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Money and Bonds: An Equivalence Theorem

Narayana R. Kocherlakota*

University of Minnesota, FRB-Minneapolis, and NBER

ABSTRACT

This paper considers four different models of asset trade. In all of them, immortal agents face idiosyncratic shocks to their ability to produce output, and they can trade a single risk-free asset over time. In the first model, the risk-free asset is a privately issued one-period bond, and in the second model, it is a publicly issued one-period bond that cannot be shorted. In the third model, it is publicly issued money (a durable barren asset). In the fourth model, it is privately issued money.

I prove the following theorem. Given an equilibrium in one of these economies, it is possible to pick the exogenous parameters in the other three economies so that there is an outcome-equivalent equilibrium in each of them. Moreover, in these outcome-equivalent equilibria, the agents' budget sets are all identical.

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1. Introduction

In this paper, I examine four different models of asset trade. In all of them, immortal agents face idiosyncratic shocks to their ability to produce output, and they can trade a single risk-free asset over time. The models differ in their specification of what this single asset is.

In the first two models, agents trade interest-bearing bonds. In the first model, agents can trade one period risk-free bonds available in zero net supply, subject to person-independent borrowing restrictions. In the second model, agents can trade one period risk-free bonds available in positive net supply, but they cannot short-sell the asset. A government pays the interest on these bonds, and regulates their supply, by using time-dependent taxes that are the same for all agents.

In the other two models, agents can trade money. Money is an asset that lasts forever, but pays no dividend. It plays no special role in transactions. In the third model, money is in positive supply. A government regulates its supply using lump-sum taxes. In the fourth model, there is no government. Agents can issue and redeem private money, subject to a period-by-period constraint on the difference between past issue and past redemption.

These models are designed to be closely related to ones already in the literature. The first model is essentially the famous Aiyagari-Bewley model of self-insurance. The second model is motivated by Aiyagari and McGrattan's (1998) study of the optimal quantity of government debt. The third model is a version of Lucas' (1980) pure currency economy. It is used by Imrohoroglu (1992) in her study of the welfare costs of inflation. The fourth model is more novel, although of course many authors have been interested in comparing the consequences of using inside instead of outside money (see, for example, Cavalcanti and

Wallace (1999)).¹

I demonstrate that these four models are essentially isomorphic. In particular, I prove the following theorem. Given an equilibrium in any of the four economies, I can find a specification of the exogenous elements in the other three economies so that there is an outcome-equivalent equilibrium in each of them. Even more strongly, I prove that in these outcome-equivalent equilibria, the budget sets of all agents are the same as in the original equilibrium.

This theorem establishes two equivalence results. First, public issue and private issue of bonds/money are equivalent to one another. To make this equivalence work, the government in the economy with public issue must be able to levy a lump-sum tax that is the same for all agents. I show that the lump-sum tax exactly equals the interest payment made by a borrower in the private-issue economy who holds the maximal level of debt in each period. In this sense, public issue and private issue are equivalent as long as the collection powers of the private sector and public sector are the same.

Second, the theorem shows that risk-free bonds and money are equivalent to one another. The key to this demonstration is that money can have a positive real rate of return even though it does not pay dividends. This price rise can occur in equilibrium in the third model if the government shrinks the supply of money. It can occur in the fourth model if the limits on net money issue are shrinking over time.

The theorem is related to Wallace's (1981) famous Modigliani-Miller theorem for open market operations. Wallace proves that the money/bond composition of a government's debt portfolio does not affect equilibrium outcomes. Like my theorem, Wallace's relies on two

¹Lucas (1992) compares versions of the first and third models to each other.

crucial assumptions. First, as noted above, the government must have access to lump-sum taxes. Second, money cannot have a transactions advantage over bonds. This assumption is not satisfied by cash-in-advance, money-in-the-utility function, or transaction cost models. Like Wallace's paper, mine is also closely related to Barro's (1974) analysis of government debt.

Taub (1994) poses the question, "Are currency and credit equivalent mechanisms?" that motivates this paper. As I do, he answers this question affirmatively. However, he confines his analysis to a rather special example (linear utility). Levine (1991) and Green and Zhou (2005) use linear utility examples to demonstrate how a government using public money issue can achieve a first-best outcome in a world in which agents experience shocks to their need for consumption. In their examples, the government achieves this good outcome by using an inflationary monetary policy is inflationary.

2. Setup

Consider an infinite horizon environment with a unit measure of agents in which time is indexed by the natural numbers. At the beginning of period 1, for each agent, Nature draws an infinite sequence $(\theta_t)_{t=1}^{\infty}$ from the set Θ^{∞} , where Θ is finite. The draws are i.i d. across agents, with measure μ . Hence, there is no aggregate risk. At the beginning of period t , a given agent observes his own realization of θ_t ; his information at date t consists of the history $\theta^t = (\theta_1, \dots, \theta_t)$.

The shocks affect individuals as follows. The typical agent has preferences of the form:

$$\sum_{t=1}^{\infty} \beta^{t-1} u(c_t, y_t, \theta_t)$$

where c_t is the agent's consumption in period t , y_t is the agent's output in period t , and

$0 < \beta < 1$. The agent's utility function u is assumed to be strictly increasing in c_t , strictly decreasing in y_t , and is a function of the realization of θ_t .

I then consider four different (possibly incomplete markets) trading structures embedded in this setting.

A. Private-Bond Economy

The first market structure is a *private-bond* economy. It is completely characterized by a *borrowing limit* sequence $B^{priv} = (B_{t+1}^{priv})_{t=1}^{\infty}$, where $B_{t+1}^{priv} \in R_+$. (Note that the borrowing limits are the same for all agents in all periods.) At each date, the agents trade one-period risk-free real bonds in zero net supply for consumption. They are initially endowed with zero units of bonds each. Each agent's bond-holdings in period t must be no smaller than $-B_{t+1}^{priv}$ (as measured in terms of consumption in that period).

In this economy, individuals take interest rates $r = (r_t)_{t=1}^{\infty}$, $r_t \in R$, as given and then choose consumption, output, and bond-holdings $(c, y, b) = (c_t, y_t, b_{t+1})_{t=1}^{\infty}$, $(c_t, y_t, b_{t+1}) : \Theta^t \rightarrow R_+^2 \times R$. Hence, the agent's problem is:

$$\begin{aligned}
& \max_{(c, y, b)} E \sum_{t=1}^{\infty} \beta^{t-1} u(c_t, y_t, \theta_t) \\
& s.t. \ c_t(\theta^t) + b_{t+1}(\theta^t) \\
& \leq \ y_t(\theta^t) + b_t(\theta^{t-1})(1 + r_{t-1}) \ \forall (\theta^t, t \geq 1) \\
& \quad b_{t+1}(\theta^t) + B_{t+1}^{priv} \geq 0, c_t(\theta^t) \geq 0 \ \forall \theta^t, t \geq 1 \\
& \quad b_1 = 0
\end{aligned}$$

An equilibrium in a private-bond economy B^{priv} is a specification of (c, y, b) and r such that

(c, y, b) solves the agent's problem and markets clear for all t :

$$\begin{aligned}\int c_t d\mu &= \int y_t d\mu \\ \int b_{t+1} d\mu &= 0\end{aligned}$$

B. Public-Bond Economy

The second is a *public-bond* economy. At each date, there is a government that sells one-period risk-free real bonds. The economy is completely characterized by a bond supply sequence $B^{pub} = (B_t^{pub})_{t=1}^\infty$, where $B_{t+1}^{pub} \in R_+$ and an initial period return r_0 . The government raises B_{t+1}^{pub} units of consumption in period t by selling one-period risk-free bonds. It collects τ_t^b units of consumption from each agent; the tax is the same for all agents. Each agent is initially endowed with bonds that pay off $B_1^{pub}(1 + r_0)$ units of consumption. At each date, agents trade consumption and the government-issued bonds. Agents are not allowed to short-sell these bonds.

In this economy, the individuals take interest rates $r = (r_t)_{t=1}^\infty$ as given and then choose consumption, output, and bond-holdings. Hence, the individual's problem is:

$$\begin{aligned}& \max_{(c, y, b)} E \sum_{t=1}^{\infty} \beta^{t-1} u(c_t, y_t, \theta_t) \\ & s.t. \ c_t(\theta^t) + b_{t+1}(\theta^t) \\ & \leq \ y_t(\theta^t) + b_t(\theta^t)(1 + r_{t-1}) - \tau_t^{bond} \ \forall \theta^t, t \geq 2 \\ & \quad c_1(\theta_1) + b_2(\theta_1) \leq y_1(\theta_1) + B_1^{pub}(1 + r_0) - \tau_1^{bond} \ \forall \theta_1 \\ & \quad b_{t+1}(\theta^t) \geq 0, c_t(\theta^t) \geq 0 \ \forall \theta^t, t \geq 1\end{aligned}$$

An equilibrium in a public-bond economy (B^{pub}, r_0) is a specification of $(c, y, b, r, \tau^{bond})$ such that (c, y, b) solves the individual's problem given (r, τ^{bond}) and markets clear for all t :

$$\begin{aligned}\int c_t d\mu &= \int y_t d\mu \text{ for all } t \\ \int b_{t+1} d\mu &= B_{t+1}^{pub} \text{ for all } t\end{aligned}$$

Together, these imply that a government budget constraint holds at each date:

$$\tau_t^b = -B_{t+1}^{pub} + B_t^{pub}(1 + r_{t-1})$$

C. Public-Money Economy

The third economy is a *public-money* economy. By money, I mean an infinitely-lived asset that pays no dividends. Each agent is initially endowed with M_1^{pub} units of money. Then, the economy is completely characterized by a money supply sequence $M^{pub} = (M_{t+1}^{pub})_{t=1}^\infty$, where $M_{t+1}^{pub} \in R_+$. In period t , the government collects τ_t^{mon} units of consumption from each agent; again, the taxes are the same for all agents. At each date, agents trade money and consumption; the government trades so as to ensure that there are M_{t+1}^{pub} units of money outstanding.

In this economy, the individuals take money prices p as given and then choose consumption, output, and money-holdings. Hence, the individual's problem is:

$$\begin{aligned}& \max_{(c, y, M)} E \sum_{t=1}^{\infty} \beta^{t-1} u(c_t, y_t, \theta_t) \\ & s.t. \ c_t(\theta^t) + M_{t+1}(\theta^t)p_t \\ & \leq \ y_t(\theta^t) + M_t(\theta^{t-1})p_t - \tau_t^{mon} \forall \theta^t, t \geq 2\end{aligned}$$

$$c_1(\theta_1) + M_2(\theta_1)p_1 \leq y_1(\theta_1) + M_1^{pub}p_1 - \tau_1^{mon} \forall \theta_1$$

$$M_{t+1}(\theta^t), c_t(\theta^t) \geq 0 \forall \theta^t, t \geq 1$$

An equilibrium in a public-money economy (M^{pub}) is a specification of (c, y, M, p, τ^{mon}) such that (c, y, M) solves the individual's problem given (p, τ^{mon}) and markets clear for all t :

$$\begin{aligned} \int c_t d\mu &= \int y_t d\mu \\ \int M_{t+1} d\mu &= M_{t+1}^{pub} \end{aligned}$$

Again, a government budget constraint is implied at each date by market-clearing:

$$\tau_t^{mon} = M_t^{pub}p_t - M_{t+1}^{pub}p_t$$

There is no cash-in-advance constraint or any transaction cost advantage associated with money in this setting.

D. Private Money Economy

The fourth and final economy is a *private-money* economy. In this economy, there is no government. Agents are able to issue their money in exchange for consumption, and redeem others' monies in exchange for consumption. However, in each history, they face an exogenous upper bound on the net amount of money issue that they have done in their lifetimes. The economy is completely characterized by the bound process $M^{priv} = (M_{t+1}^{priv})_{t=1}^{\infty}$, where $M_{t+1}^{priv} \in R_+$.

In this economy, the individuals take money prices p as given and then choose consumption, how much money to issue and how much money to redeem. (I assume that all monies are traded at the same price p ; there may be other equilibria in which this restriction

is not satisfied.) Hence, the individual's problem is:

$$\begin{aligned}
& \max_{(c, y, M^{iss}, M^{red})} E \sum_{t=1}^{\infty} \beta^{t-1} u(c_t, y_t, \theta_t) \\
& s.t. \ c_t(\theta^t) + M_{t+1}^{red}(\theta^t) p_t \\
& \leq \ y_t(\theta^t) + M_{t+1}^{iss}(\theta^t) p_t \ \forall \theta^t, \ t \geq 1 \\
& M_{t+1}^{iss}(\theta^t), M_{t+1}^{red}(\theta^t) \geq 0 \text{ for all } \theta^t, y^t, t \\
& \sum_{s=1}^t [M_{s+1}^{iss}(\theta^s) - M_{s+1}^{red}(\theta^s)] \leq M_{t+1}^{priv} \ \forall \theta^t, t \geq 1
\end{aligned}$$

An equilibrium in a private-money economy (M^{priv}) is a specification of $(c, y, M^{red}, M^{iss}, p)$ such that $(c, y, M^{red}, M^{iss}, p)$ solves the individual's problem and markets clear for all t :

$$\begin{aligned}
\int c_t d\mu &= \int y_t d\mu \\
\int M_{t+1}^{red} d\mu &= \int M_{t+1}^{iss} d\mu
\end{aligned}$$

Again, there is no cash-in-advance constraint or transaction cost advantages associated with money in this setting.

3. Example Economy

In this section, I work through an example of the above general structure that illustrates the general equivalence theorem that follows in the next section. I start with an equilibrium in a private-bond economy. I then construct a public-bond economy, a public-money economy, and a private-money economy. I show that in each of these economies, there is an equilibrium with the same consumption allocation as the original, private-bond, equilibrium. Even more strongly, agents have exactly the same budget sets in each of these

equilibria.

In the example, output is inelastically supplied. Half of the agents receive an endowment stream of the form $(1 + h, 1, 1, \dots)$ and the other half get an endowment stream of the form $(1 - h, 1, 1, \dots)$, where $1 > h > 0$. I will call the first half "rich" and the second half "poor". The agents have identical preferences of the form:

$$\sum_{t=1}^{\infty} \beta^{t-1} \ln(c_t)$$

where $1 > \beta > 0$.

A. Private-Bond Economy

Consider first a private-bond economy in which the borrowing limit B_{t+1}^{priv} is constant at $\beta\lambda(1 - \beta)^{-1}$, where $(1 - \beta) < \lambda < 1$. I claim that there is an equilibrium to this economy of the following form. The equilibrium interest rate r_t is constant at $1/\beta - 1$. Rich agents consume a constant amount c^r , where:

$$c^r = (1 + h)(1 - \beta) + \beta = 1 + h(1 - \beta)$$

Rich agents' bond-holdings b_{t+1}^r equal:

$$\beta h$$

for period $t \geq 1$. Poor agents consume a constant amount $c^p = 2 - c^r$. Poor agents' bond-holdings b_{t+1}^p equal:

$$-h\beta$$

for period $t \geq 1$. Note that the borrowing limit has been chosen so that it never binds in equilibrium.

This claim is readily verified. Markets clear. The agents' flow budget constraints are satisfied because:

$$c^r + b_2^r - 1 - h = 0$$

$$c^r + b_{t+1}^r - 1 - b_t^r \beta^{-1} = 0 \text{ for all } t \geq 2$$

and similarly for poor agents. Because the borrowing limit does not bind, the agents' Euler equations are satisfied. We need only check the agents' transversality conditions, which are satisfied because the two limits:

$$\lim_{t \rightarrow \infty} \beta^{t-1} u'(c^r)(b_{t+1}^r - 2\beta)$$

$$\lim_{t \rightarrow \infty} \beta^{t-1} u'(c^p)(b_{t+1}^p - 2\beta)$$

are both equal to zero

B. Public-Bond Economy

I now want to design a public-bond economy with an outcome-equivalent equilibrium. In a public-bond economy, agents are not allowed to borrow. Hence, to get a non-autarkic equilibrium, there must be a positive amount of debt outstanding. I set $B_t^{pub} = B_t^{priv} = \beta\lambda(1 - \beta)^{-1}$ for all t and $r_0 = 1/\beta - 1$. Hence, in this public-bond economy, the supply of bonds equals the borrowing limit in the private-bond economy described above. I claim that there is an equilibrium to this economy in which the equilibrium interest rate r_t is constant at $1/\beta - 1$. Rich agents consume c^r (as defined above) in each period, and poor agents consume c^p in each period. Each agent pays a lump-sum tax $\tau_t^b = \lambda$ at every date. Then, in period

$t \geq 1$, rich agents' bond-holdings b_{t+1}^r equal:

$$\beta h + \beta \lambda (1 - \beta)^{-1}$$

and poor agents' bond-holdings b_{t+1}^p equal:

$$-\beta h + \beta \lambda (1 - \beta)^{-1}$$

Note that the agents' bond-holdings are always positive.

Again, it is simple to verify that this claim. Markets clear. The rich agent's flow budget constraint in period 1 is satisfied because:

$$\begin{aligned} & c^r + b_2^r - 1 - h - B_1^{pub} \beta^{-1} + \tau^{bond} \\ &= 1 + h(1 - \beta) + h\beta + \beta \lambda (1 - \beta)^{-1} - 1 - h - \lambda (1 - \beta)^{-1} + \lambda \\ &= 0 \end{aligned}$$

The rich agent's flow budget constraint in period $t > 1$ is satisfied because:

$$\begin{aligned} & c^r + b_{t+1}^r - 1 - b_t^r \beta^{-1} + \tau^{bond} \\ &= 1 + h(1 - \beta) + h\beta + \beta \lambda (1 - \beta)^{-1} - 1 - h - \lambda (1 - \beta)^{-1} + \lambda \\ &= 0 \end{aligned}$$

We can check the poor agents' flow constraints in a similar fashion.

The agents' Euler equations are clearly satisfied, because their no-short-sales constraint never binds. Finally, we need to verify the transversality conditions:

$$\lim_{t \rightarrow \infty} \beta^t u'(c^r) b_{t+1}^r = 0$$

$$\lim_{t \rightarrow \infty} \beta^t u'(c^p) b_{t+1}^p = 0$$

Hence, there is an equilibrium in this public-bond economy with the same consumption allocation as the original, private-bond, equilibrium.

C. Public-Money Economy

I now want to design a public-money economy with an outcome-equivalent equilibrium. Clearly, the gross rate of return on money must be β^{-1} , in order to satisfy the agents' first-order conditions. Since money pays no dividend, this rate of return implies that the price of money must rise at rate β^{-1} . At the same time, we need the rich agents' transversality condition:

$$\lim_{t \rightarrow \infty} \beta^t u'(c^r) M_{t+1}^r p_t = 0$$

to be satisfied. This requires that the money stock must converge to zero over time.

Given these considerations, consider a public-money economy in which $M_t^{pub} = \beta^t \lambda (1 - \beta)^{-1}$. I claim that there is an equilibrium in this economy in which $p_t = \beta^{-t}$ for all t . Rich agents consume c^r in each period and poor agents consume c^p in each period. As in the public-bond economy, each agent pays a lump-sum tax $\tau_t^{mon} = \lambda$ at every date. Then, in period $t \geq 1$, rich agents' money-holdings M_{t+1}^r equal:

$$\beta^{t+1} h + \beta^{t+1} \lambda (1 - \beta)^{-1}$$

and poor agents' bond-holdings M_{t+1}^p equal:

$$-\beta^{t+1} h + \beta^{t+1} \lambda (1 - \beta)^{-1}$$

To verify this claim, note that markets clear and that individual Euler equations are satisfied. Clearly, the transversality conditions are also satisfied, because the money supply

converges to zero. Finally, we can verify that the flow budget constraints are satisfied. In period 1, the rich agents' flow constraint are:

$$\begin{aligned}
& c^r + p_1 M_2^r - 1 - h - p_1 M_1^r + \tau^{mon} \\
&= 1 + h(1 - \beta) + \beta^{-1}(\beta^2 h + \beta^2 \lambda(1 - \beta)^{-1}) - 1 - h - \lambda(1 - \beta)^{-1} + \lambda \\
&= 0
\end{aligned}$$

In period $t > 1$, the rich agents' flow constraints are:

$$\begin{aligned}
& c^r + p_t M_{t+1}^r - 1 - p_t M_t^r + \tau^{mon} \\
&= 1 + h(1 - \beta) + \beta^{-t}(\beta^{t+1} h + \beta^{t+1} \lambda(1 - \beta)^{-1}) - 1 - (h + \lambda(1 - \beta)^{-1}) + \lambda \\
&= 0
\end{aligned}$$

The poor agents' flow constraints are similar.

Money pays no dividend, and has no liquidity benefits. Nonetheless, money has a positive value in this equilibrium. This positive value seems to create the possibility for an arbitrage, in which a given agent permanently reduces his money-holdings by a small amount ε . However, this arbitrage is infeasible, because the supply of money eventually falls to zero.

D. Private-Money Economy

Finally, I design a private-money economy that induces the same equilibrium consumption allocation. Consider a private-money economy in which $M_{t+1}^{priv} = \beta^{t+1} \lambda(1 - \beta)^{-1}$. Then, I claim that there is an equilibrium in this economy in which $p_t = \beta^{-t}$ for all t . Rich agents consume c^r in each period and poor agents consume c^p in each period. In period 1,

poor agents issue $M_2^{iss,p} = \beta^2 h$ and rich agents redeem $M_2^{red,r} = \beta^2 h$. In period $t > 1$, poor agents redeem $M_{t+1}^{red,p} = h(1 - \beta)\beta^t$ and rich agents issue $M_{t+1}^{iss,r} = h(1 - \beta)\beta^t$. Note that in period t ,

$$\sum_{s=1}^t (M_{s+1}^{iss,p} - M_{s+1}^{red,p}) = \beta^2 h - \beta^2 h(1 - \beta) - \beta^3 h(1 - \beta) - \dots - \beta^t h(1 - \beta) = \beta^{t+1} h$$

which is less than $\beta^{t+1} \lambda(1 - \beta)^{-1}$. In this private-money economy, neither rich nor poor agents ever have net money issue equal to β^t . Hence, the bound on money issue never binds in equilibrium.

We can verify the validity of this equilibrium as follows. Markets clear, and the agents' Euler equations are clearly satisfied. In period 1, the rich agents' flow budget constraints are:

$$\begin{aligned} & c^r + p_1 M_2^{red,r} - 1 - h - p_1 M_2^{iss,r} \\ &= 1 + h(1 - \beta) + \beta h - 1 - h \\ &= 0 \end{aligned}$$

In period $t > 1$, the rich agents' flow budget constraints are:

$$\begin{aligned} & c^r + p_t M_{t+1}^{red,r} - 1 - p_t M_{t+1}^{iss,r} \\ &= 1 + h(1 - \beta) - h(1 - \beta) - 1 \\ &= 0 \end{aligned}$$

The poor agents' flow constraints can be checked in a similar fashion.

Finally, we need to verify the agents' transversality conditions. To do so, note that:

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \beta^t u'(c^r) p_t \left\{ - \sum_{s=1}^t [M_{s+1}^{iss} - M_{s+1}^{red}] + M_{t+1}^{priv} \right\} \\
&= \lim_{t \rightarrow \infty} \beta^t u'(c^r) p_t \{ \beta^{t+1} h \} [\lambda(1 - \beta)^{-1} - 1] \\
&= 0
\end{aligned}$$

In this economy, money has a positive value, and pays no dividends. Why is it not optimal for an agent issue ε units more money in period 1? Issuing ε more units of money in period 1 so means that the agent's net money issue would be $\varepsilon + \beta^{t+1}h$ in period t . But for large t , this amount will exceed the upper bound on money issue. Put another way, even though it never binds, the money issue constraint is structured so that agents must eventually redeem whatever currency they have issued.

E. Budget Set Equivalence

The arguments above establish that the equilibrium outcomes are identical across the four economies. But, with some algebra, it is possible to prove an even stronger equivalence: the equilibrium *budget sets* of (c, y) are the same across the four economies. Agents are confronted with exactly the same sets of possible choices in the four economies. As we shall see, this deeper isomorphism can be generalized.

4. An Equivalence Theorem

In this section, I prove the main theorem in the paper. The theorem starts with an equilibrium (c, y, b, r) in a private- bond economy defined by B^{priv} . It then shows how, by translating the bond-holdings upward by B^{priv} and crafting taxes in the right way, we can

get an outcome-equivalent equilibrium in the public-bond economy. The key to the theorem is that these taxes equal the net interest payments on the bond-holding limits.

Then, the theorem goes on to show that, by setting monetary policy in the right way, we can design an outcome-equivalent equilibrium in the public-money economy. The key here is that monetary policy must be designed so that the price level falls at the rate of interest, and the real value of aggregate money-holdings always equals aggregate bond-holdings. As we will see later, this will typically mean that aggregate money-holdings will be shrinking (but not necessarily at the Friedman Rule $1/\beta - 1$).

Finally, the theorem turns to the private-money economy. Here, I set the upper bounds on net real money issue equal to the borrowing limit in the private-bond economy. By doing so, I can induce an outcome-equivalent equilibrium in this setting.²

THEOREM 1. *Suppose $(r, p) = (r_{t-1}, p_t, B_{t+1}^{priv}, B_t^{pub}, \tau_t^{bond}, M_t^{pub}, \tau_t^{mon}, M_{t+1}^{priv})_{t=1}^{\infty}$ are sequences such that for all t, z^t :*

$$\begin{aligned} p_t &= (1 + r_{t-1})p_{t-1}; \quad p_1 = M_1^{pub}(1 + r_0)^{-1}/B_1^{pub} \\ B_{t+1}^{priv} &= B_{t+1}^{pub} = M_{t+1}^{pub}p_t = M_{t+1}^{priv}p_t \\ \tau_t^{bond} &= \tau_t^{mon} = B_t^{pub}(1 + r_{t-1}) - B_{t+1}^{pub} \end{aligned}$$

Then the following four statements are equivalent.

1. (c, y, b, r) is an equilibrium in a private-bond economy defined by B^{priv} .
2. $(c, y, b + B^{priv}, r, \tau^{bond})$ is an equilibrium in a public-bond economy defined by B^{pub} and

²The proof of this Theorem relies on techniques similar to those that I use in Kocherlakota (forthcoming).

r_0 .

3. $(c, y, (b + B^{priv})/p, p, \tau^{mon})$ is an equilibrium in a public-money economy defined by M^{pub} .

4. $(c, y, M^{red}, M^{iss}, p)$ is an equilibrium in a private-money economy defined by M^{priv} , where for all t :

$$\sum_{s=1}^t [M_{s+1}^{red}(\theta^s) - M_{s+1}^{iss}(\theta^s)] = b_{t+1}/p_t$$

for all θ^s .

Proof. I first prove the equivalence of statements 1 and 2. Pick an element $(\hat{c}, \hat{b})$ of an agent's budget set in the private-bond economy. Then, define:

$$\hat{b}'_{t+1}(\theta^t) = \hat{b}_{t+1}(\theta^t) + B_{t+1}^{priv}, t \geq 1, \text{ all } \theta^t$$

My claim is that $(\hat{c}, \hat{y}, \hat{b}')$ is in the agent's budget set in the public-bond economy. Obviously,

$\hat{b}'_{t+1}(\theta^t) \geq 0$, because $\hat{b}_{t+1}(\theta^t) \geq -B_{t+1}^{priv}$. Note that for $t \geq 2$:

$$\begin{aligned} & \hat{c}_t(\theta^t) + \hat{b}'_{t+1}(\theta^t) - \hat{y}_t(\theta^t) - (1 + r_{t-1})\hat{b}'_t(\theta^{t-1}) + \tau_t^{bond} \\ = & \hat{c}_t(\theta^t) + \hat{b}_{t+1}(\theta^t) + B_{t+1}^{priv} \\ & - \hat{y}_t(\theta^t) - (1 + r_{t-1})\hat{b}_t(\theta^{t-1}) - (1 + r_{t-1})B_t^{priv} + \tau_t^{bond} \\ = & \hat{c}_t(\theta^t) + \hat{b}_{t+1}(\theta^t) - \hat{y}_t(\theta^t) - (1 + r_{t-1})\hat{b}_t(\theta^{t-1}) \\ \leq & 0 \text{ for all } \theta^t \end{aligned}$$

We can use similar logic to check the flow budget constraint for $t = 1$:

$$\hat{c}_1(\theta_1) + \hat{b}'_2(\theta_1) - \hat{y}_1(\theta_1) - B_1^{pub}(1 + r_0) + \tau_1^{bond}$$

$$\begin{aligned}
&= \hat{c}_1(\theta_1) + \hat{b}_2(\theta_1) + B_2^{priv} - \hat{y}_1(\theta_1) - B_1^{pub}(1 + r_0) + \tau_1^{bond} \\
&= \hat{c}_1(\theta_1) + \hat{b}_2(\theta_1) - \hat{y}_1(\theta_1) \leq 0
\end{aligned}$$

for all (θ_1) . Running the steps in reverse establishes the converse. Hence, the agent's budget sets are (c, y) -equivalent in the two economies. But it is then straightforward to see that Statements 1 and 2 are equivalent.

I now prove the equivalence of statements 2 and 3. Pick an element $(\hat{c}, \hat{y}, \hat{b}')$ of an agent's budget set in the public-bond economy. Define $\widehat{M}_{t+1} = \hat{b}'_{t+1}/p_t$. I claim that $(\hat{c}, \hat{y}, \widehat{M})$ is in the agent's budget set in the public- money economy. Then for all θ^t , $t \geq 2$:

$$\begin{aligned}
&\hat{c}_t(\theta^t) + \widehat{M}_{t+1}(\theta^t)p_t - \hat{y}_t(\theta^t) - \widehat{M}_t(\theta^{t-1})p_t + \tau_t^{mon} \\
&= \hat{c}_t(\theta^t) + \hat{b}'_{t+1}(\theta^t) - \hat{y}_t(\theta^t) - (1 + r_{t-1})\widehat{M}_t(\theta^t)p_{t-1} + \tau_t^{bond} \\
&= \hat{c}_t(\theta^t) + \hat{b}'_{t+1}(\theta^t) - \hat{y}_t(\theta^t) - (1 + r_{t-1})\hat{b}'_t(\theta^t) + \tau_t^{bond} \\
&\leq 0
\end{aligned}$$

In period 1, we can verify:

$$\begin{aligned}
&\hat{c}_1(\theta_1) + \widehat{M}_2(\theta_1)p_1 - \hat{y}_1(\theta_1) - M_1^{pub}p_1 + \tau_1^{mon} \\
&= \hat{c}_1(\theta_1) + \hat{b}_2(\theta_1) - \hat{y}_1(\theta_1) - B_1^{pub} + \tau_1^{bond} \\
&\leq 0
\end{aligned}$$

for all (θ_1) . We can run the logic in reverse to check the converse. Thus, the agent's budget sets are (c, y) -equivalent; this in turn establishes that statements 2 and 3 are equivalent.

Finally, I prove the equivalence of statements 1 and 4. Pick an element $(\hat{c}, \hat{y}, \hat{b})$ of an agent's budget set in the private-bond economy. Then, define:

$$M_{t+1}^{red}(\theta^t) = \max(\hat{b}_{t+1}(\theta^t) - (1 + r_{t-1})\hat{b}_t(\theta^{t-1}), 0)/p_t, t \geq 2$$

$$M_{t+1}^{iss}(\theta^t) = \max((1 + r_{t-1})\hat{b}_t(\theta^{t-1}) - \hat{b}_{t+1}(\theta^t), 0)/p_t, t \geq 2$$

$$M_2^{red}(\theta_1) = \max(\hat{b}_2(\theta_1), 0)/p_1$$

$$M_2^{iss}(\theta_1) = \max(-\hat{b}_2(\theta_1), 0)/p_1$$

I claim that $(\hat{c}, \hat{y}, M^{red}, M^{iss})$ is in the agent's budget set in the private-money economy. In period $t > 1$:

$$\begin{aligned} & \hat{c}_t(\theta^t) + M_{t+1}^{red}(\theta^t)p_t - \hat{y}_t(\theta^t) - M_{t+1}^{iss}(\theta^t)p_t \\ &= \hat{c}_t(\theta^t) + \hat{b}_{t+1}(\theta^t) - (1 + r_{t-1})\hat{b}_t(\theta^t) - \hat{y}_t(\theta^t) \\ &\leq 0 \end{aligned}$$

for all (θ^t) and in period 1, for all (θ_1) :

$$\begin{aligned} & \hat{c}_1(\theta_1) + M_2^{red}(\theta_1)p_1 - \hat{y}_1(\theta_1) - M_2^{iss}(\theta_1)p_1 \\ &= \hat{c}_1 + \hat{b}_2(\theta_1) - \hat{y}_1(\theta_1) \\ &\leq 0 \end{aligned}$$

Note too that:

$$\sum_{s=1}^t (M_{s+1}^{red}(\theta^s) - M_{s+1}^{iss}(\theta^s))$$

$$\begin{aligned}
&= \sum_{s=1}^t [\widehat{b}_{s+1}(\theta^s) - (1 + r_{s-1})\widehat{b}_s(\theta^{s-1})]/p_s \\
&= \sum_{s=1}^t [\widehat{b}_{s+1}(\theta^s)/p_s - \widehat{b}_s(\theta^s)/p_{s-1}] \\
&= \widehat{b}_{t+1}(\theta^t)/p_t \\
&\geq -B_{t+1}^{priv}/p_t \\
&= -M_{t+1}^{priv}
\end{aligned}$$

This confirms that the budget-feasible consumptions in the private-bond economy are a subset of the budget-feasible consumption set in the private-money economy. Conversely, suppose $(\widehat{c}, \widehat{y}, M^{red}, M^{iss})$ is in the agent's budget set in the private-money economy. Then define:

$$\widehat{b}_{t+1}(\theta^s) = p_t \sum_{s=1}^t (M_{s+1}^{red}(\theta^s) - M_{s+1}^{iss}(\theta^s))$$

I claim that $(\widehat{c}, \widehat{y}, \widehat{b})$ is budget-feasible in the private-bond economy. In period $t > 1$, for any (θ^t) :

$$\begin{aligned}
&\widehat{c}_t(\theta^t) + \widehat{b}_{t+1}(\theta^t) - \widehat{y}_t(\theta^t) - \widehat{b}_t(\theta^t)(1 + r_{t-1}) \\
&= \widehat{c}_t(\theta^t) + p_t \sum_{s=1}^t (M_{s+1}^{red}(\theta^s) - M_{s+1}^{iss}(\theta^s)) - \widehat{y}_t(\theta^t) - p_t \sum_{s=1}^{t-1} (M_{s+1}^{red}(\theta^s, y^s) - M_{s+1}^{iss}(\theta^s, y^s)) \\
&= \widehat{c}_t(\theta^t, y^t) + p_t (M_{t+1}^{red}(\theta^t) - M_{t+1}^{iss}(\theta^t)) - \widehat{y}_t(\theta^t) \\
&\leq 0
\end{aligned}$$

In period 1, for any (θ_1) :

$$\begin{aligned}
&\widehat{c}_1(\theta_1) + \widehat{b}_2(\theta_1) - \widehat{y}_1(\theta_1) \\
&= \widehat{c}_1(\theta_1, y_1) + p_1 (M_2^{red}(\theta_1) - M_2^{iss}(\theta_1)) - \widehat{y}_1(\theta_1)
\end{aligned}$$

$$\leq 0$$

Agent j 's bond-holdings $\hat{b}$ clearly satisfy the borrowing limit $-B^{priv}$. Hence, $(\hat{c}, \hat{y}, \hat{b})$ is budget-feasible in the private-bond economy. The budget sets are (c, y) -equivalent. It follows that Statements 1 and 4 are equivalent. QED

5. Discussion

In this section, I discuss several aspects of Theorem 1.

A. Equivalences

Theorem 1 establishes two kinds of equivalences. The first is between private issue and public issue (of money or bonds). Consider, for example, a private-bond economy in which agents have a constant borrowing limit B^{priv} . In this economy, all agents begin with the same holdings of bonds (zero). They can run down their holdings to $-B^{priv}$. Now, consider a public-bond economy in which all agents begin their lives by holding B^{priv} units of bonds. They face taxes with present value equal to B^{priv} , and can run down their holdings to zero. In the public-bond economy, agents' initial wealths are the same as in the private debt economy. As well, they can run down their initial bond-holdings by exactly the same amount (B^{priv}) as in the private debt economy. Hence, their budget sets are the same in the two kinds of economies.³

The second kind of equivalence is between monetary economies and bond economies.

³It is possible to extend Theorem 1 to include model economies in which agents can trade both public and private debt. In particular, suppose there is an equilibrium (c, y, b, r) in a private-bond economy defined by the sequence B^{priv} . Suppose too that there is an economy in which the sequence of supplies of outside debt is given by B^{pubt} and the borrowing limit sequence is given by B^{privt} . Then, there is an equilibrium with allocation (c, y) in this latter private-public bond economy if $B^{pubt} + B^{privt} = B^{priv}$ in all periods.

Consider a public-bond economy, in which the equilibrium rate of return is constant at $r > 0$, and the value of outstanding public debt is constant at B^{pub} . Theorem 1 designs a public-money economy in which the equilibrium rate of return is also r , and value of outstanding public obligations (now in the form of money) is B^{pub} . In this public-money economy, the price of money must rise at rate r . Hence, the quantity of money must fall at this same rate. The government sucks out this money using the same taxes that it used to finance its interest payments in the public-bond economy.

It is worth pointing out that the proof of Theorem 1 establishes a stronger result than Theorem 1 itself. The statement of Theorem 1 is that the equilibrium outcomes across the four economies are the same in terms of (c, y) . Actually, the proof establishes a much stronger result: the *equilibrium budget sets of* (c, y) are the same in the four economies. Agents face exactly the same choice problems in the four equilibria of the four economies.

It is possible to extend Theorem 1 to include model economies in which agents can trade both public and private debt. In particular, suppose there is an equilibrium (c, y, b, r) in a private-bond economy defined by the sequence B^{priv} . Suppose that too there is an economy in which the sequence of supplies of outside debt is given by $B^{pub'}$ and the borrowing limit sequence is given by $B^{priv'}$. Then, there is an equilibrium with allocation (c, y) in this latter private-public bond economy if $B^{pub'} + B^{priv'} = B^{priv}$ in all periods. Public and private debt are perfect substitutes.

B. Lump-Sum Taxes

A key element of the proof of Theorem 1 is that the taxes τ^{mon} and τ^{bond} don't depend on y ; hence, they are *lump-sum*. These lump-sum taxes simply give the government similar

collection powers to those of the private sector. In particular, consider an equilibrium (c, y, b, r) in a private bond economy defined by B^{priv} . In this equilibrium, a borrower may owe as much as:

$$B_{t+1}^{priv}(1 + r_t)$$

in period $(t + 1)$. The borrower can only borrow up to B_{t+2}^{priv} to repay this loan. Hence, a lender must be able to collect:

$$B_{t+1}^{priv}(1 + r_t) - B_{t+2}^{priv}$$

in period $(t + 1)$. This collection limit is independent of decisions about (c, y) being made by the borrower. In this sense, lenders are able to levy a lump-sum tax on the borrower equal to $B_{t+1}^{priv}(1 + r_t) - B_{t+2}^{priv}$ in period $(t + 1)$.

This collection limit is exactly equal to τ^{bond} and τ^{mon} in the equivalent public bond and money economies constructed in Theorem 1. Hence, by assuming that the government can levy taxes equal to τ^{bond} and τ^{mon} , I am assuming that the government can levy the same lump-sum taxes as can a private lender.

In reality, governments can use a broader range of taxes than is assumed in the above economies. More generally, suppose that in all four economies, the government can use any element of a class C of tax schedules $\psi = \{\psi_t\}_{t=1}^{\infty}$, where an agent who has production history y_t in period t pays a tax $\psi_t(y^t)$. We can prove a version of Theorem 1 if C is closed under the addition of a sequence of constants, so that if $\alpha = \{\alpha_t\}_{t=1}^{\infty} \in R^{\infty}$, and $\psi_t(y^t) - \psi'_t(y^t) = \alpha_t$ for all t, y^t , then:

$$\psi \in C \implies \psi' \in C$$

For example, the class C may consist of the affine tax codes discussed by Werning (forthcoming).⁴

C. Multiple Equilibria

The equivalence described in Theorem 1 is limited in the following way. Suppose c is an equilibrium consumption allocation in a public-money economy defined by M^{pub} and τ . Then, Theorem 1 says that there is an outcome-equivalent equilibrium outcome in the private debt economy defined by B^{priv} . The theorem does not say that the *sets* of equilibrium outcomes are the same in the two economies.

The following example makes this point more forcefully.

Example 1. Suppose that $\Theta = \{1, 2\}$, that $\theta_t = 1$ for all t with probability $1/2$, and that $\theta_t = 2$ for all t with probability $1/2$. Suppose that $u(c, y) = \ln(c)$, so that output is inelastically supplied and $\beta > 1/2$. Finally, suppose that for all agents, $y_t = \theta_1$ if t is odd and $y_t = 3 - \theta_1$ if t is even.

Consider a public-money economy in which $M_t^{pub} = 1$ for all t . It is well-known that there are (at least) two equilibria in this economy. In the one equilibrium, $p_t = 0$ for all t , and $c_t = y_t$ for all t . In the other equilibrium:

$$p_1 = (4\beta - 2)/(5 + 2\beta)$$

$$p_t = p^* \equiv (2\beta - 1)/(2 + 2\beta), \quad t > 1$$

⁴To impose lump-sum taxes in the public-money economy, the government must be able to threaten agents with some kind of penalty if they fail to pay those taxes. Kocherlakota (2003) and Berentsen and Waller (2006) argue that these penalties could be used to enforce cross-agent transfers of resources, and thereby eliminate the need for money altogether. However, to ensure that there is no need for money, the government must be able to impose an arbitrarily large penalty, and know the realization of θ for each agent. Neither of these assumptions are implied by the government's being able to levy a particular lump-sum tax of size τ^{mon} .

$$c_1(2) = 2 - p_1; c_1(1) = 1 + p_1$$

$$c_{2t+1}(2) = c_{2t}(1) = 2 - 2p^*, t = 1, 2, 3, \dots$$

$$c_{2t}(2) = c_{2t+1}(1) = 1 + 2p^*, t = 1, 2, 3, \dots$$

(Here, $c_t(y)$ is consumption in period t if $y_1 = y$.) In this latter equilibrium, the two agents swap the money stock back and forth in exchange for consumption.

As in Theorem 1, construct a private bond economy by setting:

$$B_t^{priv} = p_t, t > 1$$

for all t . There is an equilibrium in this economy of the form:

$$r_1 = (1 - 2\beta)/(4 + 4\beta)$$

$$r_t = 0, t > 1$$

$$c_1(2) = 2 - p_1; c_1(1) = 1 + p_1$$

$$c_{2t+1}(2) = c_{2t}(1) = 2 - 2p^*, t = 1, 2, 3, \dots$$

$$c_{2t}(2) = c_{2t+1}(1) = 1 + 2p^*, t = 1, 2, 3, \dots$$

This equilibrium is consumption-equivalent to the second equilibrium in the public-money economy. (In this equilibrium, the poor agents (with income 1) borrow as much as possible in each period.) However, in the private bond economy, there is no autarkic equilibrium. Hence, the set of equilibrium outcomes is not the same as in the original public-money economy.

D. Welfare Implications

Cole and Kocherlakota (2001) consider a setup in which agents have hidden endowments and store secretly over time. They prove that, under weak conditions, the symmetric Pareto optimum is the equilibrium outcome of risk-free borrowing and lending, subject to the natural borrowing constraint. The analysis in this paper shows that this symmetric Pareto optimum could also be implemented as an equilibrium in an economy in which agents can only hold money. Note that the rate of return on money in this economy is necessarily less than the rate of time preference.

Aiyagari (1995) studies the properties of optimal capital income taxes in economies of this kind (in which agents trade only capital and risk-free bonds, and face shocks to their labor productivities). Aiyagari (1995) finds that the optimal capital income tax rate is positive, and the equilibrium interest rate in the economy is less than the rate of time preference. Using the logic of Theorem 1, one could re-do Aiyagari's analysis in a public-money economy. Aiyagari's result implies that in such a setting, the optimal rate of return on money is less than the rate of time preference.⁵

6. Conclusions

There is a fundamental asymmetry in the traditional modelling of public and private debt. With private risk-free debt, the borrower must make a repayment that is independent of the borrower's decisions or shocks. In essence, the lender is essentially able to impose a lump-sum tax on the borrower at the time of repayment. With public risk-free debt, the government makes a repayment that is independent of any aggregate shocks. However, it

⁵Aiyagari restricts attention to comparisons across steady-states. Davila, et. al. (2005) show that this restriction may lead to a misleading characterization of optimal taxes.

is typically assumed that the government must use linear taxes to collect the resources for its repayments. This restriction to linear taxes means that government repayment of public debts must distort agents' decisions in a way that is not true of private debt repayment.

In this paper, I eliminate this asymmetry. In the models of public asset issue in this paper, the government is able to levy a lump-sum tax that is the same for all agents. Given the heterogeneity in the models, this tax instrument is not enough to achieve a first-best outcome. Rather, it functions much like the intercept term in Werning's (forthcoming) affine tax systems. When set to an appropriate level, the head tax gives the government the same collection powers as private lenders.

This change in the nature of taxes leads to two important conclusions. The first is rather direct. Once the government has the same collection powers as private agents, private and public asset issue are perfect substitutes. There is no real way to discuss an optimum quantity of debt (as in Aiyagari and McGrattan (1998) or Shin (2006)).

The second lesson is more subtle. As Taub (1994) emphasizes, there is a conventional view that private credit allows agents to achieve better outcomes than does publicly provided money. According to this view, the barren asset money has value only because agents occasionally run out of it. These occasional shortages of money create intertemporal inefficiencies: the agents who have no money would like to borrow from other agents. These inefficiencies do not exist when agents can use private credit. Lucas (1980) discusses this problem in some detail, and concludes by declaring that, "There is a sense in which money is a second-rate asset." In their work, Cavalcanti and Wallace (1999) argue that private money issue has a similar advantage over public money issue.

Theorem 1 shows that this conventional view is grounded in governments' not being

able to use head taxes. For example, in the public-money equilibria described in Section 3, the government uses its power to tax to remove the money supply from the economy asymptotically. With this tax policy, money can pay a positive return and have value in equilibrium, even though agents always have positive moneyholdings and even though money pays no dividends. Money is no longer second-rate.

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