partial x} = \frac{f'(h^{-1})}{h'(h^{-1})} = \frac{\tilde{V}''}{u''} > 0$$

if $u'' < 0$, $\tilde{V}'' < 0$ and

$$\begin{aligned}
\frac{\partial^2 c}{\partial x^2} &= \frac{(f''(h^{-1})/h'(h^{-1})) (h'(h^{-1})) - (f'(h^{-1})) (h''(h^{-1})/h'(h^{-1}))}{[h'(h^{-1})]^2} \\
&= \frac{f'(h^{-1})}{[h'(h^{-1})]^2} \left[\frac{f''(h^{-1})}{f'(h^{-1})} - \frac{h''(h^{-1})}{h'(h^{-1})} \right].
\end{aligned}$$

Applying Lemma 2 we find

$$\frac{\partial^2 c}{\partial x^2} = \frac{f'(h^{-1})}{[h'(h^{-1})]^2} \frac{1}{z} \left[\underbrace{-\frac{zh''(h^{-1})}{h'(h^{-1})}}_{\geq k} - \underbrace{\frac{-zf''(h^{-1})}{f'(h^{-1})}}_{=k} \right].$$

The sign of this derivative is smaller or equal than zero if $\text{sgn}(f'(h^{-1})) < 0$. Recalling that $f'(h^{-1}) = f'(z) = 1/u'' < 0$, this is the case for a strictly concave utility function. Analogous manipulations for $d_t(x) = l_t(h_t^{-1}(x))$ prove $\partial d(x)/\partial x > 0$ and $\partial^2 d(x)/(\partial x)^2 \leq 0$.

Since $a_{t+1}(x) = x_t - c_t(x) - d_{t+1}(x)$,

$$\frac{\partial a}{\partial x} = 1 - \frac{\partial c(x)}{\partial x} - \frac{\partial d(x)}{\partial x}$$

and

$$\frac{\partial^2 a}{\partial x^2} = -\frac{\partial^2 c(x)}{\partial x^2} - \frac{\partial^2 d(x)}{\partial x^2} \geq 0.$$

Thus, financial wealth increases or decreases with x , depending on whether the marginal propensity to consume $\partial c(x)/\partial x + \partial d(x)/\partial x \gtrless 1$. The second derivative is certainly positive so that $a(x)$ is convex.

We now investigate the properties of the consumption propensities further. In particular, do we know whether $\partial c(x)/\partial x + \partial d(x)/\partial x > 1$?

Noting that

$$h' = f' + \frac{g'}{(1+r^j)} + \delta l'$$

we can write

$$\frac{\partial c}{\partial x} = \frac{f'(h^{-1})}{f'(h^{-1}) + \frac{g'(h^{-1})}{(1+r^j)} + \delta l'(h^{-1})}$$

and

$$\frac{\partial d}{\partial x} = \frac{l'(h^{-1})}{f'(h^{-1}) + \frac{g'(h^{-1})}{(1+r^j)} + \delta l'(h^{-1})}.$$

Thus,

$$\frac{\partial c}{\partial x} + \frac{\partial d}{\partial x} = \frac{f'(h^{-1}) + l'(h^{-1})}{f'(h^{-1}) + \frac{g'(h^{-1})}{(1+r^j)} + \delta l'(h^{-1})} < 1,$$

if $\delta = 1$ and $g'(h^{-1}) > 0$.

We now compute the derivative of $a(x) = g(h^{-1}(x))/(1+r^j)$:

$$\begin{aligned} \frac{\partial a}{\partial x} &= \frac{1}{1+r^j} g'(h^{-1}(x))/h'(h^{-1}(x)) \\ &= \frac{\tilde{V}_t''}{1+r^j} \left(\frac{1}{(1+r^j)\xi''} - (1-\delta) \frac{r^j + \delta}{\beta(1+r^j)w''} \right), \end{aligned}$$

which is certainly positive if $\delta = 1$ since $\tilde{V}_t'' < 0, \xi'' < 0$. For $\delta < 1$, we need to impose an additional condition on the curvature

$$\begin{aligned} \frac{1}{(1+r^j)\xi''} - (1-\delta) \frac{r^j + \delta}{\beta(1+r^j)w''} &< 0 \text{ or} \\ \frac{\xi''}{\beta w''} &< \frac{r^j + \delta}{1-\delta}. \end{aligned}$$

In general the sign of $\partial a/\partial x$ depends on the relative curvature of the value function expected tomorrow, ξ_t'' , and instantaneous utility derived from the durable, w'' . Intuitively, a larger δ makes durables less useful to transfer utility and thus increase the marginal propensity of financial assets to transfer resources.

The lemmas derived above imply Theorem 1 as in Carroll and Kimball (1996). Note that the second-order derivatives for the policy functions hold with strict equality if $k > 0$ and there is some labor income uncertainty.

Carroll and Kimball show results for a finite horizon. In a finite horizon, we have that in the last period $V_T = u(c) + w(d)$ so that prudence of $u(\cdot)$ and $w(\cdot)$ trivially also apply to V_T . Then one iterates forward using Lemma 1 and 2. To extend these results to the infinite horizon one needs to apply the contraction property of V , for $T \rightarrow \infty$. Since cash on hand is finite, agents discount and V satisfies monotonicity, $\lim_{T \rightarrow \infty} V_T(x) = V(x)$ for all x (see Stokey and Lucas, 1989, ch. 3). Pointwise convergence implies that the properties of V_t are conserved as V_t converges towards V . ■

Claim (ii): If the collateral constraint binds, $\partial a(x)/\partial x$ falls and can become negative.

Proof: Intuitively, the value function will be more concave if the collateral constraint holds. The expression for the propensities derived above, then imply that $\partial c(x)/\partial x + \partial d(x)/\partial x$ increases if $\tilde{V}''$ falls (i.e., increases in absolute value). This can imply $\partial a(x)/\partial x < 0$, which we now want to derive more formally. Adding the multiplier κ for the collateral constraint and γ for the constraint $d > 0$, the four equations used in Lemma 2 change to

$$\begin{aligned} z_t &= u'(c_t^*(x_t)) , \\ u'(c_t^*(x_t)) &= \tilde{V}_t'(x_t) , \\ u'(c_t^*(x_t)) &= (1 + r^j)(\xi_t' + \kappa) , \\ u'(c_t^*(x_t)) &= \beta w'(d_{t+1}) + (1 - \delta)(\xi_t' + \mu\kappa) + \gamma , \end{aligned}$$

so that

$$\begin{aligned} f_t(z_t) &= u'^{-1}(z_t) = c_t , \\ h_t(z_t) &= \tilde{V}_t'^{-1}(z_t) = x_t , \\ l_t(z_t) &= w'^{-1}\left(\frac{z_t - (1 - \delta)(\xi_t' + \mu\kappa) - \gamma}{\beta}\right) = d_{t+1} , \\ g_t(z_t) &= \xi_t'^{-1}\left(\frac{z_t}{1 + r^j} - \kappa\right) - (1 - \delta)l_t(z_t) = (1 + r^j)a_{t+1} . \end{aligned}$$

Observing that

$$(1 + r^j)a_{t+1} + (1 - \delta)d_{t+1} = \xi_t'^{-1}\left(\frac{z_t}{1 + r^j} - \kappa\right) ,$$

the third equation can be rewritten as

$$l_t(z_t) = w'^{-1}\left(\frac{\frac{r^j + \delta}{1 + r^j}z_t - (1 - \mu)(1 - \delta)\kappa - \gamma}{\beta}\right) = d_{t+1} .$$

Thus, a binding collateral constraint does directly affect d_{t+1} only if $\mu < 1$. Instead if the constraint $d = 0$ is expected to bind this lowers $w'(d_{t+1})$ and thus induces a larger d_{t+1} , ceteris paribus.

More interestingly, let us investigate how the marginal propensity of $a(x)$ changes if the collateral constraint is binding (we neglect the constraint $d \geq 0$ for simplicity). Recall that $a(x) = g(h^{-1}(x))/(1 + r^j)$:

$$\begin{aligned}\frac{\partial a}{\partial x} &= \frac{1}{1 + r^j} g'(h^{-1}(x))/h'(h^{-1}(x)) \\ &= \frac{\tilde{V}_t''}{1 + r^j} \left(\frac{\frac{1}{1+r^j} - \frac{\partial \kappa}{\partial z}}{\xi''} - (1 - \delta) \frac{r^j + \delta}{\beta(1 + r^j) w''} \right) .\end{aligned}$$

Since a larger $z = u'(c^*(x))$ means a smaller c and x , $\partial \kappa / \partial z > 0$, i.e. the collateral constraint is more binding for smaller x and thus larger z . Then, this derivative shows that the propensity $\partial a / \partial x$ falls if the collateral constraint binds in the next period. In particular, the propensity need no longer be positive. The intuition is that the binding collateral constraint increases the amount of financial wealth for small values of x so that the slope is flatter. ■

Claim (iii): If the Euler equations for non-durable consumption are slack, $c(x)$, $d(x)$ are no longer globally concave and $a(x)$ is not globally convex.

Proof: We show that $c(x)$, $d(x)$ are locally strictly convex and $a(x)$ is locally strictly concave in the range where $a = 0$. In particular, $\partial c(x) / \partial x|_{a=0} > \partial c(x) / \partial x$ and $\partial d(x) / \partial x|_{a=0} > \partial d(x) / \partial x$ for given x , and $w'(d')/E_y \mu'$ falls.

$$\text{If } a_{t+1}(x) = 0,$$

$$c_t = x_t - d_{t+1}$$

and thus

$$h = f + l .$$

Hence,

$$-\frac{zh''}{h'} = \underbrace{\frac{f'}{f' + l'}}_{>0} \underbrace{\left(-\frac{zf''}{f'}\right)}_{\geq k} + \underbrace{\frac{l'}{f' + l'}}_{>0} \underbrace{\left(-\frac{zl''}{l'}\right)}_{\geq k}$$

so that the curvature of $w(\cdot)$ becomes much more important for the curvature of the value function.

Also

$$\frac{\partial c}{\partial x} + \frac{\partial d}{\partial x} = \frac{f'(h^{-1}) + l'(h^{-1})}{f'(h^{-1}) + l'(h^{-1})} = 1 ,$$

so that the propensities increase since $\partial a(x) / \partial x > 0$ to the left of the range where $a(x) = 0$. The local increase of the propensities implies local convexity of the consumption functions. Moreover, $\partial a(x) / \partial x > 0$ is locally concave.

More formally, if $\partial a(x) / \partial x = 0$, the collateral constraint is certainly not binding and

$$\begin{aligned}
z_t &= u'(c_t^*(x_t)) , \\
u'(c_t^*(x_t)) &= \tilde{V}_t'(x_t) , \\
(1+r^a)\xi_t' &< u'(c_t^*(x_t)) < (1+r^b)\xi_t' \\
u'(c_t^*(x_t)) &= \beta w'(d_{t+1}) + (1-\delta)\xi_t' ,
\end{aligned}$$

so that

$$\begin{aligned}
f_t(z_t) &= u'^{-1}(z_t) = c_t , \\
h_t(z_t) &= \tilde{V}_t'^{-1}(z_t) = x_t , \\
l_t(z_t) &= w'^{-1}\left(\frac{z_t - (1-\delta)\xi_t'(\cdot)}{\beta}\right) = d_{t+1} , \\
g_t(z_t) &= \xi_t'^{-1}(z_t + \lambda^b) - (1-\delta)l_t(z_t) = (1+r^b)a_{t+1}
\end{aligned}$$

or

$$g_t(z_t) = \xi_t'^{-1}(z_t - \lambda^a) - (1-\delta)l_t(z_t) = (1+r^a)a_{t+1}$$

with $\lambda^a > 0$ and $\lambda^b > 0$.

This implies

$$\begin{aligned}
\frac{\partial a}{\partial x} &= \frac{1}{1+r^b} g'(h^{-1}(x))/h'(h^{-1}(x)) \\
&= \frac{\tilde{V}_t''}{1+r^b} \left(\frac{1 + \frac{\partial \lambda^b}{\partial z}}{(1+r^b)\xi_t''} - (1-\delta) \frac{r^b + \delta}{\beta(1+r^b)w''} \right) .
\end{aligned}$$

For the range $a_{t+1}(x) = 0$, $\partial \lambda^b / \partial z < 0$ so that $\partial a / \partial x = 0$ (Note that $\partial \lambda^b / \partial x > 0$). Similarly, for the lending Euler-equation,

$$\frac{\partial a}{\partial x} = \frac{\tilde{V}_t''}{1+r^b} \left(\frac{1 - \frac{\partial \lambda^a}{\partial z}}{(1+r^b)\xi_t''} - (1-\delta) \frac{r^b + \delta}{\beta(1+r^b)w''} \right) ,$$

with $\partial \lambda^a / \partial z > 0$ (Note that $\partial \lambda^a / \partial x < 0$). ■

II: Numerical appendix: extending the endogenous grid-point method.

Extending the method of endogenous grid-points (EGM), proposed by Carroll (2006), allows for an efficient and accurate solution of our model. Carroll's method avoids root-finding and/or maximization operations to solve for the optimal endogenous state next period given this period's state. Instead, by specifying an exogenous grid for the state variable in the next period, the first-order conditions are used to determine the endogenous grid of the state variable in this period implied by the optimal choice. The challenge of extending this idea to the problem in our paper is to ensure that the mapping from tomorrow's endogenous state to today's state is unique. It turns out that this is only the case if the collateral constraint is not binding. Moreover, the optimal mapping from tomorrow's endogenous state to today's state depends on the possible slackness of the other constraints. This implies that we need to compute the endogenous grid conditional on whether each of these other constraints binds or not, and to determine which case is optimal given this period's state.

We need to distinguish the following 8 cases:

A. Binding collateral constraint

The collateral constraint can be binding only if $a' < 0$. We need to distinguish two cases:

1. $a' < 0$ and $d' > 0$
2. $a' < 0$ and $d' = 0$

B. If the collateral constraint is not binding, we have additional 6 cases:

3. $a' < 0$ and $d' > 0$
4. $a' = 0$ and $d' > 0$
5. $a' > 0$ and $d' > 0$
6. $a' < 0$ and $d' = 0$
7. $a' = 0$ and $d' = 0$
8. $a' > 0$ and $d' = 0$.

We first show that the EGM fails if the collateral constraint is binding (cases 1 and 2). We then show that the collateral constraint is binding in a closed interval of total wealth $\tilde{x}$. Thus, we still can employ the EGM for $\tilde{x}$'s for which the collateral constraint is not binding.

In order to recover the endogenous state variable today, let us define the deterministic part of cash-on-hand, which we call total wealth, as

$$\tilde{x}' = (1 + r^j)a' + (1 - \delta)d'$$

where the realized cash-on-hand is

$$x' = \tilde{x}' + y'.$$

We rewrite the Bellman equation as a function of $\tilde{x}$ so that

$$v(\tilde{x}, y) = \max_{a', d'} \left[u(\underbrace{\tilde{x} + y - a' - d'}_c) + \beta w(d') + \hat{v}(\tilde{x}'(a', d'), y) \right],$$

where we denote the expected value of next period's value function as

$$\widehat{v}(\widetilde{x}'(a', d'), y) \equiv \beta E v(\widetilde{x}', y') .$$

Note that y is required as state variable not only for the transition probabilities of the income process, as before, but also to compute the realized cash-on-hand x . The first-order conditions without a binding constraint are

$$c = \left((1 + r^j) \frac{\partial \widehat{v}}{\partial \widetilde{x}'} \right)^{-\frac{1}{\sigma}}$$

and

$$d' = \frac{1}{\beta \phi} \left((r^j + \delta) \frac{\partial \widehat{v}}{\partial \widetilde{x}'} \right)^{-\frac{1}{\sigma}} - \underline{d} .$$

Cases 1 and 2: binding collateral constraint. If the collateral constraint binds, $a' < 0$, $d' \geq 0$. With a binding collateral constraint the two relevant Euler equations have an additional multiplier κ

$$c = \left((1 + r^j) \left(\frac{\partial \widehat{v}}{\partial x'} + \kappa \right) \right)^{-\frac{1}{\sigma}}$$

and

$$c = \left(\beta \phi (d' + \underline{d})^{-\sigma} + (1 - \delta) \frac{\partial \widehat{v}}{\partial \widetilde{x}'} + (1 - \delta) \kappa \right)^{-\sigma} .$$

We know further that

$$(1 + r^b) a' = -(1 - \delta) d' - \underline{y}$$

or

$$\widetilde{x}' = -\underline{y} .$$

Note that $\widetilde{x}' = -\underline{y}$ is consistent with a variety of values for $\widetilde{x}$ or x at which the collateral constraint binds (there can be an interval of values even conditional on y). We only know that, for all of these values of $\widetilde{x}$, $a'(\widetilde{x})$ and $d'(\widetilde{x})$ have to satisfy

$$\underbrace{(1 + r^b) a' + (1 - \delta) d'}_{\widetilde{x}'} = -\underline{y} .$$

Hence, we cannot recover the endogenous grid-points for $\widetilde{x}$ from the exogenous grid $\widetilde{x}'$.

We now show that we can still use the EGM for values of total wealth $\widetilde{x}' > -\underline{y}$ for which the collateral constraint is not binding. We exploit the theorems 4.6, 4.8 and 4.11 in Stokey and Lucas (1989), which imply that the value function is strictly concave, continuous and differentiable, and the policies are continuous functions. We also know from Remark 1 that $\widetilde{x}'(\widetilde{x})$ is monotonically increasing at each iteration over the value function if the collateral constraint is not binding (this follows from $h' < 0$, $\partial c / \partial x > 0$ and strict concavity of the instantaneous utility). At most the slope of $\widetilde{x}'(\widetilde{x})$ can be zero: by assumption, if $\widetilde{x} = -\underline{y}$ and the collateral constraint is binding so that $\widetilde{x}' = -\underline{y}$, then the slope of $\widetilde{x}'(\widetilde{x})$ is zero for $\widetilde{x} = -\underline{y}$. This allows us to prove

Remark 2: *The collateral constraint is binding in the closed interval $\tilde{x} \in [-\underline{y}, \tilde{x}^+]$.*

Proof: We prove this by contradiction using the positive monotonicity of $\tilde{x}'(\tilde{x})$ which follows directly from Remark 1. Note that $\tilde{x}^+$ is the largest value of $\tilde{x}$ for which the collateral constraint is binding so that $\tilde{x}'(\tilde{x}^+) = -\underline{y}$. Assume then that for some $\tilde{x}^* < \tilde{x}^+$ the collateral constraint is not binding. This immediately contradicts positive monotonicity because then $\tilde{x}'(\tilde{x}^*) > \tilde{x}'(\tilde{x}^+)$ but $\tilde{x}^* < \tilde{x}^+$. ■

Note that Remark 2 allows us to “solve forward” for the value function in the interval $\tilde{x} \in [-\underline{y}, \tilde{x}^+]$ using standard value function iteration. Once we find an $\tilde{x}^+$ at which the collateral constraint is no longer binding, the positive monotonicity of $\tilde{x}'(\tilde{x})$ implies that this is also the case for all larger values of $\tilde{x}^+$. We then apply the endogenous grid-point method for all $\tilde{x}' > \tilde{x}'(\tilde{x}^+)$. The outer envelope of the value function for all $\tilde{x}' > \tilde{x}'(\tilde{x}^+)$ is found by checking, for all cases and for each endogenous grid-point $\tilde{x}$, whether the reverse mapping $\tilde{x}(\tilde{x}')$ for a given case results in an optimal mapping back into the same exogenous point $\tilde{x}'(\tilde{x})$ for that very case. If not, that specific endogenous grid-point is not relevant.

We now show how we compute the endogenous grid-points for cases 3-8.

Cases 3 and 5: Specifying an exogenous grid for $\tilde{x}'$ allows us to compute c and d' using the first-order conditions

$$c = \left((1 + r^j) \frac{\partial \hat{v}}{\partial \tilde{x}'} \right)^{-\frac{1}{\sigma}}$$

and

$$d' = \frac{1}{\beta\phi} \left((r^j + \delta) \frac{\partial \hat{v}}{\partial \tilde{x}'} \right)^{-\frac{1}{\sigma}} - \underline{d}.$$

Now we can back out a' since

$$a' = \frac{\tilde{x}' - (1 - \delta)d'}{(1 + r^j)}.$$

Finally, we get the endogenous grid of $\tilde{x}$ using

$$\tilde{x} = c + a' + d' - y,$$

where y is known.

Case 4: slack financial-asset Euler equation. In this case we cannot use the first-order condition for non-durable consumption. But since $a' = 0$, the equation for total wealth now is a one-to-one mapping between planned total wealth $\tilde{x}'$ and durables d' so that for a given $\tilde{x}'$ we get

$$d' = \frac{\tilde{x}'}{1 - \delta}.$$

We can use the Euler equation for durables to determine c ,

$$c = \left(\beta\phi (d' + \underline{d})^{-\sigma} + (1 - \delta) \frac{\partial \hat{v}}{\partial \tilde{x}'} \right)^{-\frac{1}{\sigma}}.$$

Then we get the endogenous grid for $\tilde{x}$ using

$$\tilde{x} = c + d' - y .$$

Cases 6 and 8: slack durable Euler equation. In these cases $d' = 0$. Then, the Euler equation for the financial asset determines c

$$c = \left((1 + r^j) \frac{\partial \hat{v}}{\partial \tilde{x}'} \right)^{-\frac{1}{\sigma}}, j = a, b$$

and financial assets are given by

$$a' = \frac{\tilde{x}'}{(1 + r^j)}, j = a, b$$

so that the endogenous grid is determined by

$$\tilde{x} = c + a' - y .$$

Case 7: slack durable and financial-asset Euler equation. In this case, $a' = d' = 0$ and $\tilde{x}' = 0$. That is there is only one point of $\tilde{x}'$ consistent with this case, and

$$\tilde{x} = c - y .$$

That is, $c = x = \tilde{x} + y$. Because $a' = d' = 0$, there is no link between $\tilde{x}'$ and $\tilde{x}$ and the grid of $\tilde{x}$. However, we still can compute the value function for this case and check whether this case is optimal.

We thus implement the hybrid value-function-iteration-EGM algorithm in the following steps:

1. We use a starting guess for the value function over a grid of 1,000 points for total wealth.
2. Starting from a lower bound of the grid, we maximize (continuously) the right-hand side of the Bellman equation and check whether the collateral constraint is binding at each grid point. If the constraint no longer binds at the point $\tilde{x}^+$, we switch to EGM. Due to Remark 2, we can be sure that the collateral constraint does not bind for any $\tilde{x} > \tilde{x}^+$
3. Applying the EGM, we
 - (a) compute the policies c , a' , d' and the endogenous grid $\tilde{x}$ for each case.
 - (b) compute the value functions on the relevant endogenous grid for each case. Here we exploit the positive monotonicity of $\tilde{x}'(\tilde{x})$ again to find the upper bound of the endogenous grid over $\tilde{x}$, by checking which case optimally maps back the candidate upper bound for $\tilde{x}$ into the upper bound of the exogenous grid $\tilde{x}'$.
 - (c) choose the upper envelope of the computed value functions and interpolate linearly.

Since the derivative $\partial \hat{v} / \partial \tilde{x}'$ is important for determining the endogenous grid, we smooth the value function and its derivative using a shape-preserving cubic Hermite polynomial on a subset of grid-points.

4. We restart from 1. with the updated value function as guess and continue until convergence is achieved at a precision of 10^{-7} .

This combination of forward optimization of the right-hand side of the Bellman equation and the backward optimization by the EGM greatly enhances the efficiency of our algorithm because we need standard forward optimization only on a tiny fraction of grid points. The buffer-stock saving motive implies that the constraint only binds in the worst income state and even in this state only for a small range of very low values of total wealth. This pays off especially because we need to consider a larger range of total wealth to match realistic Gini indexes for wealth inequality.

III: Data appendix.

We use the SCF 1983 and 2004 and largely follow Budría Rodríguez et al. (2002) and Díaz-Giménez et al. (1997) in constructing measures for the distribution of earnings, income and wealth in the US. The variables used in the calibration are constructed in the following way:

Labor income is the sum of wage and salary income and transfer income like unemployment compensation, child support or food stamps. This transfer income is included since the autocorrelation estimates of Storesletten et al. (2004) are based on this labor income definition. Finally, as in Budría Rodríguez et al. (2002) we add a fraction of the business income where the fraction is the average share of labor income in total income in the SCF.

Financial assets are defined as the sum of money in checking accounts, savings accounts, money-market accounts, money-market mutual funds, call accounts in brokerages, certificates of deposit, mutual funds, stocks, bonds, account-type pension plans, thrift accounts, the current value of life insurance, savings bonds, other managed funds, other financial assets.

Gross financial debt is defined as the sum of mortgage and housing debt, other lines of credit and debt on residential and nonresidential property, credit-card debt, margin, installment or pension loans, debt on non-financial business assets.

Net-Financial assets are defined as *financial assets* - *gross financial debt*. This is the measure of financial assets we use in the calibration.

Durables or non-financial assets are defined as the sum of residential property, vehicles, other durables like jewelry or antiques, owned non-financial business assets.

Total wealth is defined as the sum of net-financial and durable wealth.

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<i>Parameters</i>	<i>Values</i>	<i>Sources / Targets</i>
<i>Fixed parameters</i>		
risk aversion: $\sigma = 2$		e.g., Díaz and Luengo-Prado (2005)
lending rate: $r^a = 0.04$		Caporale and Grier (2000)
loan-to-value ratio: $\mu = 0.84$		Campbell and Hercowitz (2006)
<i>Income process</i>		
transition probabilities:		
$\begin{bmatrix} 0.96500 & 0.03470 & 0.00030 \\ 0.03937 & 0.95000 & 0.01063 \\ 0.00000 & 0.08300 & 0.91700 \end{bmatrix}$		Díaz and Luengo-Prado (2005), Table 1
		First-order autocorrelation: 0.95,
		Storesletten et al. (2004) based on PSID
income levels (3 states):		
$y \in [0.2015; 1.217; 6.2395]$		Gini: 0.533, SCF 1983
		$E(y)$ normalized to 1
		Implied stationary distribution: [0.5055; 0.4355; 0.0590]
<i>Jointly calibrated parameters</i>		
borrowing spread: $\tau = 0.0097$		4.9% of households with zero net-financial wealth, SCF 1983
discount factor: $\beta = 0.9014$		Total wealth - labor earnings ratio 6.11, SCF 1983
weight of durable utility: $\phi = 3.67$		Durable wealth - labor earnings ratio 4.96, SCF 1983
minimum durable: $\underline{d} = 1.185$		8.5% of households with no durable wealth, SCF 1983
depreciation rate: $\delta = 0.029$		Ratio of non-durable consumption to durable investment $c/i = 6.25$, NIPA Tables, Bureau of Economic Analysis & DLP (2005), Table 4

Table 1: Parameter values for the calibration of the economy to the US in 1983

Targets	Data 1983	Model 1983	Data 2004	Model 2004
			(Annualized % Change 1983-2004 in brackets)	(Annualized % Change 1983-2004 in brackets)
	(1)	(2)	(3)	(4)
total wealth - lab.earn. ratio	6.11	6.11	8.37 (1.8%)	6.10 (0%)
durable wealth - lab.earn. ratio	4.96	4.96	6.54 (1.5%)	7.02 (2.0%)
non-dur. cons/ durable inv. ratio	6.25	6.25	5.89 (-0.3%)	8.29 (1.5%)
prob($a = 0$)	0.049	0.049	0.035	0.046
prob($d = 0$)	0.085	0.123	0.058	0.201
Gini earnings	0.533	0.533	0.595	0.595
Gini total wealth	0.792	0.721	0.807	0.727
Gini financial wealth	0.956	0.966	0.957	0.889
Gini durable wealth	0.756	0.607	0.737	0.63

Table 2: Data vs. model in 1983 and observed vs. predicted changes for 2004

Wealth determinants	Data 1983	Data 2004
	(1)	(2)
Gini of labor earnings (SCF 1983, 2004)	0.533	0.595
Loan-to-value ratio (Campbell and Hercowitz, 2006)	84%	89%
Real interest rate (Caballero et al., 2006)	0.04	0.02
Interest spread (Iacoviello and Pavan, 2006)	0.01	0.005
Depreciation rate (Himmelberg et al., 2005)	0.029	0.015

Table 3: Observed quantitative changes in consumer-debt determinants 1983-2004

Statistics	Data 1983	Model 1983	Data 2004	Model 2004
			(Annualized % Change 1983-2004 in brackets)	(Annualized % Change 1983-2004 in brackets)
	(1)	(2)	(3)	(4)
financial wealth - lab. earn. ratio	1.15	1.15	1.83 (2.8%)	-0.92 (-8.5%)
financial debt ratio (conditional on $a < 0$)	-1.24	-0.48	-2.06 (-3.2%)	-1.44 (-9.4%)

Table 4: Data and model predictions for consumer debt 1983-2004

Statistics	Gini $y = 0.595$	$\mu = 0.89$	$r^a = 0.02$	$\tau = 0.005$	$\delta = 0.015$
	(1)	(2)	(3)	(4)	(5)
financial wealth - lab. earn. ratio	2.32	1.14	-0.63	1.00	0.45
financial debt ratio (conditional on $a < 0$)	-0.21	-0.49	-1.20	-0.61	-0.76

Table 5: Decomposition of changes in consumer debt 1983-2004