

# The Societal Benefits of Outside Versus Inside Bonds

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PRELIMINARY

## Abstract

We use a general equilibrium model of money to compare the use of ‘illiquid’ government-issued bonds (*outside* bonds) versus credit (*inside* bonds) to alleviate buyers’ liquidity constraints. We assume all transactions must be voluntary. This implies that the central bank cannot run deflation via lump-sum taxation and that agents must voluntarily redeem their outstanding private debt obligations. When the steady state outside bond to money ratio is taken as given, the allocation with inside bonds is more likely to dominate the allocation with inside bonds. When this ratio is also a policy choice, then the allocation with illiquid outside bonds dominates the inside bond allocation.

# 1 Introduction

In monetary models, agents often face binding liquidity constraints. Two solutions have been proposed for improving on the monetary allocation. Kocherlakota (2003) shows that the presence of ‘illiquid’ nominal government bonds (*outside* bonds) would allow those who are in need of cash to sell their bonds to those with ‘idle’ cash and thus improve the allocation. Alternatively, using a different monetary model, Berentsen, Camera and Waller (2006) show how credit (or *inside* bonds) improves the allocation since it allows agents to borrow or lend cash depending on their liquidity needs. These results raise the following questions: Within a common monetary framework, when are the allocations with illiquid outside bonds and inside bonds the same? When do they differ and why? Our focus in this paper is to address these questions.<sup>1</sup>

We construct a general equilibrium monetary model in the spirit of Lagos and Wright (2005) with fiat money and one-period bonds. As in Kocherlakota, we assume all trades must be voluntary meaning the central bank cannot run a deflation since it requires lump sum taxation of money balances. Each period consumers receive idiosyncratic preference shocks such that some of them want to consume a lot while others want to consume very little. This creates differential needs for liquidity across consumers. After these shocks occur but before the goods market opens agents can trade bonds for money. Those with high liquidity needs sell bonds for money while those with low liquidity needs sell money for bonds. As in Kocherlakota (2003) we assume that the outside bonds are illiquid in the sense that they cannot be used as a medium of exchange in the goods market. With inside bonds this issue does not arise since those acquiring inside bonds do not want to consume. Using the LW framework allows us to keep the distribution of money balances tractable and study the effects of steady state inflation on the allocations as opposed to one-time changes in the stock of outside bonds as done by Kocherlakota.

For the case of illiquid outside bonds, we find that for any positive inflation rate, the existence of illiquid outside bonds improves the allocation and thus generate societal benefits. When the distribution of the idiosyncratic preference shocks is non-degenerate, the optimal policy in this economy involves having a strictly positive but arbitrarily small inflation rate. It also requires having a positive but arbitrarily small ratio of nominal outside bond to money. In the same

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<sup>1</sup>We would like to thank Neil Wallace for suggesting we pursue this line of research.

environment, we then consider the use of inside bonds as in Berentsen, Camera and Waller (2006). We show that a monetary equilibrium with inside bonds also improves the allocation for any positive inflation rate and the best policy involves a strictly positive inflation rate.

The key finding of the paper is that under the optimal policies, the best allocation with illiquid inside bonds is more likely to dominate the allocation with inside bonds when the steady-state outside bond to money stock ratio is taken as given. When this ratio is also a policy instrument in the outside bond economy, then illiquid outside bonds will tend to dominate the inside bond economy.

Several papers are related to what we do here. Kehoe and Levine (2001) compare allocations in a dynamic economy when agents can acquire consumption goods in one case by selling their capital holdings and in another case by issuing debt subject to a borrowing constraint. They show that if agents are sufficiently patient, the allocations are the same in a deterministic environment but if they are sufficiently impatient, then the debt constrained allocation leads to a better allocation. Shi (2006) examines the implications of illiquid bonds in a monetary search model where there are legal restrictions preventing bonds from being used as a medium of exchange in some transactions but not others. The legal restrictions make outside bonds illiquid relative to money. He finds that if the preference shocks are sufficiently disperse but not greatly so, then having illiquid bonds is welfare improving. However, he never discusses optimal monetary policy, in particular, whether legal restrictions are beneficial under the optimal monetary policy. In Boel and Camera (2006) bonds are illiquid in the sense that there is a transaction fee for converting them into cash. Since agents have different discount factors and trading opportunities, for some parameter configurations, there is a welfare improving role for illiquid bonds under the optimal monetary policy. Finally, our work is closely related to that of Lagos and Rocheteau (2003) who study the use of illiquid bonds in a variant of the LW model. They find that under the optimal monetary policy (zero inflation) illiquid bonds are inessential.

The structure of the paper is as follows. In Section 2 we describe the environment. Section 3 contains analysis of the economy with outside bonds. Section 4 examines the economy with inside bonds. Section 5 concludes.

## 2 The environment

The environment builds on the Lagos and Wright (2005) framework as modified by Berentsen, Camera and Waller (2005). Time is discrete and in each period three perfectly competitive markets open and close sequentially. There is a  $[0,1]$  continuum of infinitively-lived agents. All agents produce and consume one perishable good.

The timing of the model is as follows. The first market is a financial market where agents trade money for bonds. The second market is a goods market where agents trade money for market 2 goods. In the third market all agents can produce and consume goods. At the beginning of the first market, agents get a preference shock that determines whether they can produce or consume in the second market. With probability  $1 - n$  an agent can consume and cannot produce. We refer to these agents as buyers. With probability  $n$ , an agent can produce and cannot consume. These are sellers.

Moreover, buyers receive an idiosyncratic preference shock to utility. They learn that they will get utility  $\varepsilon u(q)$  from  $q$  consumption in the second market, where  $u'(q), -u''(q) > 0$ . The shock  $\varepsilon$  has a continuous distribution  $F(\varepsilon)$  with support  $[0, \varepsilon_H]$ , is serially uncorrelated and has expected value  $\bar{\varepsilon} = \int_0^{\varepsilon_H} \varepsilon dF(\varepsilon)$ .<sup>2</sup> Producers in the second market incur a utility cost  $c(q) = q$  from producing  $q$  units of output. All trades in market 2 are anonymous and agents' trading histories in this market are private information, thus no trade credit exists. Hence there is a role for money, as sellers require immediate compensation for their production effort.

Following Lagos and Wright (2005) we assume that in the last market all agents consume and produce. They receive utility  $U(x)$  from  $x$  consumption, with  $U'(x), -U''(x) > 0$ ,  $U'(0) = \infty$ , and  $U'(+\infty) = 0$ . The goods are produced solely from inputs of labor according to a constant returns to scale production technology where one unit of the consumption good is produced with one unit of labor  $h$  generating one unit of disutility.<sup>3</sup> The discount factor across periods is  $\beta$ .

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<sup>2</sup>All of our results go through with a non-zero lower bound. Setting the lower bound of  $\varepsilon$  to zero simplifies the presentation of the results.

<sup>3</sup>As in Lagos and Wright (2005), these assumptions allow us to get a degenerate distribution of money holdings at the beginning of a period. The different utility functions  $U(\cdot)$  and  $u(\cdot)$  allow us to impose technical conditions such that in equilibrium all agents produce and consume in the last market.

## 2.1 First-best allocation

The expected steady state lifetime utility of the representative agent at the beginning of the period before types are realized is

$$(1 - \beta) \mathcal{W} = \int_0^{\varepsilon_H} [(1 - n) \varepsilon u(q_\varepsilon) - n q_s] dF(\varepsilon) + U(x) - h. \quad (1)$$

where  $q_\varepsilon$  is consumption and  $q_s$  production in market 2. We use (1) as our welfare criteria.

To derive the welfare maximizing quantities we assume that all agents are treated symmetrically. The planner then maximizes (1) subject to the feasibility constraint

$$Q \equiv (1 - n) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) = n q_s. \quad (2)$$

where  $Q$  is aggregate consumption. The first-best allocation satisfies

$$\begin{aligned} U'(x^*) &= 1 \text{ and} \\ \varepsilon u'[q^*(\varepsilon)] &= 1 \text{ for all } \varepsilon. \end{aligned} \quad (3)$$

These are the quantities chosen by a social planner who could force agents to produce and consume.

## 2.2 Outside bonds versus inside bonds

We analyze equilibria of the model under two different bond markets – a market for outside bonds and one for inside bonds. We do so in order to see if one of the bond markets generates a superior allocation relative to the other. Outside bonds are nominal government debt obligations whereas inside bonds are private debt obligations. We assume that the government has a record-keeping technology over bond trades and acts as the intermediary in the bond market. Consequently, bond holdings can simply be book-keeping entries – no physical object exists. This makes bonds incapable of being used as media of exchange hence they are illiquid. However, the government has no record-keeping technology over goods exchange and a buyer's promise to deliver outside bonds to a seller in the next market is not credible. Hence, fiat money is essential for trade in market 2. Because of this structure, agents must hold non-negative quantities of outside bonds. We also

consider the case where the outside bonds are physical objects that can be passed hand-to-hand as a medium of exchange.

Inside bonds are financial claims on private agents issued in a private bond market. Consequently, issuing inside bonds is equivalent to receiving credit as in Berentsen, Camera and Waller (2006). We assume a perfectly competitive financial market exists where intermediaries have a record-keeping technology over financial trades. The intermediaries acquire nominal debt obligations from agents and issue nominal debt obligations on themselves, which are securitized by their acquired claims on private agents. In this sense, private agents are still anonymous to each other but not to the financial intermediary.<sup>4</sup> However, this record-keeping technology does not exist in the goods market thus ruling out trade credit between buyers and sellers of goods.

In any model of credit, default is a serious issue. We consider an environment where repayment is voluntary – agents cannot be forced to redeem bonds they issued. The only punishment available is that an agent who defaults on his bonds is excluded from the financial sector in all future periods. Given this punishment, we derive conditions to ensure voluntary redemption and show that this may involve binding borrowing constraints, i.e. credit rationing. We assume the financial intermediaries honor their debt obligations.

### 2.3 Government

In the model with outside bonds, we assume a government exists that controls the supply of fiat currency and issues one-period nominal bonds. Government bonds are perfectly divisible, payable to the bearer and default free. One bond pays off one unit of currency at maturity. Denote  $M_t$  the per capita money stock and  $B_t$  the per capita stock of newly issued bonds in market 3 in period  $t$ . The change in the money stock is given by

$$M_t - M_{t-1} = \tau_t M_{t-1} + B_{t-1} - \rho_t B_t + P_t G_t \quad (4)$$

where  $P_t G_t$  is the nominal amount of government spending in period  $t$  in the centralized market and  $P_t$  is the money price of goods in market 3. The total change in the money stock is comprised of three components: first, a lump-sum transfer of cash (a ‘gift’ of cash); second, the net difference

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<sup>4</sup>An example is a bank who accepts nominal deposits and makes nominal loans. While the bank knows who it trades with, borrowers do not know the identity of depositors and vice versa.

between the cash created to redeem bonds,  $B_{t-1}$ , and the net cash withdrawal from selling  $B_t$  units of bonds at the price  $\rho_t$ ; and finally, the cash printed to pay for government goods. We assume there are positive initial stocks of money and outside bonds  $M_0$  and  $B_0$ . If  $\tau_t > 0$ , agents receive lump-sum transfers of money. For  $\tau_t < 0$ , the government must be able to extract money via lump-sum taxes from the economy. Throughout the paper we assume that  $\tau_t < 0$  is not feasible.<sup>5</sup> We can then write this expression as

$$M_t = \gamma_t M_{t-1} \quad (5)$$

where  $\gamma_t$  is the gross growth rate of the money supply in period  $t$ .

To simplify the analysis, we assume  $G_t = 0$  for all  $t$ . This implies that all money creation comes from paying off net nominal bond obligations  $B_{t-1} - \rho_t B_t$  and the lump-sum gifts of money  $\tau_t M_{t-1}$ . For the case of outside bonds we assume that  $\tau_t = 0$ . Consequently, the government budget constraint (4) reduces to<sup>6</sup>

$$M_t - M_{t-1} = B_{t-1} - \rho_t B_t. \quad (6)$$

This allows us to focus solely on how ‘open market operations’ affect the equilibrium allocation and allows us to ignore optimal taxation issues to finance government spending. Using (4) and (5) yields

$$\gamma_t - 1 = \frac{B_{t-1}}{M_{t-1}} (1 - \rho_t \eta_t) \quad (7)$$

where  $\eta_t = B_t/B_{t-1}$  is the gross growth rate of bonds. This equation relates the gross growth rate of money  $\gamma_t$  to the gross growth rate of bonds  $\eta_t$ . For a given ratio of bonds to money and a given discount on bonds an increase in  $\eta_t$  requires a decrease in  $\gamma_t$ . For notational ease, variables corresponding to the next period are indexed by  $+1$ , and variables corresponding to the previous period are indexed by  $-1$ .

In the model with inside bonds, we assume that  $B_t = 0$  in all periods but the government still

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<sup>5</sup>The inability to impose lump-sum taxes occurs in environments with limited enforcement. In such environments all trades must be voluntary and so lump-sum taxes of money are not feasible because the central bank cannot impose any penalties on the agents. If she could impose such penalties there would be no role for money since "producers could be forced to produce for households" (Kocherlakota 2003, p. 185). This implies that the government cannot run the Friedman rule which would implement the first-best allocation in this environment. Note also that proportional taxes on money holdings cannot substitute for the inability to impose lump-sum taxes since such taxes are neutral.

<sup>6</sup>All our results continue to hold for  $\tau_t > 0$ . The case  $\tau_t < 0$  is not feasible. See the previous footnote.

controls the amount of fiat currency in the economy. In this case, agents receive lump-sum gifts of money  $\tau_t M_{t-1} \geq 0$  and the money supply grows according to (5).

## 2.4 Stationary equilibria

In period  $t$ , let  $\phi = 1/P$  be the real price of money in market 3. We focus on symmetric and stationary equilibria where all agents follow identical strategies and where real allocations are constant over time. In a stationary equilibrium end-of-period real money balances are time-invariant

$$\phi M = \phi_{+1} M_{+1}. \quad (8)$$

Moreover, we restrict our attention to equilibria where  $\gamma$  is time invariant which implies that  $\phi/\phi_{+1} = P_{+1}/P = M_{+1}/M = \gamma$ .

## 3 Outside bonds

In this Section we analyze the economy with outside bonds. We initially look at the case where the bonds are illiquid then the case where they are liquid. Let  $V_1(m, b)$  denote the expected value from entering market 1 with  $m$  units of money and  $b$  outside bonds,  $V_{2j}(m, b)$ ,  $j = \varepsilon, s$ , the expected value from entering market 2 with  $m$  units of money and  $b$  bonds, and  $V_3(m, b)$  the expected value from entering market 3 with  $m$  and  $b$ . For notational simplicity we suppress the dependence of the value function on the time index  $t$ . In what follows we look at a representative period  $t$  and work backward, from the third to the first market.

**The third market** In the third market, the problem of a representative agent is:

$$\begin{aligned} V_3(m, b) &= \max_{x, h, m_{+1}, b_{+1}} U(x) - h + \beta V_1(m_{+1}, b_{+1}) \\ \text{s.t. } & x + \phi m_{+1} + \phi \rho b_{+1} = h + \phi m + \phi b. \end{aligned}$$



where  $\rho$  is the money price of bonds in the third market. Using the budget constraint to eliminate  $h$  in the objective function, one obtains the first-order conditions  $U'(x) = 1$  and

$$\beta V_1^m(m_{+1}, b_{+1}) \leq \phi \quad (= 0 \text{ if } m_{+1} > 0) \quad (9)$$

$$\beta V_1^b(m_{+1}, b_{+1}) \leq \phi \rho \quad (= 0 \text{ if } b_{+1} > 0). \quad (10)$$

$V_1^m(m_{+1}, b_{+1})$  is the marginal value of taking an additional unit of money into period  $t + 1$ . Since the marginal disutility of working is 1,  $-\phi$  is the utility cost of acquiring one unit of money in the third market of period  $t$ .  $V_1^b(m_{+1}, b_{+1})$  is the marginal value of taking additional bonds into period  $t + 1$ . Since the marginal disutility of working is 1,  $-\phi\rho$  is the utility cost of acquiring a bond in the third market of period  $t$ . The implication of (9) and (10) is that all agents enter the following period with the same amount of money and the same amount of bonds.

The envelope conditions are

$$V_3^m = \phi; V_3^b = \phi \quad (11)$$

As in Lagos-Wright (2005) the value function is linear in wealth.

**The second market** Let  $q_\varepsilon$  denote the quantities consumed by a type  $\varepsilon$  buyer and  $q_s$  the quantity produced by a seller trading in market 2. Let  $p$  be the nominal price of goods in market 2.

A seller who holds  $m$  money and  $b$  bonds at the opening of the second market has expected life-time utility  $V_{2s}(m, b) = \max_{q_s} [-q_s + V_3(m + pq_s, b)]$ . Using (11), the first-order condition reduces to

$$p = 1/\phi. \quad (12)$$

Consequently, sellers are indifferent on how much they produce. Nevertheless in a symmetric equilibrium they all produce the same amount.

A type  $\varepsilon$  buyer has expected lifetime utility  $V_{2\varepsilon}(m, b) = \max_{q_\varepsilon} [\varepsilon u(q_\varepsilon) + V_3(m - q_\varepsilon/\phi, b)]$  s.t.  $q_\varepsilon \leq \phi m$  where (12) has been used to eliminate  $p$ . Using (11) and (12) the buyer's first-order conditions can be written as

$$\varepsilon u'(q_\varepsilon) = 1 + \lambda_\varepsilon \quad (13)$$

where  $\lambda_\varepsilon$  is the multiplier of the buyer's budget constraint. If the budget constraint is not binding,

$\lambda_\varepsilon = 0$ . If it is binding, then  $\varepsilon u'(q_\varepsilon) > 1$  which means trades are inefficient. In this case the buyer spends all of his money.

Using the envelope theorem, the marginal values of bonds and the marginal values of money for buyers and sellers at the beginning of the second market are

$$V_{2\varepsilon}^b = V_{2s}^b = \phi \quad (14)$$

$$V_{2\varepsilon}^m = \phi \varepsilon u'(q_\varepsilon) \text{ and } V_{2s}^m = \phi. \quad (15)$$

Finally, market clearing satisfies (2).

**The first market** Let  $y_j$  denote the quantity of outside bonds bought by an agent of type  $j = \varepsilon, s$  in market 2. Let  $a$  be the price of bonds in market 1, consequently the nominal interest rate earned by acquiring a bond in this market is  $i = (1 - a)/a$ , which is greater than zero if and only if  $a < 1$ . Note that there are two short-selling constraints: agents cannot sell more bonds or money than they hold.

An agent who holds  $m$  money and  $b$  bonds at the opening of the first market has expected lifetime utility

$$\begin{aligned} V_1(m, b) = & (1 - n) \int_0^{\varepsilon_H} V_{2\varepsilon}[m - y_\varepsilon / (1 + i), b + y_\varepsilon] dF(\varepsilon) \\ & + n V_{2s}[m - y_s / (1 + i), b + y_s] \end{aligned} \quad (16)$$

where for  $j = \varepsilon, s$   $y_j = \arg \max_y V_{2j}[m - y / (1 + i), b + y]$  s.t.  $m - y / (1 + i) \geq 0$  and  $b + y \geq 0$ .

The first-order condition is

$$-V_{2j}^m / (1 + i) + V_{2j}^b - \phi \mu_j / (1 + i) + \phi \theta_j = 0 \quad (17)$$

where  $\phi \mu_j$  is the Lagrange multiplier on the short-selling constraint  $m - y_j / (1 + i) \geq 0$  and  $\phi \theta_j$  is the Lagrange multiplier on  $b + y_j \geq 0$ . Obviously, both can not bind at the same time.

Consider first an agent who will be a producer in market 2. If  $i < 0$ , then  $y_s = -b$  and goods producers sell their bonds for money in market 1. This obviously cannot be an equilibrium and is ignored for the remainder of the paper which means that  $\theta_s = 0$ . We can then use (14) and (15) to

substitute  $V_{2j}^m$  and  $V_{2j}^b$  in (17) to get

$$\mu_s = i \quad (18)$$

If  $i > 0$ , it is optimal to sell the entire money holdings for bonds. If  $i = 0$ , the producer is indifferent on how much money to supply. Thus, a producer's bond demand is

$$\begin{aligned} y_s &\in [-b, m(1+i)] \quad \text{if } i = 0 \\ y_s &= m(1+i) \quad \text{if } i > 0 \end{aligned} \quad (19)$$

Consider next an agent who will be a buyer in market 2. Since Inada conditions are assumed on  $u(q)$  a buyer will always carry some money from market 1 to market 2. Thus,  $\mu_\varepsilon = 0$ . Accordingly, we can use (14) and (15) to write (17) as follows

$$\varepsilon u'(q_\varepsilon) = (1 + \theta_\varepsilon)(1 + i). \quad (20)$$

If  $\varepsilon u'(q_\varepsilon) = 1 + i$ , then  $\theta_\varepsilon = 0$  and  $y_\varepsilon \leq b$ . In this case the buyer is indifferent between holding bonds or money. In this case, an increase in  $i$  makes bonds more attractive relative to holding onto the cash for consumption in market 2. Consequently, buyers trade more of their money balances for illiquid outside bonds, which reduces  $q_\varepsilon$ .

Finally, if  $\varepsilon u'(q_\varepsilon) > 1 + i$ , then  $\theta_\varepsilon > 0$  and he sells all of his bonds implying  $y_\varepsilon = -b$ . Thus, a buyer's bond demand is

$$\begin{aligned} y_\varepsilon &\in [-b, m(1+i)] \quad \text{if } \varepsilon u'(q_\varepsilon) = 1 + i \\ y_\varepsilon &= -b \quad \text{if } \varepsilon u'(q_\varepsilon) > 1 + i. \end{aligned} \quad (21)$$

Because a buyer's desired consumption is increasing in  $\varepsilon$ , there is a critical value for the taste index  $\tilde{\varepsilon}$  such that

$$\tilde{\varepsilon} u'(\tilde{q}) = 1 + i \quad (22)$$

If  $\varepsilon \leq \tilde{\varepsilon}$ , he does not sell all his bonds while if  $\varepsilon \geq \tilde{\varepsilon}$  he sells all his bonds and consumes

$$\tilde{q} = \phi m + \phi b / (1 + i). \quad (23)$$

Accordingly, a buyer's consumption satisfies

$$q_\varepsilon = \begin{cases} u'^{-1}[(1+i)/\varepsilon] & \text{if } \varepsilon \leq \tilde{\varepsilon} \\ u'^{-1}[(1+i)/\tilde{\varepsilon}] & \text{if } \varepsilon \geq \tilde{\varepsilon} \end{cases} \quad (24)$$

Finally, apply the envelope theorem to equation (16) and using equations (14), (15) the marginal value of money and the marginal value of bonds satisfy

$$\frac{\partial V_1(m, b)}{\partial m} = (1-n) \int_0^{\varepsilon_H} \phi [\varepsilon u'(q_\varepsilon) + \mu_\varepsilon] dF(\varepsilon) + n\phi(1 + \mu_s) \quad (25)$$

$$\frac{\partial V_1(m, b)}{\partial b} = (1-n) \int_0^{\varepsilon_H} \phi(1 + \theta_\varepsilon) dF(\varepsilon) + n\phi(1 + \theta_s) \quad (26)$$

### 3.1 Equilibrium

To derive the symmetric stationary equilibrium we have to solve for the equilibrium quantities and prices. In any symmetric equilibrium  $m = M_{-1}$  and  $b = B_{-1}$ . Then use (25) and the first-order conditions (9) along with  $a = 1/(1+i)$  to get

$$\frac{a\gamma}{\beta} = (1-n) \int_0^{\varepsilon_H} (1 + \theta_\varepsilon) dF(\varepsilon) + n \quad (27)$$

and (10) and (26) to get

$$\frac{\rho_{-1}\gamma}{\beta} = (1-n) \int_0^{\varepsilon_H} (1 + \theta_\varepsilon) dF(\varepsilon) + n. \quad (28)$$

This implies that in any symmetric stationary equilibrium  $\rho_{-1} = a$  and so from now on we set  $\rho_{-1} = a$  and ignore the second equation. Using (18), (20) and (27) we obtain

$$\frac{\gamma - \beta}{\beta} = (1-n) \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) + ni. \quad (29)$$

Using (24) and rearranging yields

$$\frac{\gamma - \beta(1+i)}{\beta(1+i)} = (1-n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left( \frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon). \quad (30)$$

This is an equation in  $i$  and  $\tilde{\varepsilon}$ . We now derive a second equation  $i$  and  $\tilde{\varepsilon}$  which we can then use to define the equilibrium.

In any stationary equilibrium the stocks of bonds  $B$  and money  $M$  must grow at the same rate  $\gamma$  implying  $a$  is constant so  $a = 1/(1+i) = \rho$ . We can then rewrite the government budget constraint (6) to solve for  $1+i$  as a function of  $\gamma$  and  $M_{-1}/B_{-1}$

$$1+i = \frac{\gamma}{1 + (1-\gamma) \frac{M_{-1}}{B_{-1}}} \quad (31)$$

If  $\gamma = 1$ , then  $i = 0$ . In a stationary equilibrium  $M_{-1}/B_{-1} = M_0/B_0$  for all  $t$ . A non-negative nominal interest rate requires the denominator to be positive or  $1 + B_0/M_0 > \gamma$ , and  $\gamma \geq 1$ . Thus, for a given ratio of nominal outside bonds to outside money, the range of feasible  $\gamma$  is bounded by this expression. Thus, define  $\bar{\gamma} \equiv 1 + B_0/M_0$  and as  $\gamma \rightarrow \bar{\gamma}$ ,  $i \rightarrow \infty$  and all quantities go to zero. Substitute (31) into (22) and (30) to get

$$\tilde{\varepsilon} u'(\tilde{q}) = \frac{\gamma}{1 + (1-\gamma) \frac{M_0}{B_0}} \quad (32)$$

$$\frac{1 + (1-\gamma) M_0/B_0}{\beta} - 1 = (1-n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left( \frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) \quad (33)$$

For given values of  $\gamma$  and  $M_0/B_0$ , the first equation pins down  $\tilde{q}$  as a function of  $\tilde{\varepsilon}$  while the second determines  $\tilde{\varepsilon}$ . Thus, if a unique value  $0 \leq \tilde{\varepsilon} \leq \varepsilon_H$  solves this equation then we have a unique stationary equilibrium. Given  $\tilde{q}$  and  $\tilde{\varepsilon}$  we can then solve for all endogenous quantities and prices.

**Definition 1** *A stationary monetary equilibrium is an  $\tilde{\varepsilon}$  that satisfies (33).*

In what follows define  $\gamma_H \equiv 1 + (1-\beta) B_0/M_0 < \bar{\gamma}$ .

**Proposition 1** *For  $1 \leq \gamma < \bar{\gamma}$  a unique stationary monetary equilibrium exists. If  $\gamma \leq \gamma_H$ , then  $\tilde{\varepsilon} \in (0, \varepsilon_H]$ . If  $\gamma_H < \gamma < \bar{\gamma}$  then  $\tilde{\varepsilon} > \varepsilon_H$ .*

The essence of this proposition is that for sufficiently low inflation rates, high  $\varepsilon$  buyers will face binding bond sale constraints and so  $\varepsilon u'(q_\varepsilon) > 1+i$ . Thus, on the margin they would like to sell more bonds but do not have them. For sufficiently high inflation rates, no buyers face binding bond sales constraints and so  $\varepsilon u'(q_\varepsilon) = 1+i$  for all  $\varepsilon$  and from (33) we have  $1 + (1-\gamma) M_0/B_0 = \beta$  and so from (31)

$$i = \frac{\gamma - \beta}{\beta}. \quad (34)$$

Note that illiquid bonds trade at a discount if and only if  $\gamma > 1$ . If  $\gamma = 1$ , then  $i = 0$  ( $a = 1$ ) and so the question is whether the allocation with illiquid bonds differs from the allocation when they are liquid. Moreover, we also would like to know whether inflation can be welfare improving and what is the optimal monetary when bonds are illiquid. We now discuss these two issues in the following sections.

**Liquid bonds** Suppose now that instead of book-keeping entries, bonds are tangible objects that can be used as a medium of exchange in market 2, i.e., they are liquid. One can show that in this environment, the allocation is the same as the allocation without bonds. The intuition and proof for this result is straightforward and provided in Kocherlakota (2003, p. 184): "If bonds are liquid as money, then people will only hold money if nominal interest rates are zero. But then the bonds can just be replaced by money: there is no difference between the two instruments at all." An interesting implication of this result is that "any essentiality of nominal bonds can be traced directly to their (relative) illiquidity (Kocherlakota 2003, p. 184)."

It is straightforward to show that with liquid outside bonds the quantities solve

$$\frac{\gamma - \beta}{\beta} = (1 - n) \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) \quad (35)$$

Comparing (29) and (35) it is clear that for  $i = 0$  the quantities solving these expressions are the same so the allocations are identical regardless of the bonds' liquidity properties. For  $i > 0$  the right-hand side of (29) is larger than (35) because the marginal value of money is higher with illiquid outside bonds in this case. The reason is that now, if an agent brings in money and does not need it, either because they are a seller or a low  $\varepsilon$  buyer, he can effectively trade the money for an interest bearing asset that compensates them for bringing 'idle' money into market 1. This increases the demand for money in market 3 and thus the real value of money balances. As a result, there is higher expected consumption in market 2 thereby lowering the expected marginal utility of consumption in market 2. Thus, illiquid bonds will improve the allocation relative to the liquid bonds case if  $i > 0$ . However,  $i > 0$  requires  $\gamma > 1$ . We now explore whether or not it is optimal to set  $\gamma > 1$ .

**Is inflation welfare improving?** Since illiquid outside bonds do not improve the allocation at  $\gamma = 1$ , is it optimal for the monetary authority to create inflation by setting  $\gamma > 1$ ? Doing so has two effects. Raising  $\gamma$  above 1 makes illiquid bonds essential and allows those with low liquidity needs to be compensated for holding idle balances. This improves welfare. However, the higher inflation reduces real money balances, expected consumption and welfare. We would like to know under what conditions inflation is welfare improving.

**Proposition 2** *With an exogenously given value of  $M_0/B_0$ , there exists a cutoff value  $\tilde{\varepsilon} \in (0, \varepsilon_H)$  such that positive steady state inflation is welfare improving if  $\varepsilon_H > \tilde{\varepsilon} > \tilde{\varepsilon}$ .*

The intuition behind this result is the following. At  $\gamma = 1$ ,  $i = 0$  some buyers consume their first-best quantities while others do not. Consequently, if there is a sufficient measure of agents consuming the first-best quantity ( $\tilde{\varepsilon} > \tilde{\varepsilon}$ ), then there is a first-order welfare gain from moving some consumption from those buyers at their first-best quantities to those who are not. However, buyers giving up consumption on the margin have to be compensated, which can be achieved by giving them interest bearing bonds. This requires raising  $\gamma$  marginally above one so that  $i > 0$ . Thus, for sufficiently large values of  $\tilde{\varepsilon}$ , the gain from this redistribution of consumption dominates the reduction in average consumption that occurs from the erosion of real money balances. In this case, illiquid interest-bearing bonds have societal benefits. However, at a sufficiently high inflation rate,  $\gamma > \gamma_H$ , then  $\tilde{\varepsilon} = \varepsilon_H$  and no buyers are constrained on the margin since  $\varepsilon u'(q) = 1 + i$  for all  $\varepsilon$ . Thus, increasing inflation raises  $i$  lowering  $q$  for all buyers, which reduces welfare. On the contrary, if there are few agents at the first-best quantity when  $\gamma = 1$ , then there is not enough consumption to redistribute among the different buyers to offset the loss of purchasing power that occurs from the higher inflation. In that case, welfare is decreasing in inflation.

Finally, a necessary condition for the welfare improving role of illiquid bonds is that there is more than one buyer type. With only one buyer type in equilibrium all buyers to consume less than their first-best quantities for  $\gamma = 1$ . Consequently, no beneficial redistribution of consumption can occur and inflation merely lowers the quantities consumed for all buyers, thus lowering welfare.

**Optimal choice of  $M_0/B_0$**  Bonds improve the allocation when  $i > 0$ . From (31) there are two factors that affect  $i$ : the inflation rate and the relative supply of money to nominal bonds. Any

optimal allocation requires that the marginal utility of consumption is equalized across all buyers. This requires that the left-hand side of (33) equals zero implying

$$\gamma = 1 + \frac{B_0}{M_0} (1 - \beta) \quad (36)$$

Using this expression and (32) we get

$$\varepsilon u'(q_\varepsilon) = \gamma/\beta \quad \forall \varepsilon.$$

It is easy to see that the optimal policy is to choose  $\gamma$  to be as small as possible yet keep it above one. Thus, from (36) this can be achieved by letting  $B_0 \rightarrow 0$  implying  $\gamma$  is arbitrarily close to one. The reason for having  $B_0$  being very small is similar to the arguments given by Kocherlakota (2003). To summarize, under the optimal policy the limiting allocation satisfies

$$\varepsilon u'(q_\varepsilon) = 1/\beta \quad \forall \varepsilon. \quad (37)$$

Thus, when the money-bond ratio is also a policy instrument, the optimal policy makes the nominal interest rate equals the real interest rate.

## 4 Inside Bonds

In this section we analyze the model where there are no outside bonds but inside bonds can be traded in market 1. In market 1, sellers and low  $\varepsilon$  buyers can use their idle cash balances to acquire nominal bonds from the financial intermediary, which are redeemed in market 3. High  $\varepsilon$  buyers can issue nominal bonds in market 1 to the financial intermediary and redeem them in market 3. Inside bonds are perfectly divisible and one inside bond pays off one unit of fiat currency in market 3. Again, we focus on symmetric and stationary equilibria where all agents follow identical strategies and where real allocations are constant over time. In a stationary equilibrium end-of-period real money balances are time-invariant.

Since inside bonds are not held across periods, let  $V_1(m)$  denote the expected value from entering market 1 with  $m$  units of money,  $V_{2j}(m, b)$ ,  $j = \varepsilon, s$ , the expected value from entering market 2



with  $m$  units of money and  $b$  inside bonds, and  $V_3(m, b)$  the expected value from entering market 3 with  $m$  and  $b$ .

In the third market, the problem of a representative agent is:

$$\begin{aligned} V_3(m, b) &= \max_{x, h, m_{+1}} U(x) - h + \beta V_1(m_{+1}) \\ \text{s.t. } & x + \phi m_{+1} = h + \phi m + \phi b. \end{aligned}$$

again yielding the first-order conditions  $U'(x) = 1$ ,  $\phi = \beta V_1^m(m_{+1})$  and the envelope condition  $V_3^m = \phi$ .

In the second market, the agents' problems are unaffected by the types of bonds they hold so (12)-(15) are unaffected.

**The first market** The first market with inside bonds differs somewhat from the case with outside bonds so we discuss it in more detail. Let  $b_j$  denote the quantity of private bonds purchased by an agent of type  $j = \varepsilon, s$  in market 2. If  $b_j < 0$ , the agent is issuing private bonds, i.e., he is borrowing. Let  $a$  be the price of an inside bond in market 1, consequently the nominal interest rate earned by acquiring an inside bond in this market is  $i = (1 - a)/a$ , which is greater than zero if and only if  $a < 1$ . Note that there is only one short-selling constraint: agents cannot sell more money than they hold. However, because of the possibility of default agents cannot borrow more than  $\bar{b}/(1 + i)$ , which implies the constraint  $b \geq -\bar{b}$ . Agents take this constraint as given. However, in equilibrium the upper bound  $\bar{b}$  is determined endogenously.

An agent who holds  $m$  units of money at the opening of the first market has expected lifetime utility

$$\begin{aligned} V_1(m) &= (1 - n) \int_0^{\varepsilon_H} V_{2\varepsilon}[m - b_\varepsilon/(1 + i), b_\varepsilon] dF(\varepsilon) \\ &\quad + n V_{2s}[m - b_s/(1 + i), b_s] \end{aligned} \tag{38}$$

where for  $j = \varepsilon, s$   $\max_{b_j} V_{2j}[m - b_j/(1 + i), b_j]$  s.t.  $m - b_j/(1 + i) \geq 0$  and  $b_j \geq -\bar{b}$ . The first-order condition is

$$-V_{2j}^m/(1 + i) + V_{2j}^b - \phi \mu_j/(1 + i) + \phi \theta_j = 0 \tag{39}$$

where  $\phi\mu_j$  is the Lagrange multiplier on the short-selling constraint  $m - b_j / (1 + i) \geq 0$  and  $\phi\theta_j$  is the Lagrange multiplier on  $b \geq -\bar{b}$ . Obviously, both can not bind at the same time.

Consider first an agent who will be a producer in market 2. If  $i < 0$ , then  $b_s = -\bar{b}$  and goods producers issue bonds to acquire money in market 1. This obviously cannot be an equilibrium and is ignored for the remainder of the paper which means that  $\theta_s = 0$ . If  $i > 0$ , it is optimal to spend your entire money holdings to acquire private bonds. We can then use (14) and (15) to substitute  $V_{2j}^m$  and  $V_{2j}^b$  in (17) to get

$$\mu_s = i \quad (40)$$

If  $i = 0$ , a producer is indifferent on how much money to supply. Thus, a producer's bond demand is

$$\begin{aligned} b_s &\in [-\bar{b}, m(1 + i)] & \text{if } i = 0 \\ b_s &= m(1 + i) & \text{if } i > 0 \end{aligned} \quad (41)$$

Consider next an agent who will be a buyer in market 2. Since Inada conditions are assumed on  $u(q)$  a buyer will always carry some money from market 1 to market 2. Thus,  $\mu_\varepsilon = 0$ . Accordingly, we can use (14) and (15) to write (39) as follows

$$\varepsilon u'(q_\varepsilon) = (1 + \theta_\varepsilon)(1 + i). \quad (42)$$

If  $\varepsilon u'(q_\varepsilon) = 1 + i$ , then  $\theta_\varepsilon = 0$  and  $b_\varepsilon > -\bar{b}$ . Finally, if  $\varepsilon u'(q_\varepsilon) > 1 + i$ , then  $\theta_\varepsilon > 0$  and  $b_\varepsilon = -\bar{b}$ . Thus, a buyer's bond demand is

$$\begin{aligned} b_\varepsilon &\in [-\bar{b}, m(1 + i)] & \text{if } \varepsilon u'(q_\varepsilon) = 1 + i \\ b_\varepsilon &= -\bar{b} & \text{if } \varepsilon u'(q_\varepsilon) > 1 + i. \end{aligned} \quad (43)$$

As was the case in the outside bonds economy, because a buyer's desired consumption is increasing in  $\varepsilon$ , there is a critical value for the taste index  $\tilde{\varepsilon}$  such that

$$\tilde{\varepsilon} u'(\tilde{q}) = 1 + i \quad (44)$$

If  $\varepsilon \leq \tilde{\varepsilon}$ , he does not issue the maximal amount of bonds while if  $\varepsilon \geq \tilde{\varepsilon}$  he issues the maximal

amount and consumes

$$\tilde{q} = \phi m + \phi \bar{b} / (1 + i). \quad (45)$$

Accordingly, a buyer's consumption satisfies

$$q_\varepsilon = \begin{cases} u'^{-1} [(1 + i) / \varepsilon] & \text{if } \varepsilon \leq \tilde{\varepsilon} \\ u'^{-1} [(1 + i) / \tilde{\varepsilon}] & \text{if } \varepsilon \geq \tilde{\varepsilon} \end{cases} \quad (46)$$

Finally, apply the envelope theorem to equation (38) and using (15) the marginal value of money satisfies

$$\frac{\partial V_1(m, b)}{\partial m} = (1 - n) \int_0^{\varepsilon_H} \phi [\varepsilon u'(q_\varepsilon) + \mu_\varepsilon] dF(\varepsilon) + n\phi(1 + \mu_s) \quad (47)$$

#### 4.1 Stationary equilibria

To derive the symmetric stationary equilibrium we have to solve for the equilibrium quantities and prices. In any symmetric equilibrium  $m = M_{-1}$ . Then use (42), (46) and (47), to write  $\phi = \beta V_1^m(m_{+1})$  as follows

$$\frac{\gamma - \beta(1 + i)}{\beta(1 + i)} = (1 - n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left( \frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon). \quad (48)$$

Comparing (48) to (30) we see that the cutoff values and thus the allocations for the outside bond and inside bond economies will be the same if the nominal interest rate in each economy is the same. Hence, what needs to be determined is whether or not the nominal interest rates will differ across the two economies.

We now derive the value of  $\bar{b}$ . Since  $\bar{b}$  is a nominal variable what we really want is the real value  $\bar{\ell} \equiv \phi \bar{b}$ . This quantity is the maximal real amount that an agent is willing to repay in the last market. For buyers entering the last market with no money, who redeem their bonds, the expected discounted utility in a steady state is

$$V_3(m, b) = \max_{x, h, m_{+1}} U(x) - h_\varepsilon + \beta V_1(m_{+1})$$

where  $h_\varepsilon$  is a buyer's production in market 3 if he redeems his bonds. A defaulting buyer's expected

discounted utility is

$$\widehat{V}_3(m, b) = U(\widehat{x}) - \widehat{h}_\varepsilon + \beta \widehat{V}_1(\widehat{m}_{1,+1})$$

where the hat indicates the optimal choice by a defaulter. The real borrowing constraint makes the agent indifferent between redeeming his bonds or defaulting so that  $V_3(m, b) = \widehat{V}_3(m, b)$ .

**Defaulter** When no enforcement exists, agents must voluntarily redeem their bonds. The only punishment for default is permanent exclusion from the inside bond market. Let  $\widehat{q}_\varepsilon$  denote the quantity purchased by an agent with preference shock  $\varepsilon$  who is excluded from the inside bond market. It is straightforward to show that the marginal value of money for a defaulter satisfies

$$\frac{\gamma - \beta}{\beta} = (1 - n) \int_0^{\varepsilon_H} [\varepsilon u'(\widehat{q}_\varepsilon) - 1] dF(\varepsilon) \quad (49)$$

while (48) continues to determine the value of money for a non-defaulter. Since an agent who defaults can only use the money balances he brings into the period to buy goods, then there is a critical value  $\widehat{\varepsilon}$  such that

$$\widehat{\varepsilon} u'(\widehat{q}) = 1 \quad (50)$$

His consumption is

$$\widehat{q}_\varepsilon = \begin{cases} q_\varepsilon^* & \text{if } \varepsilon \leq \widehat{\varepsilon} \\ u'^{-1}(1/\widehat{\varepsilon}) & \text{if } \varepsilon \geq \widehat{\varepsilon} \end{cases} \quad (51)$$

which means that he consumes the first-best quantity  $q_\varepsilon^*$  for  $\varepsilon \leq \widehat{\varepsilon}$  and the same quantity  $u'^{-1}(1/\widehat{\varepsilon})$  for all  $\varepsilon \geq \widehat{\varepsilon}$ .

**Real borrowing constraint** Given a borrowing constraint there are two possibilities: 1) the borrowing constraint is nonbinding for all agents or 2) it binds for some agents. The following Lemma is used for the remainder of this section.

**Lemma 3** *The real borrowing constraint is*

$$\frac{\bar{\ell}}{1+i} = \frac{\beta}{(1+i)(1-\beta)} \left[ (1-n) \Psi(q_\varepsilon, \widehat{q}_\varepsilon) + \left( \frac{\gamma - \beta}{\beta} \right) (\widehat{q} - Q) \right] \quad (52)$$

where

$$\Psi(q_\varepsilon, \hat{q}_\varepsilon) = \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) - \int_0^{\varepsilon_H} [\varepsilon u(\hat{q}_\varepsilon) - \hat{q}_\varepsilon] dF(\varepsilon).$$

We can now define a monetary equilibrium with inside bonds.

**Definition 2** *A monetary equilibrium with unconstrained borrowing is a set  $\{q_\varepsilon, \hat{q}_\varepsilon, \bar{\ell}, i, \hat{\varepsilon}, \tilde{\varepsilon}\}$  satisfying  $\tilde{\varepsilon} = 0$ , (46), (51), (52) and*

$$0 < \ell_H < \bar{\ell} \tag{53}$$

$$\frac{\gamma - \beta}{\beta} = i \tag{54}$$

$$\frac{\gamma - \beta}{\beta} = (1 - n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left( \frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) \tag{55}$$

Equation (54) is obtained by using (44) in (48) while (55) results from substituting (50) in (49).

**Definition 3** *A monetary equilibrium with constrained borrowing is a set  $\{q_\varepsilon, \hat{q}_\varepsilon, \bar{\ell}, i, \hat{\varepsilon}, \tilde{\varepsilon}\}$  satisfying (46), (51), (52) and*

$$\frac{\bar{\ell}}{1 + i} = u'^{-1}[(1 + i)/\tilde{\varepsilon}] - Q \tag{56}$$

$$\frac{\gamma - \beta(1 + i)}{\beta(1 + i)} = (1 - n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left( \frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) \tag{57}$$

$$\frac{\gamma - \beta}{\beta} = (1 - n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \left( \frac{\varepsilon}{\tilde{\varepsilon}} - 1 \right) dF(\varepsilon) \tag{58}$$

Equation (56) comes from a credit constrained borrower's cash constraint in market 2 while (57) is derived using (46) for  $\varepsilon \leq \tilde{\varepsilon}$  in (48) and  $u'(q_\varepsilon) = u'(\tilde{q})$  for  $\varepsilon > \tilde{\varepsilon}$ . Note that from (57) and (58) since the right-hand sides are decreasing functions of  $\varepsilon$  that  $\tilde{\varepsilon} > \hat{\varepsilon}$  if  $i > 0$ .

**Proposition 4** *For a value  $\bar{\beta}$  sufficiently close to one, if  $\beta \in [\bar{\beta}, 1)$  then there is an  $\hat{i} > 0$  such that:*

- (i) *If  $i \geq \hat{i}$ , then a unique monetary equilibrium with unconstrained borrowing exists.*
- (ii) *If  $0 < i < \hat{i}$ , then a monetary equilibrium with constrained borrowing may exist.*
- (iii) *If  $i = 0$  a unique monetary equilibrium without borrowing exists.*

Since  $i = 0$  at  $\gamma = 1$  inside bonds are not traded and the allocation is the same as the illiquid outside bonds since they are inessential at  $i = 0$  at  $\gamma = 1$ . Furthermore, for sufficiently high inflation

rates (54) and (34) are same so the allocation in an unconstrained borrowing equilibrium is the same as the illiquid outside bond allocation when  $\gamma \geq \gamma_H$ . For low inflation rates, the allocation with inside bonds will differ from that with outside bonds.

## 4.2 Is inflation welfare improving?

In an unconstrained borrowing equilibrium, it is straightforward to show that inflation is always welfare reducing since it reduces the real value of money balances and consumption for all agents. However, in a constrained borrowing equilibrium, it may be optimal for the monetary authority to set  $\gamma > 1$ . As was the case with outside bonds, at  $\gamma = 1$  and  $i = 0$  some buyers consume their first-best quantities while others do not. Consequently, there is a first-order welfare gain from moving some consumption from those buyers at their first-best quantities to those who are not. In addition to this welfare gain, there is another positive welfare effect from raising  $\gamma$  above 1 – it increases the cost of being excluded from the banking system. This relaxes the borrowing constraint and creates a first-order welfare gain. However, the higher inflation reduces real money balances and expected consumption, which lowers welfare.<sup>7</sup> We can thus state the following

**Proposition 5** *In a constrained borrowing equilibrium, if  $\beta > \left[1 + n + (1 - n) \int_0^{\tilde{\varepsilon}} dF(\tilde{\varepsilon})\right]^{-1}$ , then a positive steady state inflation rate maximizes welfare  $\forall \tilde{\varepsilon}$ .*

The key part of this proposition is that inflation is welfare improving for any distribution of  $\varepsilon$ , even a degenerate one, as long as agents do not discount the future too much. This was not the case with illiquid bonds – inflation was only welfare improving if the distribution of the  $\varepsilon$  shocks was sufficiently dispersed. Thus, it is in this sense that the inside bond economy dominates the illiquid outside bond economy – at  $\gamma = 1$  the allocations are the same and inflation is more likely to be welfare improving in the inside bond economy.

To compare the allocations with inside and outside bonds, it is useful to examine the inside bond allocation when redemption of inside bonds can be forced on agents. In this case default is not feasible and agents are unconstrained in issuing private bonds. It is straightforward to show

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<sup>7</sup>BCW show that for a *degenerate* distribution for  $\varepsilon$ , inflation is always welfare increasing for sufficiently high values of  $\beta$ . In this section we extend those results to the case of a non-degenerate distribution of  $\varepsilon$ .

that in a stationary equilibrium, the interest rate and all of the quantities  $q_\varepsilon$  are determined by

$$\frac{\gamma - \beta}{\beta} = i \quad (59)$$

$$\varepsilon u'(q_\varepsilon) = 1 + i \quad (60)$$

The first equation comes from the agent's decision of how much money to bring into the period. In equilibrium they are indifferent between acquiring an additional unit of money in the centralized market or borrowing it in the private bond market. The second equation comes from the buyers' decisions of how much to borrow. Since  $q_\varepsilon$  is unambiguously decreasing in  $i$ , it follows that  $\gamma = 1$  generates the best allocation. This immediately implies that the best allocation with enforcement satisfies

$$\varepsilon u'(q_\varepsilon) = 1/\beta \quad \forall \varepsilon.$$

This immediately implies the following:

**Proposition 6** *The best allocation with outside bonds when  $M_0/B_0$  is a policy instrument generates higher welfare than the best allocation with inside bonds.*

The proof immediately follows from the fact that the best allocation with outside bonds replicates the best allocation with inside bonds when enforcement is possible. The best allocation when enforcement is feasible must dominate the one when redemption is voluntary because the borrowing constraint is binding for  $\gamma$  close to one.

What is the intuition for this result? In the inside bond economy, there is only one instrument,  $\gamma$ , to solve to inefficiencies – the intensive margin of consumption and the distribution of consumption across agents. In the outside bond economy where  $M_0/B_0$  is endogenously chosen means the government now has two instruments to deal with two inefficiencies. Thus, it is not surprising that the allocation with outside bonds dominates inside bonds in this case.

## 5 Conclusion

When agents are liquidity constrained, two options exist to relax this constraint: sell assets or issue debt. We have analyzed and compared the welfare properties of these two options in a model where

agents can either issue nominal inside bonds or sell nominal outside bonds. The following results emerged from our analysis. First, in the model with illiquid outside bonds, steady-state inflation is welfare improving but it may not be with illiquid outside bonds. However, the optimal policy involves inflation being positive but arbitrarily close to zero with an infinitesimally small ratio of outside bonds to outside money. Second, the allocation in the model with inside bonds, steady-state inflation is always welfare improving. Finally, the best allocation with illiquid outside bonds strictly dominates the best allocation with inside bonds. Surprisingly, this allocation is equivalent to the allocation with inside bonds when default is infeasible.



## 6 Appendix

**Proof of Proposition 1.** Consider  $\gamma < \bar{\gamma}$ . From (33) define

$$f(\tilde{\varepsilon}) \equiv (1-n) \frac{1}{\tilde{\varepsilon}} \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) - (1-n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} dF(\varepsilon).$$

We have  $f'(\tilde{\varepsilon}) < 0$  with  $\lim_{\tilde{\varepsilon} \rightarrow 0} f(\tilde{\varepsilon}) = +\infty$  and  $f(\varepsilon_H) = 0$  so if  $1 + (1-\gamma) M_0/B_0 \geq \beta$  or  $\gamma_H = 1 + (1-\beta) B_0/M_0 \geq \gamma$  then a unique  $0 < \tilde{\varepsilon} \leq \varepsilon_H$  solves (33). Otherwise, for  $\bar{\gamma} > \gamma > \gamma_H$  we have  $\tilde{\varepsilon} > \varepsilon_H$  and

$$q_\varepsilon = u'^{-1} \left( \frac{\gamma}{\varepsilon \left[ 1 + (1-\gamma) \frac{M_0}{B_0} \right]} \right)$$

for all  $\varepsilon$ . ■

**Proof of Proposition 2.** Substitute (2) into (1) and differentiate  $(1-\beta) \mathcal{W}$  with respect to  $\gamma$  to get

$$(1-\beta) \frac{d\mathcal{W}}{d\gamma} \Big|_{\gamma=1} = (1-n) \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] \frac{dq_\varepsilon}{d\gamma} \Big|_{\gamma=1} dF(\varepsilon) > 0.$$

Note first that for all  $\varepsilon \leq \tilde{\varepsilon}$ ,  $\varepsilon u'(q_\varepsilon) - 1 = 0$  at  $\gamma = 1$  since  $q_\varepsilon = q_\varepsilon^*$ . It remains to show that  $\frac{dq_\varepsilon}{d\gamma} \Big|_{\gamma=1} > 0$  for all  $\varepsilon > \tilde{\varepsilon}$  since  $\varepsilon u'(q_\varepsilon) - 1 > 0$  for all  $\varepsilon > \tilde{\varepsilon}$ . Since  $q_\varepsilon = q_{\tilde{\varepsilon}}$  for all  $\varepsilon \geq \tilde{\varepsilon}$  it is sufficient to show that  $\frac{dq_{\tilde{\varepsilon}}}{d\gamma} \Big|_{\gamma=1} > 0$ .

Totally differentiate (32)

$$u'(\tilde{q}) d\tilde{\varepsilon} + \tilde{\varepsilon} u''(\tilde{q}) d\tilde{q} = \frac{1 + (1-\gamma) \frac{M_0}{B_0} + \gamma \frac{M_0}{B_0}}{\left[ 1 + (1-\gamma) \frac{M_0}{B_0} \right]^2} d\gamma$$

so

$$\frac{d\tilde{q}}{d\gamma} = \frac{-1}{\tilde{\varepsilon} u''(\tilde{q})} \left\{ u'(\tilde{q}) \frac{d\tilde{\varepsilon}}{d\gamma} - \frac{1}{\left[ 1 + (1-\gamma) \frac{M_0}{B_0} \right]^2} \left( 1 + \frac{M_0}{B_0} \right) \right\} \quad (61)$$

Totally differentiate (33)

$$\begin{aligned} -\frac{M_0}{\beta B_0} d\gamma &= -(1-n) \frac{1}{\tilde{\varepsilon}^2} \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) d\tilde{\varepsilon} \\ \frac{d\tilde{\varepsilon}}{d\gamma} &= \frac{M_0}{\beta B_0 (1-n)} \frac{\tilde{\varepsilon}^2}{\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)} > 0 \end{aligned}$$

Substitute into (61) to get

$$\frac{d\tilde{q}}{d\gamma} = \frac{-1}{\tilde{\varepsilon}u''(\tilde{q})} \left\{ u'(\tilde{q}) \frac{M_0}{\beta B_0 (1-n)} \frac{\tilde{\varepsilon}^2}{\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)} - \frac{1}{\left[1 + (1-\gamma) \frac{M_0}{B_0}\right]^2} \left(1 + \frac{M_0}{B_0}\right) \right\}$$

Evaluate at  $\gamma = 1$  and  $\tilde{\varepsilon}u'(\tilde{q}) = 1$

$$\left. \frac{d\tilde{q}}{d\gamma} \right|_{\gamma=1} = \frac{-1}{\tilde{\varepsilon}u''(\tilde{q})} \frac{M_0}{B_0} \left[ \frac{\tilde{\varepsilon}}{\beta (1-n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)} - 1 - \frac{B_0}{M_0} \right]$$

which is positive if

$$\frac{\tilde{\varepsilon}}{\beta (1-n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)} > 1 + \frac{B_0}{M_0} \equiv \bar{\gamma}$$

Define

$$z(\tilde{\varepsilon}) \equiv \frac{\tilde{\varepsilon}}{\beta (1-n) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)}$$

with

$$z'(\tilde{\varepsilon}) \equiv \frac{\int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) + \tilde{\varepsilon}^2 dF(\tilde{\varepsilon})}{\beta (1-n) \left[ \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) \right]^2} > 0$$

Note that  $z(\tilde{\varepsilon})$  is continuous with  $z(0) = 0$  and  $z(\varepsilon_H) \rightarrow +\infty$ . Thus, a unique value  $\check{\varepsilon}$  solves  $z(\check{\varepsilon}) = \bar{\gamma}$  so if  $\tilde{\varepsilon} > \check{\varepsilon}$  then inflation is welfare improving. ■

**Proof of Lemma 3.** Consider a borrower who borrowed  $-\bar{b}$  in market 1 and is considering defaulting on his issued bonds in market 3 .

If he redeems his bonds, he gets the equilibrium expected discounted utility in a steady state is

$$V_3(m, b) = U(x^*) - h_\varepsilon + \beta V_1(m_{1,+1})$$

where  $h_\varepsilon$  is his production in the market 3 if he repays his loan. A defaulter's expected discounted utility is

$$\hat{V}_3(m, b) = U(\hat{x}) - \hat{h}_\varepsilon + \beta \hat{V}_1(\hat{m}_{1,+1})$$

where the hat indicates the optimal choice by a defaulter. The real borrowing constraint  $\bar{\ell} = \phi \bar{b}$

satisfies  $V_3(m, b) = \widehat{V}_3(m, b)$  or

$$U(x^*) - h_\varepsilon + \beta V(m_{1,+1}) = U(\widehat{x}) - \widehat{h}_\varepsilon + \beta \widehat{V}(\widehat{m}_{1,+1})$$

Since  $\widehat{x} = x^*$  we have  $h_\varepsilon - \widehat{h}_\varepsilon = \beta [V(m_{1,+1}) - \widehat{V}(\widehat{m}_{1,+1})]$ . The continuation payoffs are

$$\begin{aligned} (1 - \beta) \widehat{V}(\widehat{m}_{1,+1}) &= \int_0^{\varepsilon_H} [(1 - n) \varepsilon u(\widehat{q}_\varepsilon) - n \widehat{q}_s] dF(\varepsilon) + U(x^*) - \widehat{h} \\ (1 - \beta) V(m_{1,+1}) &= \int_0^{\varepsilon_H} [(1 - n) \varepsilon u(q_\varepsilon) - n q_s] dF(\varepsilon) + U(x^*) - h. \end{aligned}$$

where  $h$  is expected hours worked for a nondefaulter. Accordingly, we have

$$h_\varepsilon - \widehat{h}_\varepsilon = \frac{\beta}{1 - \beta} [(1 - n) \Psi(q_\varepsilon, \widehat{q}_\varepsilon) + \widehat{h} - h]. \quad (62)$$

where  $\Psi(q_\varepsilon, \widehat{q}_\varepsilon) = \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) - \int_0^{\varepsilon_H} [\varepsilon u(\widehat{q}_\varepsilon) - \widehat{q}_\varepsilon] dF(\varepsilon) > 0$ . We get this expression for  $\Psi(q_\varepsilon, \widehat{q}_\varepsilon)$  because costs are linear sellers are indifferent as to how much they produce thus we just assume that the deviator produces in the decentralized market  $\widehat{q}_s = \left(\frac{1-n}{n}\right) \int_{\varepsilon_L}^{\varepsilon_H} \widehat{q}_\varepsilon dF(\varepsilon)$  in each period. Furthermore, from market clearing we have  $n q_s = (1 - n) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon)$ .

**Deriving  $h_\varepsilon - \widehat{h}_\varepsilon$ :** If the buyer redeems his bonds he works

$$\begin{aligned} h_\varepsilon &= x^* + \phi m_{1,+1} - \phi [m_1 - \bar{b}/(1 + i) - p q_\varepsilon] - \phi \tau M_{-1} - \phi \bar{b} \\ &= x^* - \bar{\ell}/(1 + i) + \phi p q_\varepsilon + \bar{\ell} = x^* + i \bar{\ell}/(1 + i) + \phi p q_\varepsilon \end{aligned}$$

where we use the equilibrium condition  $m_{1,+1} = m_1 + \tau M_{-1} = \gamma m_1$  If he defaults on his bonds, he works

$$\begin{aligned} \widehat{h}_\varepsilon &= x^* + \phi \widehat{m}_{1,+1} - \phi [m_1 - \bar{b}/(1 + i) - p q_\varepsilon] - \phi \tau M_{-1} \\ &= x^* + \phi (\widehat{m}_{1,+1} - m_{1,+1}) - \bar{\ell}/(1 + i) + \phi p q_\varepsilon \\ &= x^* + \phi \gamma (\widehat{m}_1 - m_1) - \bar{\ell}/(1 + i) + \phi p q_\varepsilon \end{aligned}$$

where we use the equilibrium condition that a defaulter's money balances must grow at the rate  $\gamma$

so  $\widehat{m}_{1,+1} = \widehat{m}_1 + \tau M_{-1} = \gamma \widehat{m}_1$ . Thus

$$\begin{aligned}
h_\varepsilon - \widehat{h}_\varepsilon &= x^* + i\bar{\ell}/(1+i) + \phi p q_\varepsilon \\
&\quad - x^* - \phi(\widehat{m}_{1,+1} - m_{1,+1}) + \bar{\ell}/(1+i) - \phi p q_\varepsilon \\
&= \bar{\ell} - \phi\gamma(\widehat{m}_1 - m_1)
\end{aligned} \tag{63}$$

**Deriving  $\widehat{h} - h$ :** Once the agent defaults, as a buyer he spent  $p\widehat{q}_\varepsilon$  units of money so his hours worked are

$$\begin{aligned}
\widehat{h}_\varepsilon &= x^* + \phi\widehat{m}_{1,+1} - \phi(\widehat{m}_1 - p\widehat{q}_\varepsilon) - \phi\tau M_{-1} \\
&= x^* + \phi(\widehat{m}_{1,+1} - \widehat{m}_1) + \phi p\widehat{q}_\varepsilon - \phi(m_{1,+1} - m_1) \\
&= x^* + (\gamma - 1)\phi(\widehat{m}_1 - m_1) + \phi p\widehat{q}_\varepsilon
\end{aligned}$$

For a seller we have

$$\begin{aligned}
\widehat{h}_s &= x^* + \phi\widehat{m}_{1,+1} - \phi(\widehat{m}_1 + p\widehat{q}_s) - \phi\tau M_{-1} \\
&= x^* + (\gamma - 1)\phi(\widehat{m}_1 - m_1) - \phi p \left( \frac{1-n}{n} \right) \int_0^{\varepsilon_H} \widehat{q}_\varepsilon dF(\varepsilon)
\end{aligned}$$

So for a defaulter expected hours worked are

$$\widehat{h} = (1-n)\widehat{h}_\varepsilon + n\widehat{h}_s = x^* + (\gamma - 1)\phi(\widehat{m}_1 - m_1)$$

while if he does not deviate he works  $h = x^*$  and so

$$\widehat{h} - h = (\gamma - 1)\phi(\widehat{m}_1 - m_1) \tag{64}$$

**Solving for  $\bar{\ell}$ :** Using (62)-(64) we get

$$\bar{\ell} = \frac{\beta}{1-\beta} \left[ (1-n) \Psi(q_\varepsilon, \widehat{q}_\varepsilon) + \left( \frac{\gamma-\beta}{\beta} \right) (\phi\widehat{m}_1 - \phi m_1) \right]$$

In equilibrium  $\phi \hat{m}_1 = \hat{q}$  and  $\phi m_1 = \phi M_{-1} = Q$ . So we have

$$\bar{\ell} = \frac{\beta}{1-\beta} \left[ (1-n) \Psi(q_\varepsilon, \hat{q}_\varepsilon) + \left( \frac{\gamma-\beta}{\beta} \right) (\hat{q} - Q) \right]$$

In market 1, the maximal amount of cash a bond issuer can acquire is

$$\frac{\bar{\ell}}{(1+i)} = \frac{\beta}{(1+i)(1-\beta)} \left[ (1-n) \Psi(q_\varepsilon, \hat{q}_\varepsilon) + \left( \frac{\gamma-\beta}{\beta} \right) (\hat{q} - Q) \right]$$

To know whether  $\bar{\ell} > 0$  we need to determine the sign of right hand side. Substituting for  $\Psi(q_\varepsilon, \hat{q}_\varepsilon)$  and  $Q$  we need

$$(1-n) \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) - (1-n) \int_0^{\varepsilon_H} [\varepsilon u(\hat{q}_\varepsilon) - \hat{q}_\varepsilon] dF(\varepsilon) + \left( \frac{\gamma-\beta}{\beta} \right) \left[ \hat{q} - (1-n) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) \right] > 0$$

We have

$$\frac{\gamma-\beta}{\beta} = (1-n) \int_0^{\varepsilon_H} [\varepsilon u'(\hat{q}_\varepsilon) - 1] dF(\varepsilon) \quad (65)$$

Substitute in to get

$$\begin{aligned} & (1-n) \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) - (1-n) \int_0^{\varepsilon_H} [\varepsilon u(\hat{q}_\varepsilon) - \hat{q}_\varepsilon] dF(\varepsilon) \\ & + (1-n) \int_0^{\varepsilon_H} [\varepsilon u'(\hat{q}_\varepsilon) - 1] dF(\varepsilon) \left[ \hat{q} - (1-n) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) \right] > 0 \end{aligned}$$

$$\begin{aligned} & \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) - \int_0^{\varepsilon_H} [\varepsilon u(\hat{q}_\varepsilon) - \hat{q}_\varepsilon] dF(\varepsilon) \\ & + \int_0^{\varepsilon_H} [\varepsilon u'(\hat{q}_\varepsilon) - 1] dF(\varepsilon) \left[ \hat{q} - (1-n) \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) \right] > 0 \end{aligned}$$

In an unconstrained borrowing equilibrium we have

$$\frac{\gamma-\beta}{\beta} = \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) \quad (66)$$

So (65) and (66) yield

$$1-n = \frac{\int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon)}{\int_0^{\varepsilon_H} [\varepsilon u'(\hat{q}_\varepsilon) - 1] dF(\varepsilon)} < 1$$

Substitute this in

$$\begin{aligned} & \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) - \int_0^{\varepsilon_H} [\varepsilon u(\widehat{q}_\varepsilon) - \widehat{q}_\varepsilon] dF(\varepsilon) \\ & + \int_0^{\varepsilon_H} [\varepsilon u'(\widehat{q}_\varepsilon) - 1] dF(\varepsilon) \left[ \widehat{q} - \frac{\int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon)}{\int_0^{\varepsilon_H} [\varepsilon u'(\widehat{q}_\varepsilon) - 1] dF(\varepsilon)} \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) \right] > 0 \end{aligned}$$

$$\begin{aligned} & \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) - \int_0^{\varepsilon_H} [\varepsilon u(\widehat{q}_\varepsilon) - \widehat{q}_\varepsilon] dF(\varepsilon) \\ & + \widehat{q} \int_0^{\varepsilon_H} [\varepsilon u'(\widehat{q}_\varepsilon) - 1] dF(\varepsilon) - \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) > 0 \end{aligned}$$

Rewrite as

$$\begin{aligned} & \int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) - \int_0^{\varepsilon_H} [\varepsilon u(\widehat{q}_\varepsilon) - \widehat{q}_\varepsilon] dF(\varepsilon) \\ & > \int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) - \widehat{q} \int_0^{\varepsilon_H} [\varepsilon u'(\widehat{q}_\varepsilon) - 1] dF(\varepsilon) \end{aligned}$$

Divide by sides by  $\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) - \int_0^{\varepsilon_H} \widehat{q}_\varepsilon dF(\varepsilon)$  to get

$$\begin{aligned} & \frac{\int_0^{\varepsilon_H} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon) - \int_0^{\varepsilon_H} [\varepsilon u(\widehat{q}_\varepsilon) - \widehat{q}_\varepsilon] dF(\varepsilon)}{\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) - \int_0^{\varepsilon_H} \widehat{q}_\varepsilon dF(\varepsilon)} \\ & > \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) \frac{\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) - \widehat{q} \frac{\int_0^{\varepsilon_H} [\varepsilon u'(\widehat{q}_\varepsilon) - 1] dF(\varepsilon)}{\int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon)}}{\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) - \int_0^{\varepsilon_H} \widehat{q}_\varepsilon dF(\varepsilon)} \end{aligned}$$

which always holds because the LHS is greater than  $\int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon)$  and

$$\frac{\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) - \widehat{q} \frac{\int_0^{\varepsilon_H} [\varepsilon u'(\widehat{q}_\varepsilon) - 1] dF(\varepsilon)}{\int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon)}}{\int_0^{\varepsilon_H} q_\varepsilon dF(\varepsilon) - \int_0^{\varepsilon_H} \widehat{q}_\varepsilon dF(\varepsilon)} < 1$$

So  $\bar{\ell} > 0$  in an unconstrained equilibrium. ■

**Proof of Proposition 4.** In an unconstrained equilibrium we have unique values for  $q_\varepsilon$  and  $i = (\gamma - \beta) / \beta$ . All that is left is to show is that  $\ell_H \leq \bar{\ell}$  or

$$\frac{\ell_H}{1+i} \leq \frac{\beta}{(1+i)(1-\beta)} (1-n) \Psi(q_\varepsilon, \widehat{q}_\varepsilon) + i\beta(\widehat{q} - Q)$$

Since all agents (including the one with  $\varepsilon_H$ ) are unconstrained we have  $\ell_H / (1 + i) = q_{\varepsilon_H} - \phi M_{-1}$  and  $\phi M_{-1} = Q$  so we have

$$(1 - \beta)(1 + i)(q_H - Q) \leq \beta(1 - n)\Psi(q_\varepsilon, \hat{q}_\varepsilon) + i\beta(\hat{q} - Q) \quad (67)$$

Define

$$\Delta(i, \beta) \equiv (1 - \beta)(1 + i)(q_{\varepsilon_H} - Q) - \beta(1 - n)\Psi(q_\varepsilon, \hat{q}_\varepsilon) - i\beta(\hat{q} - Q)$$

So we need  $\Delta(i, \beta) \leq 0$  in an unconstrained borrowing equilibrium. Note that  $\Delta(0, \beta) = (1 - \beta)(q_H^* - Q^*) > 0$  since  $\Psi(q_\varepsilon^*, \hat{q}_\varepsilon^*) = 0$  at  $i = 0$ . Thus, (67) is violated at the Friedman rule.

Define  $\Delta(i, \beta) \equiv g(i, \beta) - h(i, \beta)$  and consider solutions to  $\Delta(i, \beta) = 0$ . Note that  $\Delta(0, 1) = 0$  since  $q_\varepsilon|_{(0,1)} = \hat{q}_\varepsilon|_{(0,1)} = q_\varepsilon^*$  and  $\Psi(q_\varepsilon^*, \hat{q}_\varepsilon^*) = 0$ . We have

$$\begin{aligned} \frac{\partial \Delta(i, \beta)}{\partial i} &= (1 - \beta)(q_{\varepsilon_H} - Q) + (1 - \beta)(1 + i) \left( \frac{\partial q_{\varepsilon_H}}{\partial i} - \frac{\partial Q}{\partial i} \right) \\ &\quad - \beta(1 - n) \frac{\partial \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{\partial i} - \beta(\hat{q} - Q) + \beta i \left( \frac{\partial \hat{q}}{\partial i} - \frac{\partial Q}{\partial i} \right) \end{aligned}$$

Since the partial derivatives in this expression are all continuous  $\Delta_i(i, \beta)$  is continuous and non-zero with

$$\Delta_i(0, 1) = -\beta(\hat{q} - Q)|_{(0,1)} = - \left[ q_{\varepsilon_H}^* - (1 - n) \int_0^{\varepsilon_H} q_\varepsilon^* dF(\varepsilon) \right] < 0.$$

since  $\frac{\partial \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{\partial i} \Big|_{(0,1)} = 0$  and  $\hat{q}|_{(0,1)} = q_{\varepsilon_H}^*$ .

We also have

$$\begin{aligned} \frac{\partial \Delta(i, \beta)}{\partial \beta} &= -(1 + i)(q_{\varepsilon_H} - Q) + (1 - \beta)(1 + i) \left( \frac{\partial q_{\varepsilon_H}}{\partial \beta} - \frac{\partial Q}{\partial \beta} \right) \\ &\quad - \frac{h(i, \beta)}{\beta} - \beta(1 - n) \frac{\partial \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{\partial \beta} - i\beta \left( \frac{\partial \hat{q}}{\partial \beta} - \frac{\partial Q}{\partial \beta} \right) \end{aligned}$$

Therefore  $\Delta_\beta(0, 1)$  is continuous and

$$\Delta_\beta(0, 1) = - \left[ q_{\varepsilon_H}^* - (1 - n) \int_0^{\varepsilon_H} q_\varepsilon^* dF(\varepsilon) \right] < 0.$$

since  $\frac{\partial \Psi(q_\varepsilon, \hat{q}_\varepsilon)}{\partial \beta} \Big|_{(0,1)} = 0$  and  $h(0, 1) = 0$ . By the implicit function theorem, it follows that, for

$\beta$  arbitrarily close to one, the expression  $\Delta(i, \beta) = 0$  defines  $i$  as an implicit function of  $\beta$ , i.e.,  $i = \hat{i}(\beta)$ .

Furthermore, we have

$$\left. \frac{di}{d\beta} \right|_{(0,1)} = -\frac{\Delta_\beta(0,1)}{\Delta_i(0,1)} = -1,$$

so that as  $\beta$  falls  $i$  grows. It follows from the implicit function theorem that  $\Delta(\hat{i}, \beta) = 0$  for a unique  $\hat{i} > 0$  and  $\beta$  sufficiently close to one.

Establishing existence and uniqueness of the unconstrained credit equilibrium for  $i > \hat{i}$ . Above we established that  $\Delta(0, \beta) > 0$  for all  $0 < \beta < 1$ . Fix  $\beta$  close to 1. We have established that  $\Delta(\hat{i}, \beta) = 0$  for some  $\hat{i} > 0$ . By continuity, we have that if  $i > \hat{i}$  then  $\Delta(0, \beta) < 0$  and so an unconstrained equilibrium exists. For  $0 \leq i < \hat{i}$ , then  $\Delta(i, \beta) \geq 0$  which violates (67). This establishes the first part of Proposition 4.

Consider  $0 \leq i < \hat{i}$ . In general we cannot prove existence or uniqueness. We now characterize the properties of (52) and (56)-(58). At  $i = 0$ , (57)-(58) imply  $\tilde{\varepsilon} = \hat{\varepsilon}$  so  $\tilde{q} = \hat{q}$  and  $\Psi(q_\varepsilon, \hat{q}_\varepsilon) = 0$ . Then from (52) and (56) we have  $\gamma = 1$ . This implies there is one and only one monetary policy consistent with a nominal interest rate of zero and also satisfies (56). Thus, a monetary equilibrium with credit does not exist at  $\gamma = 1$ . Furthermore, we have

$$\left. \frac{di}{d\gamma} \right|_{\gamma=1} = \frac{1}{1-\beta} > 0 \quad (68)$$

To obtain this, substitute  $\tilde{q} = u'^{-1}[(1+i)/\tilde{\varepsilon}]$  into (56) and the resulting expression into (52) then totally differentiate:

$$\begin{aligned} & (1+i)(d\tilde{q} - dQ) + (\tilde{q} - Q)di \\ = & \frac{\beta(1-n)}{(1-\beta)} \left\{ \left[ \int_0^{\varepsilon_H} \varepsilon u'(q_\varepsilon) dq_\varepsilon - dq_\varepsilon \right] dF(\varepsilon) - \left[ \int_0^{\varepsilon_H} \varepsilon u'(\hat{q}_\varepsilon) d\hat{q}_\varepsilon - d\hat{q}_\varepsilon \right] dF(\varepsilon) \right\} \\ & + \frac{\beta}{(1-\beta)} \left\{ \left( \frac{\gamma-\beta}{\beta} \right) (d\hat{q} - dQ) + \frac{1}{\beta(1+i)} (\hat{q} - Q) d\gamma \right\} \end{aligned}$$



This can be written as

$$\begin{aligned}
& (1+i)(d\tilde{q} - dQ) + (\tilde{q} - Q) di \\
= & \frac{\beta(1-n)}{(1-\beta)} \left\{ \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) dq_\varepsilon - \int_0^{\varepsilon_H} [\varepsilon u'(\hat{q}_\varepsilon) - 1] dF(\varepsilon) d\hat{q}_\varepsilon \right\} \\
& + \frac{\beta}{(1-\beta)} \left\{ \left( \frac{\gamma - \beta}{\beta} \right) (d\hat{q} - dQ) + \frac{1}{\beta(1+i)} (\hat{q} - Q) d\gamma \right\}.
\end{aligned}$$

Evaluate at  $i = 0$ ,  $\gamma = 1$ , use  $\tilde{q} = \hat{q}$  and (57)-(58) to get

$$\begin{aligned}
d\tilde{q} - dQ + (\hat{q} - Q) di &= \frac{\beta}{(1-\beta)} \left\{ \frac{1-\beta}{\beta} d\tilde{q} - \frac{1-\beta}{\beta} d\hat{q}_\varepsilon \right\} \\
& + \frac{\beta}{(1-\beta)} \left\{ \left( \frac{1-\beta}{\beta} \right) (d\hat{q} - dQ) + \frac{1}{\beta} (\hat{q} - Q) d\gamma \right\}
\end{aligned}$$

This expression reduces to (68). So for  $\gamma > 1$ ,  $i > 0$ . ■

**Proof of Proposition 5.** Since no agent deviates in equilibrium, welfare is given by (1). Again, using (2) differentiate (1) with respect to  $\gamma$  to get

$$(1-\beta) \frac{d\mathcal{W}}{d\gamma} \Big|_{\gamma=1} = (1-n) \int_0^{\varepsilon_H} [\varepsilon u'(q_\varepsilon) - 1] \frac{dq_\varepsilon}{d\gamma} \Big|_{\gamma=1} dF(\varepsilon) > 0.$$

Since  $\varepsilon u'(q_\varepsilon) - 1 = 0$  for all  $\varepsilon \leq \tilde{\varepsilon}$  at  $\gamma = 1$  for all  $\varepsilon$ , and  $q_\varepsilon = \tilde{q}$  for all  $\varepsilon \geq \tilde{\varepsilon}$ , welfare will be increasing in  $\gamma$  if  $\frac{d\tilde{q}}{d\gamma} \Big|_{\gamma=1} > 0$ .

Using  $\varepsilon u'(q_\varepsilon) = 1 + i$  for all  $\varepsilon \leq \tilde{\varepsilon}$  and  $q_\varepsilon = \tilde{q}$  for all  $\varepsilon \geq \tilde{\varepsilon}$  (48) can be written as

$$\frac{\gamma - \beta}{\beta} = (1-n) u'(\tilde{q}) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) + (1-n)(1+i) \int_0^{\tilde{\varepsilon}} dF(\varepsilon) - 1 + n + ni \quad (69)$$

Totally differentiate (69):

$$\begin{aligned}
\frac{1}{\beta} d\gamma &= (1-n) u''(\tilde{q}) d\tilde{q} \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) - (1-n) [\tilde{\varepsilon} u'(\tilde{q}) - (1+i)] dF(\tilde{\varepsilon}) d\tilde{\varepsilon} \\
&+ \left[ n + (1-n) \int_0^{\tilde{\varepsilon}} dF(\varepsilon) \right] di
\end{aligned}$$

Using  $\tilde{\varepsilon}u'(\tilde{q}) = 1 + i$  we have

$$\frac{1}{\beta}d\gamma = (1-n)u''(\tilde{q})d\tilde{q} \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) + \left[ n + (1-n) \int_0^{\tilde{\varepsilon}} dF(\varepsilon) \right] di \quad (70)$$

Substituting (68) into (70) gives

$$\frac{1}{\beta}d\gamma = \left[ (1-n)u''(\tilde{q}) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon) \right] d\tilde{q} + \left[ n + (1-n) \int_0^{\tilde{\varepsilon}} dF(\varepsilon) \right] \frac{1}{1-\beta}d\gamma$$

Thus

$$\left. \frac{d\tilde{q}}{d\gamma} \right|_{\gamma=1} = \frac{1-\beta-\beta \left[ n + (1-n) \int_0^{\tilde{\varepsilon}} dF(\varepsilon) \right]}{\beta(1-\beta)(1-n)u''(\tilde{q}) \int_{\tilde{\varepsilon}}^{\varepsilon_H} \varepsilon dF(\varepsilon)}$$

The denominator is negative. So  $\left. \frac{d\tilde{q}}{d\gamma} \right|_{\gamma=1} > 0$  and  $(1-\beta) \left. \frac{dW}{d\gamma} \right|_{\gamma=1} > 0$  if

$$\beta > \frac{1}{1+n+(1-n) \int_0^{\tilde{\varepsilon}} dF(\varepsilon)}.$$

■

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