Technical Appendix to Collateralized Debt as the Optimal Contract

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Abstract

This appendix contains proofs for the main article (Lacker 2001).

1. Proposition 1

First observe that if (q, R) does not satisfy (3.5) with equality, either (3.5) is violated, in which case (q, R) is not feasible, or the inequality is strict. In the latter case the borrower can be made better off either by reducing R or by increasing q.

Suppose then that (q, R) satisfies (3.5) with equality. Then (A.7) below is necessary and sufficient for (q, R) to solve (P1). Condition (A.7) implies (3.6).

To show that (A.7) is necessary and sufficient, reformulate (P1) by fixing q, and restricting attention to the subproblem of finding optimal repayment schedules y_1 and y_2. Define $x_1(\theta) = u_1(\theta - y_1(\theta))$, $x_2(\theta) = u_2(k - y_2(\theta))$, and $x(\theta) = (x_1(\theta), x_2(\theta))'$. Let $\Theta = [\theta_0, \theta_1]$. A contract is now a function $\mathbf{x} : \Theta \to \mathbb{R}^2$. Restrict attention to the set A of absolutely continuous functions from Θ to $\mathbb{R}^2$. In the working paper version (Lacker 1998) I proved that the collateralized debt contract is optimal within the broader class of functions of bounded variation. Define ϕ_i as the inverse of u_i for $i = 1, 2$. Given $\mathbf{x}$, we can recover $y_1(\theta) = \theta - \phi_1(x_1(\theta))$ and $y_2(\theta) = k - \phi_2(x_2(\theta))$. The incentive constraints (3.4) can be replaced with the weaker local incentive constraints:

$$-y_1'(\theta)u_1'(\theta - y_1(\theta)) - y_2'(\theta)u_2'(k - y_2(\theta)) \geq 0, \qquad \theta \in (\theta_0, \theta_1) \tag{A.1}$$

which must hold everywhere the derivative exists. We can rewrite this as

$$x_1'(\theta) + x_2'(\theta) \geq u_1'(\phi_1(x_1(\theta))) \equiv g(x_1(\theta)), \qquad \theta \in (\theta_0, \theta_1). \tag{A.2}$$

Among absolutely continuous contracts, (A.2) is necessary and sufficient for (3.4) due to nonincreasing absolute risk aversion, but for discontinuous contracts sufficiency fails. Since

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collateralized debt contracts satisfy (3.4) by construction, it suffices to prove that the debt contract is optimal under the weaker condition (A.2). Let

$$X(\theta) = [\underline{x}_1, \overline{x}_1(\theta)] \times [\underline{x}_2, \overline{x}_2],$$

where $\underline{x}_1 \equiv u_1(0)$, $\overline{x}_1(\theta) \equiv u_1(\theta + e_{l21})$, $\underline{x}_2 \equiv u_2(0)$, $\overline{x}_2 \equiv u_2(k)$, and let

$$Z(\theta, x) = \{z \in \mathbb{R}^2 | z_1 + z_2 \geq g(x_1)\}.$$

For the collateralized debt contract, $x_1^*(\theta) = \underline{x}_1$ for $\theta \in [\theta_0, R]$, and $x_1^*(\theta) \in (\underline{x}_1, \overline{x}_1(\theta))$ for $\theta \in (R, \theta_1]$, while $x_2^*(\theta) \in (\underline{x}_2, \overline{x}_2)$ for $\theta \in (\theta_0, R)$, and $x_2^*(\theta) = \overline{x}_2$ for $\theta \in [R, \theta_1]$. If $x_2^*(\theta_0) = \underline{x}_2$, then the contract is "collateral constrained" and $R = \overline{R}$. Let φ be the nonnegative multiplier on the constraint (A.2).

Define the Lagrangian function as

$$L(\theta, x, z) = -(x_1 + x_2) f(\theta) - \varphi[e_{l21} + \theta - \phi_1(\theta) + \lambda(k - \phi_2(x_2))] f(\theta)$$

if $x \in X(\theta)$ and $x' \in Z(\theta, x)$, and $+\infty$ otherwise. Define the functional

$$J_L(\mathbf{x}) = \int_{\theta_0}^{\theta_1} L(\theta, x(\theta), x'(\theta)) d\theta.$$

The problem is now: (P′) Choose $\mathbf{x} \in A$ to minimize $J_L(\mathbf{x})$. This falls in a class of problems studied by Rockafellar (1972).

I will now show that if (q, R) satisfies (3.5) with equality and (A.7) below, then it satisfies the Hamiltonian and endpoint conditions for (P′). The costate function for this problem is a pair of mappings from Θ to $\mathbb{R}$, but for a nontrivial solution these functions must lie in the normal cone of $Z(\theta, x(\theta))$ for each θ, which requires that the two functions be everywhere equal and nonpositive. Thus the Hamiltonian conditions can be written in terms of a single costate function $\mathbf{p} : \Theta \to \mathbb{R}$. Evaluated at the collateralized debt contract, the Hamiltonian conditions are:

$$\begin{gathered} p'(\theta) - \rho^*(\theta) p(\theta) \in L_1^*(\theta) \\ p'(\theta) \in L_2^*(\theta) \end{gathered} \tag{A.3}$$

where

$$\rho^*(\theta) \equiv -\frac{u_1''(\phi_1(x_1^*(\theta)))}{u_1'(\phi_1(x_1^*(\theta)))}$$

$$L_i^*(\theta) = \begin{cases} (-\infty, \nu_i(\theta)] & \text{if } x_i^*(\theta) = \underline{x}_i, \\ \{\nu_i(\theta)\} & \text{if } x_i^*(\theta) \in (\underline{x}_i, \overline{x}_i(\theta)), \\ [\nu_i(\theta), +\infty) & \text{if } x_i^*(\theta) = \overline{x}_i(\theta) \end{cases} \qquad i = 1, 2,$$

$$\nu_1(\theta) = \left(-1 + \frac{\varphi}{u_1'(\phi_1(x_1^*(\theta)))}\right) f(\theta),$$

$$\nu_2(\theta) = \left(-1 + \frac{\varphi\lambda}{u_2'(\phi_2(x_2^*(\theta)))}\right) f(\theta),$$

and p is absolutely continuous. The endpoint conditions are

$$\begin{aligned} R < \overline{R} &\Rightarrow p(\theta_0) = 0, \\ p(\theta_0) < 0 &\Rightarrow R = \overline{R}, \\ R > e_{l21} &\Rightarrow p(\theta_1) = 0, \\ p(\theta_1) < 0 &\Rightarrow R = e_{l21}. \end{aligned} \tag{A.4}$$

The collateralized debt contract is optimal if there exists an absolutely continuous and nonpositive $\mathbf{p}$ and a nonnegative φ that satisfy (A.3) and (A.4).

The function $\mathbf{p}$ satisfies (A.3) if and only if

$$p'(\theta) = \begin{cases} \nu_2(\theta) & \text{for } \theta < R \\ \rho^*(\theta)p(\theta) & \text{for } \theta > R \end{cases} \tag{A.5}$$

and

$$\rho^*(\theta)p(\theta) + \nu_1(\theta) \geq \nu_2(\theta). \tag{A.6}$$

Solving (A.5) for $p(\theta)$ with the endpoint $p(\theta_1) = 0$ (since $R > e_{l21}$) yields

$$p(\theta)u_1'(\theta - y_1^*(\theta, R)) = p(\theta_0)u_1'(0) + E[w_b(\widehat{\theta}, R)|\widehat{\theta} \leq \theta]F(\theta) + \varphi E[w_l(\widehat{\theta}, R)|\widehat{\theta} \leq \theta]F(\theta),$$

where

$$\begin{aligned} w_b(\theta, R) &= -u_1'(\theta - y_1^*(\theta, R)) \\ w_l(\theta, R) &= \begin{cases} \frac{\lambda u_1'(\theta - y_1^*(\theta, R))}{u_2'(k - y_2^*(\theta, R))} & \text{for } \theta < R \\ \lambda & \text{for } \theta \geq R \end{cases} \end{aligned}$$

are the derivatives, respectively, of the borrowers and the lenders ex post utilities at θ with respect to R. Setting $p(\theta_0) = 0$ and substituting for φ yields

$$\begin{aligned} -\frac{p(\theta)u_1'(\theta - y_1^*(\theta, R))}{\varphi f(\theta)} &= \frac{F(\theta)}{f(\theta)}\left[E[w_b(\widehat{\theta}, R)|\widehat{\theta} < \theta]\frac{E[w_l(\widehat{\theta}, R)]}{E[w_b(\widehat{\theta}, R)]} - E[w_l(\widehat{\theta}, R)|\widehat{\theta} < \theta]\right] \\ &\equiv \phi(\theta, R). \end{aligned}$$

It is straightforward to verify that $\varphi > 0$ and $p(\theta) < 0$ for all θ. Rearranging (A.6) and substituting yields

$$\lambda \leq \frac{u_2'(k - y_2^*(\theta, R))}{u_1'(\theta - y_1^*(\theta, R))}\left[1 + \rho^*(\theta)\left(\frac{p(\theta)u_1'(\theta - y_1^*(\theta, R))}{\varphi f(\theta)}\right)\right]$$

or

$$\lambda \leq \mu_b^*(\theta, q, R) - \rho^*(\theta)\mu_b^*(\theta, q, R)\phi(\theta, R). \tag{A.7}$$

This proves that if the collateralized debt contract satisfies (A.7) then it satisfies the necessary and sufficient conditions for the problem P′. Since the term subtracted from the right hand side is positive, condition (A.7) implies Proposition 1.

To establish uniqueness, suppose that $\mathbf{x}^*$ is an optimal collateralized debt contract and $\widehat{\mathbf{x}}$ is some optimal contract that differs from $\mathbf{x}^*$ on a set of positive measure. Since $\widehat{\mathbf{x}}$ is feasible, $\widehat{\mathbf{x}}'(\theta) \in \mathbf{Z}(\theta, \widehat{x}(\theta))$ for all θ, and

$$\begin{aligned} z_1(\widehat{x}_1(\theta) - x_1^*(\theta)) + z_2(\widehat{x}_2(\theta) - x_2^*(\theta)) &\le L(\theta, \widehat{x}(\theta), \widehat{x}'(\theta)) - L(\theta, x^*(\theta), x^{*\prime}(\theta)) \\ \forall(z_1, z_2) &\in L_1^*(\theta) \times L_2^*(\theta). \end{aligned}$$

From the properties of the collateralized debt contract we have

$$\begin{aligned} &(p'(\theta) - \rho^*(\theta)p(\theta))(\widehat{x}_1(\theta) - x_1^*(\theta)) + p'(\theta)(\widehat{x}_2(\theta) - x_2^*(\theta)) \\ &\le \nu_1(\theta)(\widehat{x}_1(\theta) - x_1^*(\theta)) + \nu_2(\theta)(\widehat{x}_2(\theta) - x_2^*(\theta)) \\ &\le L(\theta, \widehat{x}(\theta), \widehat{x}'(\theta)) - L(\theta, x^*(\theta), x^{*\prime}(\theta)). \end{aligned} \tag{A.8}$$

Also note that

$$\begin{aligned} &\widehat{x}_1'(\theta) + \widehat{x}_2'(\theta) \ge u_1'(\phi_1(\widehat{x}_1(\theta))) \\ &\ge u_1'(\phi_1(x_1^*(\theta))) - \rho^*(\theta)(\widehat{x}_1(\theta) - x_1^*(\theta)) \\ &= x_1^*(\theta) + x_2^*(\theta) - \rho^*(\theta)(\widehat{x}_1(\theta) - x_1^*(\theta)). \end{aligned} \tag{A.9}$$

Therefore,

$$\begin{aligned} J_L(\widehat{\mathbf{x}}) - J_L(\mathbf{x}^*) &= \int_{\theta_0}^{\theta_1} [L(\theta, \widehat{x}(\theta), \widehat{x}'(\theta)) - L(\theta, x^*(\theta), x^{*\prime}(\theta))]\, d\theta \\ &\ge \int_{\theta_0}^{\theta_1} [(p'(\theta) - \rho^*(\theta)p(\theta))(\widehat{x}_1(\theta) - x_1^*(\theta)) + p'(\theta)(\widehat{x}_2(\theta) - x_2^*(\theta))]\, d\theta \\ &= -\int_{\theta_0}^{\theta_1} p(\theta)\, [\widehat{x}_1'(\theta) + \widehat{x}_2'(\theta) + \rho^*(\theta)(\widehat{x}_1(\theta) - x_1^*(\theta)) - x_1^{*\prime}(\theta) - x_2^{*\prime}(\theta)]\, d\theta \\ &\ge 0. \end{aligned} \tag{A.10}$$

If $\widehat{x}_1(\theta) \neq x_1^*(\theta)$ for a set of positive measure in $[\theta_0, R)$, then (A.8) is strict on that set and the first inequality in (A.10) is strict. If $\widehat{x}_2(\theta) \neq x_2^*(\theta)$ for a set of positive measure in $(R, \theta_1]$, then (A.8) is strict on that set and the first inequality in (A.10) is strict. Thus $J_L(\widehat{\mathbf{x}}) > J_L(\mathbf{x}^*)$, $\mathbf{x}^*$ is the unique optimal contract, and (A.7) is both necessary and sufficient.

2. Proposition 2

Suppose (3.6) holds for the debt contract that satisfies (3.5) with equality. Then the concavity of u implies that $\lambda < u_2'(c_2)/u_1'(c_1)$ for all feasible (c_1, c_2). Consider an arbitrary contract in which $y_1(\theta) < \theta$ and $y_2(\theta) > 0$ for some particular θ. If $u_1(\theta - y_1(\theta)) + u_2(k - y_2(\theta)) < u_1(0) + u_2(k)$, then there is a $z_2 \in [0, k)$ such that $u_1(\theta - z_1) + u_2(k - z_2) = u_1(\theta - y_1(\theta)) + u_2(k - y_2(\theta))$ and $e_{l21} + z_1 + \lambda z_2 > e_{l21} + y_1(\theta) + \lambda y_2(\theta)$ for $z_1 = \theta$. This implies that $(y_1(\theta), y_2(\theta))$ is not renegotiation-proof at θ. Similarly, if $u_1(\theta - y_1(\theta)) + u_2(k - y_2(\theta)) \ge u_1(0) + u_2(k)$, then there is a $z_1 < \theta$ such that z_1 and $z_2 = 0$ dominates $(y_1(\theta), y_2(\theta))$. Therefore, if (3.6) holds then renegotiation-proofness requires that either $y_1(\theta) = \theta$ or $y_2(\theta) = 0$.

Reformulate (P2) by defining $x(\theta) = u_1(\theta - y_1(\theta)) + u_2(k - y_2(\theta))$. Since either $y_1(\theta) = \theta$ or $y_2(\theta) = 0$, we can write $y_1(\theta)$ and $y_2(\theta)$ as functions of $x(\theta)$ and θ. The method used in the proof of Proposition 1 then can be used to show that (3.6) is necessary and sufficient for the debt contract to be optimal.

3. Proposition 3

Suppose an arbitrary contract (q, π, y_{10}, y_{1k}) satisfies (4.1), (4.3), (4.4), and (4.5) with equality. Suppose there exists a positive measure subset A of Θ on which $\pi(\theta) > 0$ and

$$\begin{gathered} u_1(0) + u_2(k) \leq \\ (1 - \pi(\theta))\left[u_1(\theta - y_{10}(\theta)) + u_2(0)\right] + \pi(\theta)\left[u_1(\theta - y_{1k}(\theta)) + u_2(k)\right] \equiv v_b(\theta). \end{gathered} \tag{A.11}$$

Then there exists an alternative contract with $\pi(\theta) = 0$ for all θ in A, and everything else unchanged. Since

$$u_1(\theta - y_{10}(\theta)) + u_2(0) = u_1(\theta - y_{1k}(\theta)) + u_2(k)$$

(if $\pi(\theta) = 1$ then the choice of $y_{10}(\theta)$ is without loss of generality), the borrower is indifferent between the two contracts in every state. Given (4.6), the lender is strictly better off in every state in A. Therefore, an optimal contract must have $\pi(\theta) = 0$ whenever (A.11) holds. If

$$v_b(\theta) < u_1(0) + u_2(k), \tag{A.12}$$

then there is no feasible $y_{10}(\theta)$ for which the borrower is indifferent between transferring collateral and keeping it, as would be required by (4.1) if $\pi(\theta) < 1$. Therefore, $\pi(\theta) = 1$ whenever (A.12) holds.

Since $v_b(\theta)$ is strictly increasing, we can write a contract in terms of a cutoff value θ' : $\pi(\theta) = 1$ for $\theta < \theta'$ and $\pi(\theta) = 0$ for $\theta > \theta'$. Incentive compatibility requires that y_{10} and y_{1k} are nonincreasing on $[\theta_0, \theta')$ and $[\theta', \theta_1]$ respectively. In addition, we require $\theta' - y_{1k}(\theta') \leq c_k$, where $y_{1k}(\theta')$ is the left hand limit of $y_{1k}(\theta)$ at θ', since otherwise θ' could be reduced with no change in $v_b(\theta)$ but an increase in $v_l(\theta)$ for some θ. Consider an arbitrary contract (q, π, y_{10}, y_{1k}) satisfying these conditions and resource feasibility. Replace y_{10} and y_{1k} by their certainty equivalents. The resulting contract is feasible, incentive compatible, and Pareto-improving, since the borrower is risk averse. We have now reduced (P3) to choosing three numbers: θ', $y_{1k}(\theta) = R_k$, and $y_{10}(\theta) = R$. These must satisfy

$$\begin{gathered} \theta_0 \geq R_k \geq \theta' - c_k, \text{ and} \\ \theta' \geq R \geq -e_{l21} \end{gathered} \tag{A.13}$$

I claim that $\theta' = R$ in an optimal contract. Take an arbitrary (θ', R, R_k) that satisfy (A.13) and (4.5) with equality. For $\delta > 0$ define $\theta'(\delta) = \theta' - \delta$, and define $R(\delta)$ by

$$e_{l11} - q + e_{l21} + R(\delta)(1 - F(\theta'(\delta))) + (R_k + \lambda k)F(\theta'(\delta)) = \overline{v}_l.$$

Hold R_k constant. Then

$$\begin{aligned} \frac{\partial E[v_b(\theta)]}{\partial \delta} = \{ & u_1(\theta'(\delta) - R_k) + u_2(0) - u_1(\theta'(\delta) - R(\delta)) - u_2(k) \\ & + (R_k + \lambda k - R(\delta))E[u_1'(\theta - R(\delta))|\theta \geq \theta']\} f(\theta'). \end{aligned}$$

Condition (4.6) implies that the right side is strictly positive for feasible values of δ. Therefore, contracts with $\theta' > R$ can be improved upon by increasing δ. Only when $\theta' = R$ is $\delta > 0$ infeasible. This proves Proposition 3.

4. Proposition 4

In view of Proposition 3, (P3) can be reduced to choosing two numbers, R and R_k, to maximize borrower's expected utility subject to

$$\theta_0 - R_k \geq 0 \qquad (\mu_0)$$
$$R_k - R + c_k \geq 0 \qquad (\mu_k)$$

where μ_0 and μ_k are the nonnegative multipliers associated with their respective constraints. Let $R_k(R)$ be the implicit function defined by (4.5) holding with equality. The first order necessary conditions can then be written

$$\begin{aligned} &\lambda k + R_k(R) - R - \frac{u_1(0) + u_2(k) - u_1(R - R_k(R)) - u_2(0)}{E[u_1'(\theta - R_k(R))|\theta \leq R]} \\ &\quad + \left[1 - \frac{E[u_1'(\theta - R)|\theta \geq R]}{E[u_1'(\theta - R_k(R))|\theta < R]}\right]\left(\frac{1 - F(R)}{f(R)}\right) \\ &= \frac{[1 + (R_k(R) + \lambda k - R)f(R)]\mu_k}{f(R)F(R)E[u_1'(\theta - R_k)|\theta < R]} + \frac{[1 - F(R) + (R_k(R) + \lambda k - R)f(R)]\mu_0}{f(R)F(R)E[u_1'(\theta - R_k(R))|\theta < R]}. \end{aligned} \tag{A.14}$$

$$\mu_k \geq 0, \quad \mu_k(R_k(R) - R + c_k) = 0 \tag{A.15}$$

$$\mu_0 \geq 0, \quad \mu_0(\theta_0 - R_k(R)) = 0 \tag{A.16}$$

Proposition 7: *If in addition to* (4.7),
(i) $(c_k - \lambda k)f(R) + F(R) \leq 1$,
(ii) $E[u_1'(\theta - R)|\theta \geq R]$ *is nondecreasing in* R, *and*
(iii) $f(R)/(1 - F(R))$ *is nondecreasing,*
then there is a unique optimal contract (R, R_k) *determined by* (A.14)-(A.16).

The proof (omitted) hinges on showing that under these assumptions $R_k(R)$ is decreasing, the left side of (A.14) is strictly decreasing in R, and the coefficients on μ_k and μ_0 are both positive. Either the left side of (A.14) is positive at $R_k(R) = \theta_0$, or the left side of (A.14) is zero (pinning down R), or the left side of (A.14) is negative at $R_k(R) = R - c_k$.

5. Proposition 5

Take an arbitrary contract (q, π, y_{10}, y_{1k}), fix θ, and suppose that $\pi(\theta) \in (0, 1)$. Together (4.1) and (4.8) imply that $y_{10}(\theta) > y_{1k}(\theta) + \lambda k$. The borrower is indifferent to setting $\pi(\theta) = 0$ instead, and the lender is strictly better off. Thus the original contract was not renegotiation-proof at θ and we must have $\pi(\theta) \in \{0, 1\}$ everywhere.

Suppose $\pi(\theta) = 1$ and $u_1(0) + u_2(k) \leq u_1(\theta - y_{1k}(\theta)) + u_2(0)$, so that the borrower is at least as well off as if the collateral is kept and all of the corn is paid. Then there exists a feasible $y_{10}(\theta)$ such that $u_1(\theta - y_{10}(\theta)) + u_2(k) = u_1(\theta - y_{1k}(\theta)) + u_2(0)$. Setting $\pi(\theta) = 0$ now dominates the original contract, which thus could not have been renegotiation-proof. Therefore, if $\pi(\theta) = 1$ then

$$u_1(0) + u_2(k) > u_1(\theta - y_{1k}(\theta)) + u_2(0).$$

Since ex post borrower utility is strictly increasing in θ, there exists a cutoff θ' such that $\pi(\theta) = 1$ for $\theta < \theta'$ and $\pi(\theta) = 0$ for $\theta > \theta'$. At θ' we must have $\theta' - y_{1k}(\theta') \leq c_k$, where $y_{1k}(\theta')$ is the left hand limit of $y_{1k}(\theta)$ at θ'.

Suppose $y_{1k}(\theta'') + \lambda k < \theta''$ for some $\theta'' < \theta'$. This contact is not renegotiation-proof at θ''. For any $y_{10} \in (y_{1k}(\theta'') + \lambda k, \theta'')$, changing $\pi(\theta)$ to 0 with $y_{10}(\theta'') = y_{10}$ makes the lender strictly better off. An implication of (4.6) is that $u_1(c) + u_2(k) > u_1(c + \lambda k) + u_2(0)$ for all $c \geq 0$, and thus

$$u_1(\theta'' - y_{1k}(\theta'') - \lambda k) + u_2(k) > u_1(\theta'' - y_{1k}(\theta'')) + u_2(0).$$

Since y_{10} can be made arbitrarily close to $y_{1k}(\theta'') + \lambda k$, there exists a y_{10} such that $u_1(\theta'' - y_{10}) + u_2(k) \geq u_1(\theta'' - y_{1k}(\theta'')) + u_2(0)$, as is required to make the borrower better off. Therefore, the contract is renegotiated at θ and (RP$'$) implies that $y_{1k}(\theta'') + \lambda k \geq \theta''$ for all $\theta'' < \theta'$.

Incentive compatibility again implies that $y_{10}(\theta)$ is nonincreasing on $\{\theta | \pi(\theta) = 0\}$, and $y_{1k}(\theta)$ is nonincreasing on $\{\theta | \pi(\theta) = 1\}$. In addition, feasibility requires that $y_{1k}(\theta_0) \leq \theta_0$ and $y_{10}(\theta_1) \geq -e_{l21}$. Collecting results, I have shown that (4.1), (4.3), (4.4) and (RP$'$) together imply that $\pi(\theta) = 1$ for $\theta < \theta'$, $\pi(\theta) = 0$ for $\theta > \theta'$, $y_{10}(\theta)$ and $y_{1k}(\theta)$ are nonincreasing, $y_{1k}(\theta_0) \leq \theta_0$, $y_{10}(\theta_1) \geq -e_{l21}$, and $y_{1k}(\theta) + \lambda k \geq \theta$ for all $\theta < \theta'$. The converse is easily proved as well. The rest of the proof follows that of Proposition 3, except that in constraint (A.14), $R_k \geq \theta' - c_k$ is replaced by $R_k \geq \theta' - \lambda k$.

6. Proposition 6

The proof parallels the proof of Proposition 4, with (A.15) replaced by

$$\mu_k \geq 0, \quad \mu_k(R_k(R) - R + \lambda k) = 0.$$

References

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