Entry, Exit, Embodied Technology, and Business Cycles: Computational Appendix

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Abstract

This appendix provides a detailed exposition of the computational method applied to the model of Campbell(1997). Heterogeneity in the production sector of that model implies that its prices and quantities are continuous functions on the real line rather than scalars. The computational method used is an extension of standard log–linearization methods. It uses quadrature approximation of integrals to approximate a system of functional equations with a standard system of vector equations. Similar computational problems arise in other economic models which incorporate heterogeneity, such as that of Merz(1995), so the methods of this appendix may be of independent interest to other researchers.

1. Overview

This appendix provides a detailed exposition of the computational method applied to the model of Campbell(1997). Heterogeneity in the production sector of that model implies that its prices and quantities are continuous functions on the real line rather than scalars. The computational method used is an extension of standard log–linearization methods. It uses quadrature approximation of integrals to approximate a system of functional equations with a standard system of

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vector equations. Similar computational problems arise in other economic models which incorporate heterogeneity, such as that of Merz(1995), so the methods of this appendix may be of independent interest to other researchers.

To compute the model's competitive equilibrium, it is useful to exploit its equivalence with the solution to a social planning problem. Because the aggregate production function is Cobb–Douglas, the capital augmenting technical change can be written in labor augmenting form. Written this way, the economy satisfies the balanced growth restrictions of King, Plosser, and Rebelo(1987), so there exists an economy with stationary production possibilities which is equivalent to the economy of interest. To compute the equilibrium path of the economy's endogenous variables, King, Plosser, and Rebelo(1987) log–linearize the stationary social planning problem's Euler equations around their non–stochastic steady state and solve the resulting linear dynamic system.

The computational method used here extends this approach to the case where the state variable is a function, rather than a scaler. This complicates both the computation of the both the non–stochastic steady state and the near steady state dynamics. The Euler equations defining the steady state and their log–linear approximations are functional equations. As such, they involve integrals of both the state and costate variables. To approximate these systems, their integrals are replaced with quadrature approximations. The quadrature approximation uses only the integrand's value on a finite set of predetermined points. This reduces an infinite dimensional problem, solving for a continuous function, down to a finite dimensional one, solving for the function's value on a finite set of points. Standard linear algebraic methods can be applied to the finite dimensional systems which result.

Similar computational challenges naturally arise in models with heterogeneity in the production sector. Merz(1995) employs the methods of this appendix to study a framework in which idiosyncratic productivity shocks effect job matches. Although the computation of the model's competitive equilibrium takes much more time than the computation for a standard real business cycle model, the methods are still very fast. Computing the model economy's first and second moments takes just under three minutes using a 100 Mhz 486 computer using Gauss for Windows NT/95 version 3.2.25. Therefore, these methods may find practical application to other similar problems. A copy of the programs used for the computation may be obtained from the world wide web at

`ftp://troi.cc.rochester.edu/pub/economics/Jeff_Campbell/research/e_and_e.`

The next section of this appendix details the transformation of the economy into one with stationary production possibilities. The following sections describe the computation of the non–stochastic steady state and the near steady state dynamics. The final section describes the sources of data used in the paper. The notation of the appendix conforms as closely as possible to that of the paper.

2. A Stationary Economy

Because the first and second welfare theorems apply to the model economy, the problem of calculating the model's competitive equilibrium corresponds to the maximization of a social welfare function. This appendix will approach the computational problem from this perspective. The economy's social planning problem is

$$\max_{\substack{\{C_t\}_{t=0}^{\infty},\, \{N_t\}_{t=0}^{\infty},\, \{I_t^j\}_{t=0}^{\infty} \\ \{\overline{K}_t\}_{t=0}^{\infty},\, \{\underline{v}_t\}_{t=0}^{\infty},\, \{K_t(u_t)\}_{t=1}^{\infty}}} E_0 \sum_{t=0}^{\infty} \beta^t (\ln(C_t) + \kappa(1 - N_t))$$

subject to

$$
\begin{aligned}
C_t + I_t^1 &= \overline{K}_t^{1-\alpha} N_t{}^{\alpha} + \eta \int_{-\infty}^{\underline{v}_t} K_t\left(v_t\right) dv_t \\
\overline{K}_t &= \int_{-\infty}^{\infty} e^{v_t} K_t\left(v_t\right) dv_t \\
I_t^j &= I_{t-1}^{j-1} \quad j = 2, \ldots, T^i \\
K_{t+1}(v_{t+1}) &= \int_{\underline{v}_t}^{\infty} \frac{1}{\sigma} \phi\left(\frac{v_{t+1} - v_t}{\sigma}\right) K_t\left(v_t\right) dv_t \\
&\quad + \frac{1}{\sigma_e} \phi\left(\frac{v_{t+1} - z_{t-T^i}}{\sigma_e}\right) I_t^{T^i}
\end{aligned}
$$

The variable I_t^j is the mass of investment projects which were begun $j-1$ periods ago in period t, so that I_t^1 is new investment and $I_t^{T^i}$ is the mass of projects which will begin operation next period.

2.1 Transforming the Original Economy

Linearization methods approximate the social planning problem's Euler equations near the economy's steady state, so applying these methods requires a stationary economic environment. Improvement in the leading edge production process causes the model economy to experience long term growth. The first step towards solving the model is accommodating this growth. To do so, first express all technical change in labor augmenting form. This can be done because the aggregate production function is Cobb–Douglas. To do so, define $u_t = v_t - z_{t-1}$, $\underline{u}_t = \underline{v}_t - z_{t-1}$, and $K_t^T\left(u_t\right) = K_t\left(u_t + z_{t-1}\right)$. Written this way, the economy's social planning problem is

$$
\max_{\substack{\{C_t\}_{t=0}^{\infty}, \{N_t\}_{t=0}^{\infty}, \{I_t^j\}_{t=0}^{\infty} \\ \{\overline{K}_t^T\}_{t=0}^{\infty}, \{\underline{u}_t\}_{t=0}^{\infty}, \{K_t^T(u_t)\}_{t=1}^{\infty}}} E_0 \sum_{t=0}^{\infty} \beta^t (\ln(C_t) + \kappa(1 - N_t))
$$

subject to

$$
\begin{aligned}
C_t + I_t^1 &= \overline{K}_t^{T1-\alpha}(e^{\left(\frac{1-\alpha}{\alpha}\right)z_{t-1}}N_t)^\alpha + \eta \int_{-\infty}^{\underline{u}_t} K_t^T(u_t)\, du_t \\
\overline{K}_t^T &= \int_{-\infty}^{\infty} e^{u_t} K_t^T(u_t)\, du_t \\
I_t^j &= I_{t-1}^{j-1} \quad j = 2, \ldots, T^i \\
K_{t+1}^T(u_{t+1}) &= \int_{\underline{u}_t}^{\infty} \frac{1}{\sigma}\phi\left(\frac{u_{t+1} - u_t + \mu_z + \varepsilon_t^z}{\sigma}\right) K_t^T(u_t)\, du_t \\
&\quad + \frac{1}{\sigma_e}\phi\left(\frac{u_{t+1} + (T^i - 1)\,\mu_z + \sum_{j=1}^{T^i-1}\varepsilon_{t-j}^z}{\sigma_e}\right) I_t^{T^i}
\end{aligned}
$$

Written this way, the model economy satisfies the conditions of King, Plosser, and Rebelo(1987), so a scaled version of the economy exists which experiences no growth. The equilibrium quantities (excepting labor) of the scaled economy are those of the original economy divided by the state of labor-augmenting technology. Following Christiano(1988) , this is called the star economy. To derive the star economy's problem, divide consumption, investment, and the capital stock by the rate of labor-augmenting technological progress, $e^{\left(\frac{1-\alpha}{\alpha}\right)z_{t-1}}$.

$$
\begin{aligned}
C_t^\star &= e^{-\left(\frac{1-\alpha}{\alpha}\right)z_{t-1}} C_t \\
I_t^\star &= e^{-\left(\frac{1-\alpha}{\alpha}\right)z_{t-1}} I_t^j \quad j = 1, \ldots, T^i \\
K_t^\star(u_t) &= e^{-\left(\frac{1-\alpha}{\alpha}\right)z_{t-1}} K_t^T(u_t) \\
\overline{K}_t^\star &= \int_{-\infty}^{\infty} e^{u_t} K_t^\star(u_t)\, du_t
\end{aligned}
$$

Rewriting the social planning problem using this notation yields

$$
\max_{\substack{\{C_t^\star\}_{t=0}^\infty,\, \{N_t\}_{t=0}^\infty,\, \{I_t^{\star j}\}_{t=0}^\infty \\ \{\overline{K}_t^\star\}_{t=0}^\infty,\, \{\underline{u}_t\}_{t=0}^\infty,\, K_t^\star(u_t)\}_{t=1}^\infty}} E_0 \sum_{t=0}^{\infty} \beta^t(-\left(\frac{1-\alpha}{\alpha}\right) z_t + \ln(C_t^\star) + \kappa(1 - N_t))
$$

subject to

$$
\begin{aligned}
C_t^\star + I_t^{\star 1} &= \overline{K}_t^{\star 1-\alpha} N_t^\alpha + \eta \int_{-\infty}^{\underline{u}_t} K_t^\star\left(u_t\right) du_t \\
\overline{K}_t^\star &= \int_{-\infty}^{\infty} e^{u_t} K_t^\star\left(u_t\right) du_t \\
e^{\left(\frac{1-\alpha}{\alpha}\right)\left(\mu_z+\varepsilon_{t-1}^z\right)} I_t^{\star j} &= I_{t-1}^{\star j-1} \quad j = 2, \ldots, T^i \\
e^{\left(\frac{1-\alpha}{\alpha}\right)\left(\mu_z+\varepsilon_t^z\right)} K_{t+1}^\star(u_{t+1}) &= \int_{\underline{u}_t}^{\infty} \frac{1}{\sigma}\phi\left(\frac{u_{t+1}-u_t+\mu_z+\varepsilon_t^z}{\sigma}\right) K^\star{}_t\left(u_t\right) du_t \\
&\quad + \frac{1}{\sigma_e}\phi\left(\frac{u_{t+1}+\left(T^i-1\right)\mu_z+\sum_{j=1}^{T^i-1}\varepsilon_{t-j}^z}{\sigma_e}\right) I_t^{\star T^i}.
\end{aligned}
$$

The first term in the objective function is a constant and so can be ignored. The remainder is the social planning problem for the star economy. Its production possibilities are stationary, so it experiences no long term growth.

2.2 First Order Necessary Conditions

The Lagrangian for the star economy's social planning problem is

$$
\begin{aligned}
&\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \{\ln(C_t^\star) + \kappa(1-N_t) + \\
&\mathcal{M}_t(N_t^\alpha \overline{K}_t^{\star 1-\alpha} + \eta \textstyle\int_{-\infty}^{\underline{u}_t} K_t^\star\left(u_t\right) du_t - C_t^\star - I_t^{\star 1}) + \\
&\mathcal{M}_t \mathcal{D}_t(\textstyle\int_{-\infty}^{\infty} e^{u_t} K_t^\star\left(u_t\right) du_t - \overline{K}_t^\star) + \\
&\textstyle\int_{-\infty}^{\infty} \mathcal{M}_t \mathcal{Q}_t(u_t)[e^{-\left(\frac{1-\alpha}{\alpha}\right)\left(\mu_z+\varepsilon_{t-1}^z\right)}(1-\delta) \times \\
&\textstyle\int_{\underline{u}_t}^{\infty} \frac{1}{\sigma}\phi\left(\frac{u_t-u_{t-1}+\mu_z+\varepsilon_{t-1}^z}{\sigma}\right) K_{t-1}^\star\left(u_{t-1}\right) du_{t-1} + \\
&e^{-\left(\frac{1-\alpha}{\alpha}\right)\left(\mu_z+\varepsilon_{t-1}^z\right)} \frac{1}{\sigma_e}\phi\left(\frac{u_t+\left(T^i-1\right)\mu_z+\sum_{j=1}^{T^i-1}\varepsilon_{t-j}^z}{\sigma_e}\right) I_{t-1}^{\star T^i} - K_t^\star(u_t)]\, du_t + \\
&\textstyle\sum_{j=2}^{T^i} \mathcal{M}_t \mathcal{Q}_t^i(j)(e^{-\left(\frac{1-\alpha}{\alpha}\right)\left(\mu_z+\varepsilon_{t-1}^z\right)} I_{t-1}^{\star j-1} - I_t^{\star j})\}.
\end{aligned}
$$

The Lagrange multiplier on the static budget constraint, $\mathcal{M}_t$, is the undiscounted price for a contingent claim on one unit of consumption in time t. Writing the other Lagrange multipliers as multiples of $\mathcal{M}_t$ expresses them as prices for capital services, $\mathcal{D}_t$ and capital goods, $\mathcal{Q}_t(u_t)$, $\mathcal{Q}_t(j)$ in terms of current period consumption.

In addition to the star economy's resource constraints, the Lagrangian's first order necessary conditions are

Contingent Claims Pricing

$$\mathcal{L}_{C_t^\star} = \frac{1}{C_t^\star} - \mathcal{M}_t = 0$$

Labor Supply/Demand

$$\mathcal{L}_{N_t} = -\kappa + \mathcal{M}_t \alpha \left(\frac{\overline{K}_t^\star}{N_t} \right)^{1-\alpha} = 0$$

Capital Services Pricing

$$\mathcal{L}_{\overline{K}_t} = (1-\alpha) \left(\frac{\overline{K}_t^\star}{N_t} \right)^{-\alpha} - \mathcal{D}_t = 0$$

Optimal Investment/Entry

$$\mathcal{L}_{I_t^{\star 1}} = -\mathcal{M}_t + \mathbf{E}_t \beta \mathcal{M}_{t+1} e^{-\left(\frac{1-\alpha}{\alpha}\right)(\mu_z + \varepsilon_t^z)} \mathcal{Q}_{t+1}^i (2) = 0$$

Investment Project Asset Pricing, $j = 2, \ldots T^i - 1$

$$\mathcal{L}_{I_t^{\star j}} = -\mathcal{M}_t \mathcal{Q}_t (j) + \mathbf{E}_t \beta \mathcal{M}_{t+1} e^{-\left(\frac{1-\alpha}{\alpha}\right)(\mu_z + \varepsilon_t^z)} \mathcal{Q}_{t+1}^i (j+1) = 0$$

Investment Project Asset Pricing, $j = T^i$

$$\begin{aligned}
\mathcal{L}_{I_t^{\star T^i}} &= -\mathcal{M}_t \mathcal{Q}_t\left(T^i\right) + \mathbf{E}_t \beta \mathcal{M}_{t+1} e^{-\left(\frac{1-\alpha}{\alpha}\right)\left(\mu_z+\varepsilon_t^z\right)} \\
&\quad \times \int_{-\infty}^{\infty} \frac{1}{\sigma_e} \phi\left(\frac{u_{t+1} + \left(T^i - 1\right)\mu_z + \sum_{i=1}^{T^i-2} \varepsilon_{t-i}^z}{\sigma_e}\right) \mathcal{Q}_{t+1}\left(u_{t+1}\right) du_{t+1} \\
&= 0
\end{aligned}$$

Plant Asset Pricing

$$\begin{aligned}
\mathcal{L}_{K_t^\star(u_t)} &= -\mathcal{M}_t \mathcal{Q}_t\left(u_t\right) + \mathcal{M}_t \mathcal{D}_t e^{u_t} + 1\left\{u_t \leq \underline{u}_t\right\} \mathcal{M}_t (1-\delta)\eta \\
&\quad +1\left\{u_t > \underline{u}_t\right\} \mathbf{E}_t \mathcal{M}_{t+1} \beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\left(\mu_z+\varepsilon_t^z\right)} \\
&\quad \times \int_{-\infty}^{\infty} \frac{1}{\sigma} \phi\left(\frac{u_{t+1} - u_t + \mu_z + \varepsilon_t^z}{\sigma}\right) \mathcal{Q}_{t+1}\left(u_{t+1}\right) du_{t+1} \\
&= 0
\end{aligned}$$

Optimal Exit

$$\begin{aligned}
\mathcal{L}_{\underline{u}_t} &= \eta \mathcal{M}_t - \mathbf{E}_t \mathcal{M}_{t+1} \beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\left(\mu_z+\varepsilon_t^z\right)} \\
&\quad \times \int_{-\infty}^{\infty} \frac{1}{\sigma} \phi\left(\frac{u_{t+1} - \underline{u}_t + \mu_z + \varepsilon_t^z}{\sigma}\right) \mathcal{Q}_{t+1}\left(u_{t+1}\right) du_{t+1} \\
&= 0
\end{aligned}$$

The relevant transversality condition is

$$\lim_{t\to\infty} \beta^t \mathcal{M}_t \left(\sum_{j=2}^{T^i} \mathcal{Q}^i(j) I_t^j + \int_{-\infty}^{\infty} \mathcal{Q}_t\left(u_t\right) K_t^\star\left(u_t\right) du_t \right) = 0$$

The transversality condition imposes a no-bubble constraint on asset prices. Any sequence which satisfies the Euler equations and the transversality condition is a solution to the social planning problem.

3. Nonstochastic Steady State

The first step in computing a linear approximation to the Euler equations is the computation of the economy's nonstochastic steady state. The nonstochastic steady state versions of the Lagrangian's first order conditions are

Contingent Claims Pricing

$$\frac{1}{C_0^\star} = \mathcal{M}_0$$

Labor Supply/Demand

$$v'\left(1 - N_0\right) = \mathcal{M}_0 \alpha \left(\frac{\overline{K}_0^\star}{N_0}\right)^{1-\alpha}$$

Capital Services Pricing

$$\mathcal{D}_t = \left(1 - \alpha\right) \left(\frac{\overline{K}_0^\star}{N_0}\right)^{-\alpha}$$

Optimal Investment/Entry

$$1 = \beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z} \mathcal{Q}_0^i\left(2\right)$$

Investment Project Asset Pricing, $j = 2, \ldots T^i - 1$

$$\mathcal{Q}_0\left(j\right) = \beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z} \mathcal{Q}_0^i\left(j+1\right) = 0$$

Investment Project Asset Pricing, $j = T^i$

$$\mathcal{Q}_0\left(T^i\right) = \beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z} \int_{-\infty}^{\infty} \frac{1}{\sigma_e} \phi\left(\frac{u_{t+1} + \left(T^i - 1\right)\mu_z}{\sigma_e}\right) \mathcal{Q}_0\left(u_{t+1}\right) du_{t+1}$$

Plant Asset Pricing

$$\begin{aligned}\mathcal{Q}_{\mathbf{0}}\left(u_t\right) &= \mathcal{D}_{\mathbf{0}} e^{u_t}+1\left\{u_t \leq \underline{u}_{\mathbf{0}}\right\} \eta+1\left\{u_t>\underline{u}_{\mathbf{0}}\right\} \beta e^{-\left(\frac{1-\alpha}{\alpha}\right) \mu_z} \\ &\quad \times \int_{-\infty}^{\infty} \frac{1}{\sigma} \phi\left(\frac{u_{t+1}-u_t+\mu_z+\varepsilon_t^z}{\sigma}\right) \mathcal{Q}_{\mathbf{0}}\left(u_{t+1}\right) d u_{t+1}\end{aligned}$$

Optimal Exit

$$\eta=\beta e^{-\left(\frac{1-\alpha}{\alpha}\right) \mu_z} \int_{-\infty}^{\infty} \frac{1}{\sigma} \phi\left(\frac{u_{t+1}-\underline{u}_{\mathbf{0}}+\mu_z}{\sigma}\right) \mathcal{Q}_{\mathbf{0}}\left(u_{t+1}\right) d u_{t+1}$$

The steady state resource constraints are

Static Resource Constraint

$$C_{\mathbf{0}}^{\star}+I_{\mathbf{0}}^{\star 1}=N_{\mathbf{0}}^{\alpha} \overline{K}_{\mathbf{0}}^{\star 1-\alpha}+\eta \int_{-\infty}^{\underline{u}_0} K_{\mathbf{0}}^{\star}\left(u_t\right) d u_t$$

Capital Services Supply/Aggregation

$$\overline{K}_{\mathbf{0}}^{\star}=\int_{-\infty}^{\infty} e^{u_t} K_{\mathbf{0}}^{\star}\left(u_t\right) d u_t$$

Plant Accumulation

$$\begin{aligned}K_{\mathbf{0}}^{\star}\left(u_t\right) &= e^{-\left(\frac{1-\alpha}{\alpha}\right) \mu_z} \int_{\underline{u}_0}^{\infty} \frac{1}{\sigma} \phi\left(\frac{u_t-u_{t-1}+\mu_z}{\sigma}\right) K_{\mathbf{0}}^{\star}\left(u_{t-1}\right) d u_{t-1} \\ &\quad +e^{-\left(\frac{1-\alpha}{\alpha}\right) \mu_z} \frac{1}{\sigma_e} \phi\left(\frac{u_t+\left(T^i-1\right) \mu_z}{\sigma_e}\right) I_{\mathbf{0}}^{\star T^i}\end{aligned}$$

Investment Project Accumulation

$$I_{\mathbf{0}}^{\star j}=e^{-\left(\frac{1-\alpha}{\alpha}\right) \mu_z} I_{\mathbf{0}}^{\star j-1} \text{ for } j=2, \ldots, T^i$$

In the standard one-sector model, the steady state intertemporal Euler equation determines the marginal product of capital, this and the predetermined steady state labor supply determine the steady state capital stock and output. The steady state capital stock and the capital accumulation equation determine steady state investment and consumption. The utility function's coefficient on leisure can then be chosen to satisfy the steady state labor supply/demand restriction.

Unlike the standard one sector growth model, the equations defining this model's nonstochastic steady state have no analytic solution, however a numerical solution procedure exists which parallels that of the simpler model. First, the investment project asset pricing equations, the plant asset pricing equations, and the optimal exit condition are solved to determine the marginal product of a unit of effective capital and the steady state exit threshold. The marginal product of effective capital and the predetermined steady state labor supply determine the steady state effective capital stock and output. The effective capital stock, the exit threshold, and the steady state plant and construction project accumulation equations then determine the stock of construction projects and the steady state productivity distribution across plants.

3.1 Quadrature Approximation

The steady state computation method begins with a standard integral approximation. Consider computing $\int_a^b g(x)\, dx$ for a continuous integrable function $g(x)$. If no analytic expression for this integral exists, quadrature methods, which approximate the integral with a weighted sum, provide a computational approximation. Fix a set of grid points, $x_1 < x_2 < \ldots, < x_{N-1} < x_N$. The quadrature approxi-

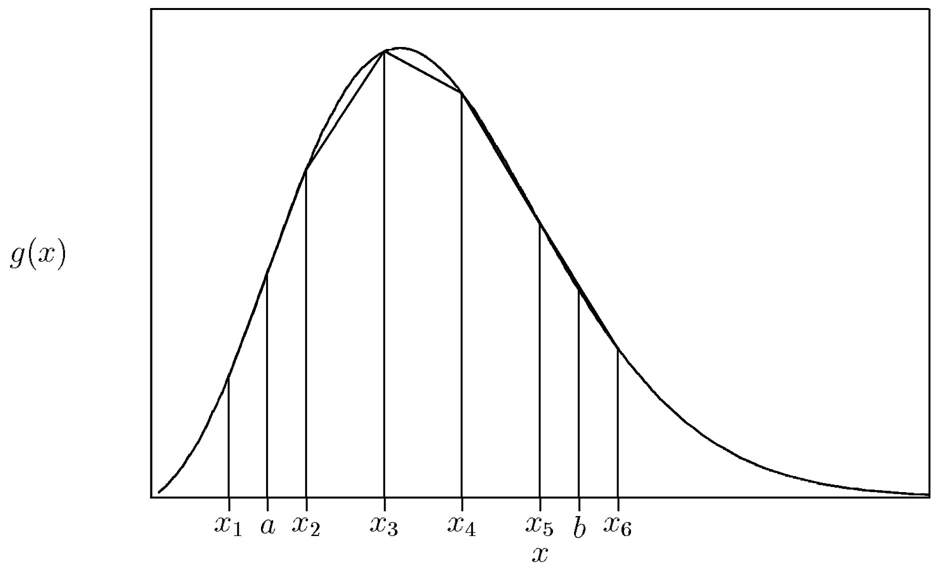

Figure 1: Quadrature Approximation

mation is a weighted sum of the function values at the grid points.

$$\int_a^b g(x)\, dx \approx \sum_{i=1}^{N} w_i(a,b) g(x_i)$$

The weights, $w_i(a,b)$ are chosen so that the sum equals the area under the adjoining trapezoids as in figure 1. The Gauss procedure `quadwts` computes these weights for any given grid and limits of integration. For a more detailed description of quadrature approximation of integrals, see Press, Teukolsky, Vetterling, and Flannery (1992)

Although more accurate integration procedures are available, this simple approach has an important advantage: The weights do not depend on the function being integrated. This allows its application in the solution of functional fixed point equations. For example, suppose for known functions $b(y)$ and $A(y,x)$, we would like to know the function $g(y)$ which satisfies

$$g(y) = b(y) + \int_a^b A\left(y, x\right) g\left(x\right)\, dx.$$

An exact solution to this functional equation is generally not available. However, consider solving the approximate functional equation which replaces integration with the above simple quadrature scheme.

$$g(y) = b(y) + \sum_{i=1}^{N} w_i(a, b) A(y, x_i) g(x_i)$$

Evaluating the left hand side of this equation at each of the grid points yields a system of N linear equations and N unknowns. Let $\vec{g} = (g(x_1), g(x_2), \ldots, g(x_N))'$ denote the vector of unknowns, the function evaluated at each of the grid points; $\vec{b} = (b(x_1), b(x_2), \ldots, b(x_N))'$ denote the constant vector; and A denote a matrix with ij'th element equal to $w_j(a, b)A(x_i, x_j)$. Then this system of equations can be written as

$$\vec{g} = \vec{b} + A\vec{g}$$

If the matrix $I_N - A$ is of full rank, then this has a simple solution.

$$\vec{g} = (I_N - A)^{-1}\vec{b}$$

To solve for $g(y)$ at any other value of y, plug the solution for $\vec{g}$ back into the quadrature approximation of the original functional equation.

3.2 Steady State Solution Procedure

To solve for the marginal product of effective capital, $\mathcal{D}_0$, first suppose that it and the steady state exit threshold, $\underline{u}_0$ are known. Given these two scalars, the plant asset pricing condition defines a functional fixed point problem like that studied above. The steady state plant asset prices, $\mathcal{Q}_0(u_t)$ satisfy this fixed point problem. Although the limits of integration in the original equations are infinite, the probability of a plant acquiring extremely high or low productivity is small. Therefore, approximating positive and negative infinity with suitably large and small numbers should not be problematic. The implementation of the quadrature solution method for this problem uses a grid $\vec{u} = (u_1, u_2, \ldots, u_N)$. The grid points are evenly spaced and centered around zero. The limits of integration are chosen to be the lowest and highest grid points, u_1 and u_N. The density and number of grid points, $u_j - u_{j-1}$ and N, are both increased to ensure the solution's robustness.

Let $\vec{\mathcal{Q}}_0 = (\mathcal{Q}_0(u_1), \ldots, \mathcal{Q}_0(u_N))$ denote the approximate plant asset prices, conditional on $\mathcal{D}_0$ and $\underline{u}_0$. This solution will have the form

$$\vec{\mathcal{Q}}_0 = C\mathcal{D}_0 + D.$$

The $N \times 1$ matrices, C and D only depend on $\underline{u}_0$ and the model's parameters, so they can be calculated independently of $\mathcal{D}_0$.

Now, suppose that only $\underline{u}_0$ is known. Substituting the T^i construction project asset pricing conditions into one another yields the following restriction on plant asset prices.

$$1 = (\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z})^{T^i} \int_{-\infty}^{\infty} \frac{1}{\sigma_e} \phi \left(\frac{u_{t+1} + (T^i - 1)\, \mu_z}{\sigma_e} \right) \mathcal{Q}_0\left(u_{t+1}\right) du_{t+1}$$

This is analogous to the steady state intertemporal Euler equation of the standard one sector model. To use this to solve for $\mathcal{D}_0$, replace the integral on its right hand side with a quadrature approximation using the grid points $\vec{u}$.

$$1 = (\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z})^{T^i} \sum_{i=1}^{N} w_i(u_1, u_N) \phi \left(\frac{u_i + (T^i - 1)\, \mu_z}{\sigma_e} \right) \mathcal{Q}_0\left(u_i\right)$$

or

$$1 = F' \vec{\mathcal{Q}}_0$$

where the ith element of the $N \times 1$ vector F equals

$$(\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z})^{T^i} w_i(u_1, u_n) \phi \left(\frac{u_i + (T^i - 1)\, \mu_z}{\sigma_e} \right).$$

Combining this with the approximate solution for $\vec{\mathcal{Q}}_0$ yields

$$1 = F'C\mathcal{D}_0 + F'D,$$

from which the solution for $\mathcal{D}_0$ is immediate.

Finally, consider the steady state optimal exit condition. If the chosen value of $\underline{u}_0$ is the steady state exit threshold, then it and the asset prices calculated above must satisfy the optimal exit condition. To evaluate the optimal exit condition, replace its integral with a quadrature approximation using the grid points $\vec{u}$. Fixing an exit threshold, $\underline{u}$, calculating the asset prices with the above procedure, and then evaluating the optimal exit condition defines a nonlinear function.

$$g(\underline{u}) = \eta - \beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z} \sum_{i=1}^{N} w_i(u_1, u_N) \frac{1}{\sigma} \phi\left(\frac{u_i - \underline{u} + \mu_z}{\sigma}\right) Q_0\left(u_i\right)$$

The root of $g(\underline{u})$ is the approximate steady state exit threshold, $\underline{u}_0$. The Gauss procedure `ss_p` uses a bisection technique to find the root of $g(\underline{u})$. It then computes the associated plant asset prices, construction project asset prices, and the capital rental rate.

The second step in computing the economy's nonstochastic steady state is to use the steady state asset prices and exit threshold to compute the steady state quantities. Given the steady state labor supply, N_0 and the marginal product of effective capital, $\mathcal{D}_0$, the capital services pricing equation yields the steady state effective capital stock, $\overline{K}_0^\star$. Substituting $\overline{K}_0^\star$ and N_0 into the aggregate production function determines the steady state level of output.

To solve for the steady state measure of plants, $K_0^\star(u_t)$ and the steady state quantity of construction projects, $I_0^{\star j}$, $j = 1, \ldots, T^i$, consider the steady state plant accumulation constraint. Given $\underline{u}_0$ and $I_0^{\star T^i}$, the plant accumulation constraint defines a functional fixed point problem, the solution to which is $K_0^\star(u_t)$.

Applying the quadrature solution method using the grid $\vec{u}$ to this problem yields a solution of the form

$$\vec{K}_0^\star = G I_0^{\star T^i},$$

where $\vec{K}_0^\star = (K_0^\star(u_1), K_0^\star(u_2), \ldots, K_0^\star(u_N))$ and G is an $N \times 1$ matrix which only depends on the model's parameters and $\underline{u}_0$.

With this solution for $K_0^\star(u_t)$, $\overline{K}_0^\star$ and the capital services supply equation determine $I_0^{\star T^i}$. Replacing the integral of the capital services supply equation with a quadrature approximation using the grid $\vec{u}$ yields

$$\begin{aligned} \overline{K}_0^\star &= \sum_{i=1}^{N} w_i(u_1, u_N) e^{u_i} K_0^\star(u_i) \\ &= H' \vec{K}_0^\star. \end{aligned}$$

Substituting the solution for $\vec{K}_0^\star$ into the approximate capital services supply equation produces

$$\overline{K}_0^\star = H' G I_0^{\star T^i},$$

which implies that $I_0^{\star T^i} = \overline{K}_0^\star/(H'G)$. With this in hand, the solution to the plant accumulation equation yields $\vec{K}_0^\star$ and the construction project accumulation equations yield $I_0^{\star j}$ for $j = 1, \ldots, T^i - 1$. The static resource constraint then implies the steady state level of consumption, $C_0^\star$. Finally, κ can be chosen to satisfy the steady state labor demand/supply restriction. The Gauss procedure `ss_q` implements these computations.

4. Near Steady-State Dynamics

With the nonstochastic steady state prices and quantities, log linearization of the star economy's Euler equations is tedious but straightforward. Let lower case letters denote percentage deviations of the star economy's endogenous variables from their steady state values. That is, $c_t = \ln(C_t^\star / C_0^\star)$, $n_t = \ln(N_t/N_0)$, $i_t^j = \ln\left(I_t^{\star j}/I_0^{\star j}\right)$, $k_t(u_t) = \ln(K_t^\star(u_t)/K_0^\star(u_t))$, $\overline{k}_t = \ln\left(\overline{K}_t^\star/\overline{K}_0^\star\right)$, $d_t = \ln(\mathcal{D}_t/\mathcal{D}_0)$, $q_t(u_t) = \ln(\mathcal{Q}_t(u_t)/\mathcal{Q}_0(u_t))$, $q_t(j) = \ln(\mathcal{Q}_t(j)/\mathcal{Q}_0(j))$ for $j = 2, \ldots, T^i$, and $m_t = \ln(\mathcal{M}_t/\mathcal{M}_0)$. Furthermore, define $\widetilde{u}_t = \underline{u}_t - \underline{u}_0$. With this notation, the log-linear approximations to the Euler equations are

Contingent Claims Pricing

$$m_t = -c_t$$

Static Budget Constraint

$$C_0^\star c_t + I_0^{\star 1} i_t^1 = Y_0^\star \left(\alpha n_t + (1-\alpha)\overline{k}_t\right) + \eta \int_{-\infty}^{\underline{u}_0} K_0^\star(u_t)\, k_t(u_t)\, du_t + \eta K_0^\star(\underline{u}_0)\,\widetilde{u}_t$$

Labor Supply / Labor Demand

$$m_t + (1-\alpha)\left(\overline{k}_t - n_t\right) = 0$$

Capital Services Production / Aggregation

$$\frac{1}{\overline{K}_0^\star} \int_{-\infty}^{\infty} e^{u_t} K_0^\star(u_t)\, k_t(u_t)\, du_t = \overline{k}_t$$

Optimal Exit

$$\begin{aligned}
0 &= m_{t+1} - m_t - \left(\frac{1-\alpha}{\alpha}\right)\varepsilon_t^z \\
&\quad + \frac{\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{\eta} \int_{-\infty}^{\infty} \frac{1}{\sigma}\phi'\left(\frac{u_{t+1} - \underline{u}_0 + \mu_z}{\sigma}\right) \mathcal{Q}_0(u_{t+1})\; du_{t+1}\left(\varepsilon_t^z - \widetilde{u}_t\right) \\
&\quad + \frac{\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{\eta} \int_{-\infty}^{\infty} \frac{1}{\sigma}\phi\left(\frac{u_{t+1} - \underline{u}_0 + \mu_z}{\sigma}\right) \mathcal{Q}_0(u_{t+1})\, q_{t+1}(u_{t+1})\, du_{t+1}
\end{aligned}$$

Capital Services Pricing

$$d_t = -\alpha \left(\overline{k}_t - n_t\right)$$

Plant Asset Pricing: $u_t \leq \underline{u}_0$

$$q_t\left(u_t\right) = \left(\mathcal{D}_0 e^{u_t} / \mathcal{Q}_0\left(u_t\right)\right) d_t$$

Plant Asset Pricing: $u_t > \underline{u}_0$

$$\begin{aligned}
q_t\left(u_t\right) &= \left(\mathcal{D}_0 e^{u_t} / \mathcal{Q}_0\left(u_t\right)\right) d_t \\
&\quad + \frac{\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{\mathcal{Q}_0\left(u_t\right)} \int_{-\infty}^{\infty} \frac{1}{\sigma} \phi\left(\frac{u_{t+1} - u_t + \mu_z}{\sigma}\right) \mathcal{Q}_0\left(u_{t+1}\right) du_{t+1} \\
&\quad \times \left(m_{t+1} - m_t - \left(\frac{1-\alpha}{\alpha}\right) \varepsilon_t^z\right) \\
&\quad + \frac{\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{\mathcal{Q}_0\left(u_t\right)} \int_{-\infty}^{\infty} \frac{1}{\sigma^2} \phi'\left(\frac{u_{t+1} - u_t + \mu_z}{\sigma}\right) \mathcal{Q}_0\left(u_{t+1}\right) du_{t+1} \varepsilon_t^z \\
&\quad + \frac{\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{\mathcal{Q}_0\left(u_t\right)} \int_{-\infty}^{\infty} \frac{1}{\sigma} \phi\left(\frac{u_{t+1} - u_t + \mu_z}{\sigma}\right) \mathcal{Q}_0\left(u_{t+1}\right) q_{t+1}\left(u_{t+1}\right) du_{t+1}
\end{aligned}$$

Investment Project Asset Pricing: $j = 2, \ldots, T^i - 1$

$$q_t(j) = m_{t+1} - m_t - \left(\frac{1-\alpha}{\alpha}\right) \varepsilon_t^z + \frac{\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z} \mathcal{Q}_0(j+1)}{\mathcal{Q}_0(j)} q_{t+1}(j+1)$$

Investment Project Asset Pricing: $j = T^i$

$$q_t\left(T^i\right) = m_{t+1} - m_t - \left(\frac{1-\alpha}{\alpha}\right) \varepsilon_t^z$$

$$+\frac{\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{\mathcal{Q}_{\mathbf{0}}\left(T^i\right)}\int_{-\infty}^{\infty}\frac{1}{\sigma_e}\phi\left(\frac{u_{t+1}+\left(T^i-1\right)\mu_z}{\sigma_e}\right)\mathcal{Q}_{\mathbf{0}}\left(u_{t+1}\right)q_{t+1}\left(u_{t+1}\right)du_{t+1}$$
$$+\frac{\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{\mathcal{Q}_{\mathbf{0}}\left(T^i\right)}\int_{-\infty}^{\infty}\frac{1}{\sigma_e^2}\phi'\left(\frac{u_{t+1}+\left(T^i-1\right)\mu_z}{\sigma_e}\right)\mathcal{Q}_{\mathbf{0}}\left(u_{t+1}\right)du_{t+1}\sum_{i=\mathbf{0}}^{T^i-2}\varepsilon_{t-i}^z$$

Optimal Investment / Entry

$$0=m_{t+1}-m_t-\left(\frac{1-\alpha}{\alpha}\right)\varepsilon_t^z+\beta e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}\mathcal{Q}_{\mathbf{0}}\left(2\right)q_{t+1}\left(2\right)$$

Plant Accumulation

$$\begin{aligned}
k_{t+1}\left(u_{t+1}\right) &= \left(\frac{1-\alpha}{\alpha}\right)\varepsilon_t^z \\
&+\frac{e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{K_{\mathbf{0}}^{\star}\left(u_{t+1}\right)}\int_{\underline{u}_0}^{\infty}\frac{1}{\sigma}\phi\left(\frac{u_{t+1}-u_t+\mu_z}{\sigma}\right)K_{\mathbf{0}}^{\star}\left(u_t\right)k_t\left(u_t\right)\,du_t \\
&+\frac{e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{K_{\mathbf{0}}^{\star}\left(u_{t+1}\right)}\int_{\underline{u}_0}^{\infty}\frac{1}{\sigma^2}\phi'\left(\frac{u_{t+1}-u_t+\mu_z}{\sigma}\right)K_{\mathbf{0}}^{\star}\left(u_t\right)\,du_t\times\varepsilon_t^z \\
&+\frac{e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{K_{\mathbf{0}}^{\star}\left(u_{t+1}\right)}\frac{1}{\sigma}\phi\left(\frac{u_{t+1}-\underline{u}_{\mathbf{0}}+\mu_z}{\sigma}\right)K_{\mathbf{0}}^{\star}\left(\underline{u}_{\mathbf{0}}\right)\widetilde{u}_t \\
&+\frac{e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{K_{\mathbf{0}}^{\star}\left(u_{t+1}\right)}\frac{1}{\sigma_e}\phi\left(\frac{u_{t+1}+\left(T^i-1\right)\mu_z}{\sigma}\right)I_{\mathbf{0}}^{\star T^i}i_t^{T^i} \\
&+\frac{e^{-\left(\frac{1-\alpha}{\alpha}\right)\mu_z}}{K_{\mathbf{0}}^{\star}\left(u_{t+1}\right)}\frac{1}{\sigma_e^2}\phi'\left(\frac{u_{t+1}+\left(T^i-1\right)\mu_z}{\sigma}\right)I_{\mathbf{0}}^{\star T^i}\times\sum_{i=\mathbf{0}}^{T^i-2}\varepsilon_{t-i}^z
\end{aligned}$$

Investment Project Accumulation

$$i_{t+1}^j=i_t^j-\left(\frac{1-\alpha}{\alpha}\right)\varepsilon_t^z\text{ for }j=2,\ldots,T^i$$

4.1 Dynamic Solution Procedure

To compute an approximate solution to this system of difference equations, fix a grid, $\vec{u} = (u_1, u_2, \ldots, u_N)$ for the quadrature approximation. This need not be the same grid used to compute the steady state. Replacing the system's integrals with quadrature approximations and evaluating the plant asset pricing equation and the plant accumulation equation at each of the grid points produces a finite dimensional linear difference equation. To conveniently analyze this system, define four vectors: a vector of state variables, $\vec{k}_t$, a vector of costate variables, $\vec{q}_t$, a vector of control variables, $\vec{c}_t$, and a vector of exogenous forcing variables $\vec{y}_t$.

$$\begin{aligned}
\vec{k}_t &= (k_t(u_1), k_t(u_2), \ldots, k_t(u_N), i_t^2, \ldots, i_t^{T^i})' \\
\vec{q}_t &= (q_t(u_1), q_t(u_2), \ldots, q_t(u_N), q_t(2), \ldots, q_t(T^i), m_t)' \\
\vec{c}_t &= (c_t, i_t^1, n_t, \overline{k}_t, \widetilde{u}_t, d_t)' \\
\vec{y}_t &= (\varepsilon_t^z, \varepsilon_{t-1}^z, \ldots, \varepsilon_{t-T^i+1}^z)'
\end{aligned}$$

Group the approximate Euler equations into four groups. The first is the control equations: the contingent claims pricing equation, the static budget constraint, the labor supply/demand condition, the capital services production constraint, the optimal exit condition, and the capital services pricing condition. Using the vectors defined above, they can be written as

$$M_{cc}\vec{c}_t + M_{ck}\vec{k}_t + M_{ck}^1\vec{k}_{t+1} + M_{cq}\vec{q}_t + M_{cq}^1\vec{q}_{t+1} + M_{cy}\vec{y}_t + M_{cy}^1\vec{y}_{t+1} = 0.$$

The second group is the asset pricing equations. It includes the plant and investment project asset pricing equations.

$$M_{qq}\vec{q}_t + M_{qq}^1\vec{q}_{t+1} + M_{qk}\vec{k}_t + M_{qc}\vec{c}_t + M_{qy}\vec{y}_t = 0$$

The plant accumulation equation and the investment project accumulation equations are the capital accumulation equations.

$$M^1_{kk}\vec{k}_{t+1} + M_{kk}\vec{k}_t + M_{kc}\vec{c}_t + M_{ky}\vec{y}_t = 0$$

Finally, the law of motion for the exogenous forcing variables can be written as

$$M^1_{yy}\vec{y}_{t+1} + M_{yy}\vec{y}_t = 0,$$

where

$$\begin{aligned} M^1_{yy} &= I_{T^i} \\ M_{yy} &= \begin{bmatrix} 0 & \cdots & 0 \\ & & 0 \\ & I_{T^i-1} & \vdots \\ & & 0 \end{bmatrix} \end{aligned}$$

To further simplify the system, solve the control equations for $\vec{c}_t$ and eliminate the control variables from the remaining equations. Define the vector $\vec{s}_t = (\vec{q}_t', \vec{k}_t', \vec{y}_t')'$. The asset pricing equations, capital accumulation equations, and the laws of motion for the exogenous driving variables may then be combined to produce a first order difference equation in $\vec{s}_t$.

$$A\vec{s}_{t+1} + B\vec{s}_t = 0$$

The Gauss procedure `loglin` implements this log-linearization procedure, computing the matrices A and B as well as the rule determining $\vec{c}_t$ as a function of $\vec{s}_t$ and $\vec{s}_{t+1}$.

Solving for a sequence $\{\vec{s}_t\}_{t=0}^{\infty}$ which satisfies the above linear difference equation is complicated by the singularity of A. The source of this singularity is the asset pricing relationship for plants with productivity below the steady state exit threshold. Because a plant with such low productivity exits before the next period, its asset price depends only on its current dividends and its scrap value. The quadrature approximation of the asset pricing equations will involve one such equation for each grid point below $\underline{u}_0$. Let N_s be the number of grid points at or below $\underline{u}_0$, so that $u_{N_s} \leq \underline{u}_0 < u_{N_s+1}$. The first N_s rows of A, those corresponding to these "static asset pricing equations", will equal zero. This implies that each of the first N_s rows of B multiplied by $\vec{s}_t$ must equal zero. These conditions can be used to solve for the first N_s elements of $\vec{q}_t$ as functions of the other elements of $\vec{s}_t$. Let $\tilde{s}_t = (q_t(u_{N_s+1}), \ldots, q_t(u_N), m_t, \vec{k}_t', \vec{y}_t')'$ denote the vector of these remaining elements. Using the solution for $(q_t(u_1), q_t(u_2), \ldots, q_t(u_{N_s}))$ to eliminate it from the difference equation yields

$$\tilde{A}\tilde{s}_{t+1} + \tilde{B}\tilde{s}_t = 0.$$

The matrix $\tilde{A}$ is of full rank, so this has a simple recursive solution for any initial condition $\tilde{s}_0$.

$$\tilde{s}_{t+1} = -\tilde{A}^{-1}\tilde{B}\tilde{s}_t = \Pi\tilde{s}_t$$

At the beginning of a period, the initial capital stock, $\vec{k}_t$, and the exogenous driving variables, $\vec{y}_t$, are predetermined; but the asset prices, $(q_t(u_{N_s+1}), \ldots, q_t(u_N))$ are not. Therefore, for any initial capital stock and driving variables, there exist an infinite number of sequences which satisfy the above linear difference equation. Each sequence corresponds to a choice of the initial asset prices. Only one of these sequences also satisfies the transversality condition. The now familiar methods of King, Plosser, and Rebelo (1987) select this sequence.

The Gauss procedure `rules` implements this computation, producing decision rules for the state and costate variables.

$$\begin{aligned} \vec{k}_{t+1} &= R_{kk}\vec{k}_t + R_{ky}\vec{y}_t \\ \vec{q}_t &= R_{qk}\vec{k}_t + R_{qy}\vec{y}_t \end{aligned}$$

4.2 Impulse Response Functions and Second Moments

To compute the moving average representations for the star economy's state and costate, the decision rules are iterated forwards. The Gauss procedure `ar2ma` perfoms this recursion. Applying the linear solution for the control variables in terms of the state and costate then yields the moving average representations for the star economy's other endogenous variables. The definitions of the star economy's variables can then be used to find the impulse response functions of the star economy's endogenous variables. After performing a suitable stationarity inducing transformation, computing the first and second moments for any variable of interest is straightforward. The Gauss procedures `imps` and `moments` implement these computations.

5. Data Sources

The empirical work used eight quarterly time series: The United States' population, real GDP, The NIPA measure of the capital stock, hours worked from the establishment survey, hours worked from the household survey, labor's share of output, job creation at new establishments in the manufacturing sector, and job destruction at dying establishments in the manufacturing sector.

The population measure is the civilian noninstitutional population over age 16. It is produced by the Bureau of Labor Statistics. Real GDP is measured as seasonally adjusted total GDP in 1987 dollars less the output of farms and government. The capital stock, as reported in the NIPA, is the fixed, nonresidential private capital for all industries measured at constant cost less the analogous measure for farms. Establishment hours were constructed by multiplying the number of employees of private, nonfarm establishments by the average weekly hours worked production or nonsupervisory employees at these same establishments. This data is compiled from a survey of employer payrolls by the Bureau of Labor Statistics. The resulting time series was seasonally adjusted using the EZ-X11 computer program. Household hours were constructed in the same manner as establishment hours. The data were compiled from the current population survey by the Bureau of Labor Statistics. Steve Davis, John Haltiwanger, and Scott Schuh provided the job creation and destruction data. In the notation of Davis, Haltiwanger, and Schuh (1996), the series are $POSB_t$, and $NEGD_t$. Further details concerning their construction can be found in Davis, Haltiwanger, and Schuh (1996)

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