

Equilibrium Default with Limited Commitment*

Hugo Hopenhayn

UCLA

hopen@econ.ucla.edu

Iván Werning

MIT

iwerning@mit.edu

May 2006

Abstract

We model the optimal financing of a firm with limited commitment. Entrepreneurs need outside funding for projects, but can default at any moment and take some outside opportunity. The value of this opportunity is random and not observable by the lender. We show that the optimal dynamic contract may allow default along the equilibrium path. Focusing on the dynamics of default, debt and capital accumulation, we find that over the life of the project the probability of default declines, long-term debt falls and capital rises.

Introduction

This paper derives the optimal dynamic contract that finances a firm with limited commitment. An entrepreneur has a project that needs outside funds to cover an initial fixed cost as well as a stream of capital investments. Lending is constrained by voluntary repayment: the entrepreneur can, at any moment, walk away for some outside opportunity. The value of the outside opportunity is random and privately observed by the entrepreneur. Within this environment, we study constrained-efficient long-term contracts between the outside lender and the firm's entrepreneur. That is, we place no ad hoc assumptions on contracting,

*Werning is grateful for the hospitality of Harvard University.

but impose the constraints arising from the entrepreneur's private information and lack of commitment.

We assume that outside opportunities are unattractive relative to the investment project, so that it is never efficient to take them. With commitment the outside opportunities are completely irrelevant. However, without commitment, the availability of the outside opportunities does constrain lending because the entrepreneur may take leave for them because when they are privately, but not socially, efficient.

When the value of the outside option is non random or is publicly observed, as in [Albuquerque and Hopenhayn \(2004\)](#), then it is never optimal to exercise it. Default never occurs along the equilibrium path. This property is also characteristic of other limited commitment models, such as in applications to consumption insurance and smoothing among individuals ([Kocherlakota, 1996](#); [Alvarez and Jermann, 2000](#)), or international borrowing and lending by sovereign countries ([Thomas and Worrall, 1994](#); [Kehoe and Perri, 2002](#)). While all these models capture the friction that ex post *potential* default introduces on ex ante lending, they are completely silent on *actual* default behavior.

In our model, in contrast, default occurs along the equilibrium path. Whenever the value of the outside opportunity is random, and its realization is not observed by the lender, we show that the optimal contract allows firms to exercise the outside option. The contract treats all firms identically, but over time some firms default and choose to exit the contract, while others choose to stay.

Our focus is on the dynamics of capital, debt and default. The optimal contract offered by the lender is forward looking, so it specifies and anticipates all these variables. We find that the probability of default is highest at the beginning of the lending relationship and declines over time. Capital rises over time as the entrepreneur pays back the long-term debt to the lender for initiating the project. The lender may play an increasing role in the financing of capital investments through short-term debt.

Our environment is closest to [Albuquerque and Hopenhayn's \(2004\)](#) model of firm finance. Their model features limited commitment, but all uncertainty is publicly observable by the lender. [Clementi and Hopenhayn \(2006\)](#), on the other hand, considers unobservable shocks to revenue, but with no commitment problems. In both papers, if liquidation is assumed valuable enough to the lender, then exit may occur after a sufficiently bad history of shocks, but there is no default. For contrast, we assume in our model that the liquidation value to the lender is zero: the interaction of private information and limited commitment is then key for our results on default.

1 An Investment Project in Need of Funding

Time is discrete and the horizon infinite, $t = 0, 1, \dots$. There are two agents: an entrepreneur and a lender. Both are risk neutral and discount the future using the discount factor $\beta \equiv (1 + r)^{-1}$, where $r > 0$. The entrepreneur has access to an investment project in need of financing, but has insufficient resources to fund it himself. The lender, on the other hand, cannot operate the project, but has an infinite amount of liquid resources. Formally, the difference in liquidity is captured by imposing a limited liability constraint on the entrepreneur, but not on the lender. Letting τ_t^i denote transfers to agent i , where $i = e$ stands for the entrepreneur and $i = l$ for the lender, then we impose: $\tau_t^e \geq 0$ and $\tau_t^l \in \mathbb{R}$.

The project requires an initial investment of size I_0 . Thereafter, the project yields revenue $R(k_t)$ at the end of any period where k_t was invested at the beginning of the period.¹

First-Best. The dynamics of the first-best solution is purposefully trivial, involving a constant capital investment

$$k^* \equiv \arg \max_k \{R(k) - (1 + r)k\}.$$

The total value of the project is then

$$W^* \equiv \sum_{t=1}^{\infty} (1 + r)^{-t} R(k^*) - \sum_{t=0}^{\infty} (1 + r)^{-t} k^* = \frac{R(k^*) - (1 + r)k^*}{r}.$$

We assume that $W^* > I_0$ so that it is efficient to start the project, when it can be operated fully efficiently. The first-best allocation is attainable with full commitment.

Outside Opportunities. Unfortunately, the entrepreneur lacks the commitment to stay in the relationship. Each period the entrepreneur has some outside opportunity that requires abandoning the project. The value from this opportunity, O_t , may depend on a random variable θ_t and current capital k_t , so that

$$O_t = O(\theta_t, k_t).$$

We shall consider a few specifications for the stochastic process of the shocks $\{\theta_t\}$, as well as for the way current capital k_t affects the outside opportunity. We assume that the

¹Goods produced at the end of any period are only available for consumption at the beginning of the next, and must be discounted accordingly.

value of a liquidated firm is zero to the lender.

Timing. The timing within any period is as follows. At the beginning of the period the lender advances k_t funds to finance the current capital investment. The entrepreneur then observes the realization of θ_t and decides whether to take the outside opportunity, $x_t = 0$, or remain in the relation, $x_t = 1$. If he stays then, at the end of the period, the lender collects revenue $R(k_t)$, while the entrepreneur is given some non-negative transfer $\tau_t^e \geq 0$ to consume.

Contracts. The most general possible contracts specify how all these variables depend, at each moment in time, on the entire history of shocks to the outside opportunity.² Thus, a contract is a sequence of functions $\mathcal{C} = \{k_t(\theta^t), x_t(\theta^t), \tau_t^e(\theta^t)\}$. To allow for randomization over the separation outcome, we let $x(\theta^t) \in [0, 1]$ denote the probability of remaining in the relationship.

Any contract induces a sequence of transfers for the lender given by

$$\tau_t^l(\theta^t) = R(k_{t-1}(\theta^{t-2})) - k_t(\theta^{t-1}) - \tau_t^e(\theta^t), \quad (1)$$

if the relation was not terminated, and equal to $-k_t(\theta^{t-1})$ otherwise. In addition, a contract implies lifetime utility values, $V_t(\theta^{t-1})$ and $B_t(\theta^{t-1})$, for the entrepreneur and lender, that satisfy

$$V_t(\theta^{t-1}) = \mathbb{E}_{t-1} [x_t(\theta^t)(\tau_t^e(\theta^t) + \beta V_{t+1}(\theta^t)) + (1 - x_t(\theta^t))O(\theta_t, k_t(\theta^{t-1}))]$$

and

$$B_t(\theta^{t-1}) = \mathbb{E}_{t-1} [x_t(\theta^t)(\tau_t^l(\theta^t) + \beta B_{t+1}(\theta^t)) - (1 - x_t(\theta^t))k_t(\theta^{t-1})].$$

for all $t = 0, 1, \dots$ and histories θ^t . We can also define the total surplus $W_t(\theta^{t-1})$ at the beginning of the period, satisfying

$$W_t(\theta^{t-1}) = \mathbb{E}_{t-1} [-k_t(\theta^{t-1}) + x_t(\theta^t)(R(k_t(\theta^{t-1})) + \beta W_{t+1}(\theta^t)) + (1 - x_t(\theta^t))O(\theta_t, k_t(\theta^{t-1}))]$$

for all $t = 0, 1, \dots$ and histories θ^t . Note that, using [equation \(1\)](#):

$$W_t(\theta^{t-1}) = V_t(\theta^{t-1}) + B_t(\theta^{t-1}).$$

²The Revelation Principle ensures that this is without loss in generality.

Due to private information, the contract must rely on the reports made by the entrepreneur, and incentives must be provided to ensure that these reports are truthful. At the same time, due to the lack of commitment, an entrepreneur cannot be retained if he prefer to leave for the outside option. Feasible contracts must satisfy both incentive-compatibility and no-commitment constraints.

These constraints can be formalized as follows. Let $\sigma_\theta \equiv \{\sigma_{\theta,t}\}$ denote a reporting strategy, where $\sigma_{\theta,t} : \Theta^{t+1} \rightarrow \Theta$ denotes the report regarding the current shock as a function of the history of true shocks; similarly let $\sigma_\theta^t : \Theta^{t+1} \rightarrow \Theta^{t+1}$ denote the history of reports as a function of the true history. Likewise let $\sigma_x \equiv \{\sigma_{x,t}\}$ denote an exit strategy for the entrepreneur, where $\sigma_{x,t} : \Theta^{t+1} \rightarrow [0, 1]$, so that $\sigma_{x,t}(\theta^t)$ represents the probability of exiting after true history θ^t . Possible exit strategies can only make leaving more likely than stipulated by the lender in the contract given the reported history $\sigma_\theta^t(\theta^t)$, so that $\sigma_{x,t}(\theta^t) \leq x_t(\sigma_\theta^t(\theta^t))$. For any contract $\mathcal{C} = \{k_t(\theta^t), x_t(\theta^t), \tau_t^e(\theta^t)\}$ define the non-empty set of possible reporting and exit strategy pairs $\sigma = (\sigma_\theta, \sigma_x)$ by $\Sigma(\mathcal{C})$. This set always contains a very special strategy: the one with truth telling with no additional exit, defined by

$$\begin{aligned} \sigma^* &= (\sigma_\theta^*, \sigma_x^*), \\ \sigma_{\theta,t}^*(\theta^t) &= \theta_t \quad \text{and} \quad \sigma_{x,t}^*(\theta^t) = x_t(\theta^t) \end{aligned} \tag{2}$$

for all $t = 0, 1, \dots$ and histories θ^t .

Define the ex-ante utility obtained by the entrepreneur from any contract $\mathcal{C}$ when following strategy σ to be $U_0(\sigma; \mathcal{C})$. Feasibility requires σ^* , the strategy with truth telling and no additional exit, to be optimal.

Definition 1 *A contract $\mathcal{C} = \{k_t(\theta^t), x_t(\theta^t), \tau_t^e(\theta^t)\}$ is feasible if it satisfies*

$$U(\sigma^*; \mathcal{C}) \geq U(\sigma; \mathcal{C}) \quad \forall \sigma \in \Sigma(\mathcal{C}), \tag{3}$$

where the truth-telling with no-additional-exit strategy σ^ is defined by condition (2).*

Note that, in particular, whenever the contract requests staying with some positive probability the entrepreneur must prefer not to leave for the outside opportunity (setting $\sigma_{x,t}(\theta^t) = 0$). The continuation utility obtained from staying is $\tau_t(\theta^t) + V_{t+1}(\theta^t)$, while the utility from leaving is $O(\theta_t, k_t(\theta^{t-1}))$. Thus, we require that

$$x_t(\theta^t) > 0 \Leftrightarrow \tau_t(\theta^t) + V_{t+1}(\theta^t) \geq O(\theta_t, k_t(\theta^{t-1})) \quad \forall t, \theta^t. \tag{4}$$

Optimal contracts maximize the utility of the entrepreneur while ensuring that the lender does not make a loss.

Definition 2 *A feasible contract $\mathcal{C}^* = \{k_t^*(\theta^t), x_t^*(\theta^t), \tau_t^{e*}(\theta^t)\}$ is optimal if $B_0^* \geq I_0$ and there exists no other feasible contract $\mathcal{C} = \{k_t(\theta^t), x_t(\theta^t), \tau_t^e(\theta^t)\}$ satisfying $B_0 \geq I_0$ yielding higher utility for the entrepreneur $V_0 \geq V_0^*$.*

Clearly, optimal contracts will feature the lender breaking even so that $B_0 = I_0$.

Default and No Separation. A leading case for us is when shocks to the outside opportunity are independently distributed over time. Our first result states that, in this case, the optimal contract does not separate entrepreneurs according to their outside opportunity shocks. The contract can then be written as a simple deterministic one.

Lemma 1 *Suppose the shock to outside opportunities θ_t is independently distributed over time. The optimal contract is independent of the history of shocks to the outside opportunity θ^t , and transfers and investments are deterministic $\{\tau_t^e, k_t\}$. Terminations are not randomized and are voluntary by the entrepreneur:*

$$\begin{aligned} x_t(\theta_t) = 1 & \Leftrightarrow \tau_t + V_{t+1} \geq O(\theta_t, k_t) \\ x_t(\theta_t) = 0 & \Leftrightarrow \tau_t + V_{t+1} < O(\theta_t, k_t) \end{aligned} \quad t = 0, 1, \dots \quad (5)$$

The intuition for this result is that, with a deterministic contract, it is simply not possible to separate entrepreneurs according to their outside opportunities. In making their decision to stay or exit, entrepreneurs compare their outside opportunity to the value of staying within the contract. Looking forward all entrepreneurs are the same, so they all agree on the best value of staying and cannot be induced to separate. Randomization can help separate different types, but lotteries turns out not to be optimal: $x_t \in \{0, 1\}$. Finally, it is always optimal to keep the entrepreneur if he so wishes: forced terminations cannot increase value since the relationship can be maintained at no cost to the lender. Based on the lemma, we now turn to characterize deterministic contracts.

The lemma implies that we can focus on deterministic contracts that impose the no-commitment constraints (5) that determine exit. Note that the second part of constraint (5) incorporates the fact that it is never optimal to force out an entrepreneur that wishes to stay.

2 Purely Random Outside Opportunities

Consider the simple case where the outside opportunity is independent of capital so that

$$O(\theta, k) = \theta,$$

and assume θ_t is independently and identically distributed, with distribution function $F(\theta)$ on the bounded support $\Theta \equiv [\underline{\theta}, \bar{\theta}]$. We assume that $\bar{\theta} < W^*$ so that the outside opportunity is never more valuable than the total value of the project, if the latter were operated at full efficiency.

Confronted with any contract, the entrepreneur will employ a reservation policy, taking any outside opportunities above some threshold and rejecting the rest. The lifetime utility for the entrepreneur V_t evolves according to

$$V_t = \int_{\Theta} \max\{\theta, \tau_t^e + \beta V_{t+1}\} dF(\theta), \quad (6)$$

with $\tau_t^e \geq 0$.

Note that if $V_t \geq \bar{\theta}$ then the first-best allocation k^* is achievable from period t onwards. For example, the entrepreneur can be paid the constant amount $\tau_{t+s}^e = V_t r / (1 + r)$ for $s = 0, 1, \dots$ ensuring the entrepreneur will always prefer to stay: $V_{t+s} \geq O_{t+s}$.

For lower values so that $V_t < \bar{\theta}$ some entrepreneurs will default in equilibrium. The entrepreneur defaults whenever $\theta > \tau_t^e + \beta V_{t+1}$. As long as $V_t < \bar{\theta}$ some default occurs.³ The fraction of entrepreneurs exiting by default at time t is

$$p_t \equiv \Pr[x_t = 1 \mid \theta^{t-1}] = \Pr[\theta > \tau_t^e + \beta V_{t+1} \mid \theta^{t-1}] = 1 - F(\tau_t^e + \beta V_{t+1}), \quad (7)$$

which is a decreasing function of the value of staying $\tau_t^e + \beta V_{t+1}$.

Whenever $V_t < \bar{\theta}$ it is optimal to set dividends to zero. This allows V_t to grow faster, reducing default and eventually enabling the first-best allocation. Setting $\tau_t^e = 0$ in [equation \(6\)](#) gives

$$V_t = \int_{\beta V_{t+1}} \theta dF(\theta) + F(\beta V_{t+1}) \beta V_{t+1}.$$

³ That is, it cannot be the case that $\theta \leq \tau_t^e + \beta V_{t+1}$, since this would imply by [equation \(6\)](#) that $V_t = \tau_t^e + \beta V_{t+1}$ but then $\bar{\theta} > V_t$ implies that $\theta > \tau_t^e + \beta V_{t+1}$, a contradiction.

Since the right-hand side is strictly increasing in V_{t+1} we can solve

$$V_{t+1} = g(V_t), \quad (8)$$

for some function $g(v)$ with

$$g'(v) = \frac{1}{\beta F(\beta g(v))} > \frac{1}{\beta} > 1.$$

The unique fixed point $\bar{V} = g(\bar{V})$ represents the value of the best searching for the best outside opportunity. For $V_0 < \bar{V}$ we have V_t falling eventually below $E[\theta]$ which is not possible since eventually $V_t < E[\theta]$, which violates [equation \(6\)](#). Thus, it must be that the initial value satisfies $V_0 \geq \bar{V}$.

The result that $V_0 \geq \bar{V}$ I think is true; the current proof is not. It is possible to force the entrepreneur into exit in some period $x_t = 0$, the expected utility of that is precisely $E[\theta]$.

For any $V_0 > \bar{V}$, the V_t is rises monotonically and at some point crosses $\bar{\theta}$ at which point the first-best becomes available. Let T be the lowest period such that $V_T \geq \bar{\theta}$. Note that as V_t rises, the probability of default, given by [equation \(7\)](#),

$$p_t = 1 - F(\beta V_{t+1}) = 1 - F(\beta g(V_t))$$

declines monotonically and eventually vanishes in period T , i.e. $p_{T+s} = 0$ for $s = 0, 1, \dots$

Optimal investment is a function of the current anticipated probability of default p :

$$\kappa(p) \equiv \arg \max_k \{(1-p)R(k) - (1+r)k\} \quad (9)$$

Optimal capital $\kappa(p)$ is an decreasing function of the probability of default p . Intuitively, default lowers the expected marginal product of investments because revenues are lost whenever default occurs.

Since we have seen that the optimal contract has the default probability $p_t = \kappa(p_t)$ falling over time, it follows that capital investment $k_t = \kappa(p_t)$ rises over time and reaches k^* at $t = T$ and then stays there, i.e. $k_{T+s} = k^*$ for $s = 0, 1, \dots$

The initial level of utility for the entrepreneur is pinned down as follows. For each value of $V_0 \geq \bar{V}$ we can solve forward for the sequence $\{V_t, p_t, k_t\}$ using [equations \(8\)–\(9\)](#), and compute the implied value for the lender, $B_0 = b(V_0)$, as a function of the initial utility for the entrepreneur. Then the optimal V_0 is the highest value such that the lender value equals

the initial investment: $b(V_0) = I_0$. If there is no such V_0 then the project is not funded.

Can we show that $V_0 \neq \bar{V}$ so that $V_0 > \bar{V}$?

We summarize these results in the next proposition.

Proposition 1 *If the project is funded, the optimal contract has zero dividends up to some period $T < \infty$. During this time, for $t \leq T$, the default probability p_t declines, while lifetime utility V_t and capital k_t rises. For $t \geq T$ there is no default, $p_t = 0$, and investment is fully efficient $k_t = k^*$.*

The randomness in outside opportunities is responsible for the non trivial dynamics of the optimal allocation. To see this, consider again the special case where outside opportunities are not random, so that the distribution of θ is degenerate. Then, it follows from the construction above that either the project is financed with the first-best investment k^* from the very start, or the project is never initiated. In contrast, when outside opportunities are random, the project may be undertaken but started at an inefficient scale for some time.

It follows that the non-trivial dynamics in [Albuquerque and Hopenhayn \(2004\)](#) emerge because of the dependency of the outside opportunity on capital k . Indeed, here the dynamics are non trivial even without this dependence. In our model, letting V_t rise over time promotes efficiency by lowering the amount of default, which increases the average productivity of investments, making higher investments worthy. This contrasts with [Albuquerque and Hopenhayn \(2004\)](#), where the reason V_t is increased is to allow higher capital investments while ensuring that default never takes place.

Note that without randomness, better outside opportunities hurt investment. In contrast, when outside opportunities are random a first-order stochastic improvement in $F(\theta)$ may facilitate investment. This may occur if the new distribution increases the option value for the entrepreneur of waiting for a still better outside opportunity draw. Since only the very best draws are taken this can lower default rates early in the project, facilitating investment. However, default is likely to be higher at latter stages of the project. For example, in the extreme case where the upper bound on the support $\bar{\theta}$ has increased, it may now take longer to reach the first best. Overall, one expects a flatter default probability path.

3 A Tradeoff Between Investment and Default

We now return to the general case where capital affects the value of the outside opportunity. The amount invested will now influence the current level of default. This effect leads to a

nontrivial dynamic optimization problem.

Lifetime utility within the contract evolves according to

$$V_t = \int_{\Theta} \max\{O(\theta_t, k_t), \tau_t^e + \beta V_{t+1}\} dF(\theta), \quad (10)$$

Setting $\tau_t^e = 0$ we implicitly solve for $\hat{\theta}$

$$\hat{\theta} = \phi(k, V_{t+1})$$

where ϕ is increasing in V_{t+1} and decreasing in k . Substituting this into the first equation:

$$V = \int_{\psi(k, V)} O(\theta, k) F(d\theta) + F(\phi(k, V_{t+1})) \beta V_{t+1}$$

The right hand side is increasing in V_{t+1} and k

$$\begin{aligned} \frac{\partial RHS}{\partial V_{t+1}} &= \underbrace{\frac{\partial \phi}{\partial V_{t+1}}(k, V_{t+1}) f(\hat{\theta}) [\beta V_{t+1} - O(\hat{\theta}, k)]}_{=0} \\ &\quad + F(\phi(k, V_{t+1})) \beta > 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial RHS}{\partial k} &= - \underbrace{\frac{\partial \phi}{\partial k}(k, V_{t+1}) O(\hat{\theta}, k) f(\hat{\theta}) + \frac{\partial \phi}{\partial k}(k, V_{t+1}) f(\hat{\theta}) \beta V_{t+1}}_{=0} \\ &\quad - \int_{\psi(k, V)} O_k(\theta, k) F(d\theta) > 0 \end{aligned}$$

So there is an implicit negative relationship between k and V_{t+1} . We denote this relationship by

$$\begin{aligned} V_{t+1} &= \varphi(k, V) \\ \hat{\theta} &= \psi(k, V) \end{aligned}$$

the function φ is decreasing in k and increasing in V ; the function ψ is decreasing in k and increasing in V .

Thus the problem can be written as:

$$W(V) = \max_k \{ F(\psi(k, V))R(k) + \int_{\psi(k, V)} O(\theta, k) dF(d\theta) - k + \beta F(\psi(k, V))W(\varphi(k, V)) \}$$

The f.o.c. with respect to k is:

$$\begin{aligned} [R(k) + \beta W(\psi(k, V)) - O(\psi(k, V), k)]\psi_k(k, V)f(\psi(k, V)) + F(\psi(k, V))R'(k) \\ + \int_{\psi(k, V)} O_k(\theta, k) dF(d\theta) - 1 + \beta F(\psi(k, V))W'(\varphi(k, V))\varphi_k(k, V) = 0 \end{aligned}$$

We would like to know if from this f.o.c. we can say that k is increasing in V . This looks difficult since there are many ways V and k enter and we don't really know all the properties of the ψ and φ functions.

4 Persistent Shocks

Assume now that the shock is fully persistent but that it is revealed slowly over time in the following sense. Every period agents that have not had a chance in the past receive with probability α an outside option and learn their θ which is distributed according to $F(\theta)$.

Then we have the following equilibrium relations:

$$\begin{aligned} \hat{\theta} &= \beta(V'_s + \tau) \\ V_s &= \beta(V'_s + \tau) \\ V_L &= \alpha \left[\int_{\hat{\theta}} \theta dF(\theta) + \beta F(\hat{\theta})(V'_s + \tau) \right] + (1 - \alpha)V'_L \end{aligned}$$

Setting $\tau = 0$ we obtain:

$$\begin{aligned} \hat{\theta} &= \beta V'_s = V_s \\ V_s &= \beta V'_s \\ V_L &= \alpha \left[\int_{\hat{\theta}} \theta dF(\theta) + \beta F(\hat{\theta})V'_s \right] + (1 - \alpha)V'_L \end{aligned}$$

For any initial V_{s0} there is a unique V_{L0} that is consistent, this can be constructed as

follows: V_s grows at rate $\beta^{-1} - 1$ each period. At the first period that it crosses $\bar{\theta}$, say T , there will no longer be default. Thus at this time V_{LT} must equal V_{sT} . Working backwards using

$$V_{Lt-1} = \alpha \left[\int_{V_{st}} \theta dF(\theta) + \beta F(\hat{\theta}) V_{st} \right] + (1 - \alpha) V_{Lt}$$

we can find the initial V_{L0} consistent with these equations. Doing this for every initial value of V_s we obtain a function relating V_s and V_L :

$$\begin{aligned} V_L &= \phi(V_s) \\ V_s &= \psi(V_L) \end{aligned}$$

it is easy to see that ψ is increasing.

Then the optimum policy is as follows: for a given value of V_{L0} we compute the initial value of V_{s0} consistent with it. We then know that V_s and therefore $\hat{\theta}$ grow at a constant positive rate: the hazard rate is thus falling over time and reaches zero in finite time.

What about the sequence of capital? Due to the tight domain constraints between V_L and V_s we can index the surplus function by V_L and λ the fraction of stayers. Given α and $\hat{\theta}$ the number of stayers, S , limbo-agents, L , and goers, G are given by:

$$\begin{aligned} S' &= S + L\alpha F(\hat{\theta}) \\ L' &= L - L\alpha = (1 - \alpha)L \\ G' &= G + L\alpha[1 - F(\hat{\theta})] \end{aligned}$$

Then we have that

$$\begin{aligned} \lambda' &\equiv \frac{S'}{S' + L'} \\ &= \frac{S + L\alpha F(\hat{\theta})}{S + L\alpha F(\hat{\theta}) + (1 - \alpha)L} \\ &= \frac{\lambda + (1 - \lambda)\alpha F(\hat{\theta})}{\lambda + (1 - \lambda)\alpha F(\hat{\theta}) + (1 - \alpha)(1 - \lambda)} \\ &= \phi(\lambda, \alpha F(\hat{\theta})) \end{aligned}$$

The default hazard here is $\alpha(1 - F(\hat{\theta}))$.

We can write the objective function for the static optimization as:

$$\pi(\hat{\theta}) \equiv \max_k [\lambda + (1 - \lambda)\alpha F(\hat{\theta})]R(k) - k$$

The surplus function is then:

$$W(\cdot, \lambda) = \max_k [\pi(\hat{\theta}) + (1 - \lambda)\alpha \int_{\hat{\theta}} \theta F(d\theta) + \beta[1 - \alpha(1 - F(\hat{\theta}))]W(\cdot, \lambda')]$$

$$\lambda' = g(\lambda, \hat{\theta})$$

5 Open Questions / To Do List

1. Work out the case with observable outside option. Do some entrepreneurs leave? Or do they all stay?
2. what does the implementation look like in the simple case? A simple debt contract?
3. what if shocks to revenue?
4. what if positive liquidation value?
5. how general is the idea that we can't/don't wish to separate

References

- Albuquerque, Rui and Hugo A. Hopenhayn**, “Optimal Lending Contracts and Firm Dynamics,” *Review of Economic Studies*, 2004, 71 (2), 285–315. [2](#), [9](#)
- Alvarez, Fernando and Urban J. Jermann**, “Efficiency, Equilibrium, and Asset Pricing with Risk of Default,” *Econometrica*, 2000, 68 (4), 775–798. [2](#)
- Clementi, Gian Luca and Hugo Hopenhayn**, “A Theory of Financing Constraints and Firm Dynamics,” *Quarterly Journal of Economics*, 2006, 121 (1), 229–265. [2](#)
- Kehoe, Patrick J. and Fabrizio Perri**, “International Business Cycles with Endogenous Incomplete Markets,” *Econometrica*, 2002, 70 (3), 907–928. [2](#)
- Kocherlakota, Narayana R.**, “Implications of Efficient Risk Sharing without Commitment,” *Review of Economic Studies*, 1996, 63 (4), 595–609. [2](#)
- Thomas, Jonathan and Tim Worrall**, “Foreign Direct Investment and the Risk of Expropriation,” *Review of Economic Studies*, 1994, 61 (1), 81–108. [2](#)