Online Appendix to "Household Income Inequality and Optimal Trend Inflation"

1 Tractable HANK (THANK) model

In this section, we present the full set of equilibrium conditions of the model.

1.1 Households

The household joint welfare maximization problem reads

$$U(B_t^S, B_t^H) = \max_{C_t^S, C_t^H, Z_{t+1}^S, Z_{t+1}^H} (1-\lambda)\frac{(C_t^S)^{1-\sigma}-1}{1-\sigma} + \lambda\frac{(C_t^H)^{1-\sigma}-1}{1-\sigma} - \chi\frac{N_t^{1+\phi}}{1+\phi} + \beta\mathbf{E}_t\left[U(B_{t+1}^S, B_{t+1}^H)\right],$$

subject to the flows of bonds and budget constraints:

$$\begin{aligned}
C_t^S + Z_{t+1}^S + TS_t + TG_t^S &= W_t N_t + \frac{1-\tau_d\lambda}{1-\lambda}D_t + \frac{R_{t-1}q_{t-1}}{\Pi_t}B_t^S + \mathcal{T}^S \\
C_t^H + Z_{t+1}^H + TS_t + TG_t^H &= W_t N_t + \tau_d D_t + \frac{R_{t-1}q_{t-1}}{\Pi_t}B_t^H + \mathcal{T}^H \\
B_{t+1}^S &= sZ_{t+1}^S + (1-s)Z_{t+1}^H \\
B_{t+1}^H &= (1-h)Z_{t+1}^S + hZ_{t+1}^H.
\end{aligned}$$

The first-order conditions of the above dynamic optimization problem are:

$$\begin{aligned}
\Lambda_t^S &= (1-\lambda)(C_t^S)^{-\sigma} \\
\Lambda_t^H &= \lambda(C_t^H)^{-\sigma} \\
\Lambda_t^S &= \beta s\mathbf{E}_t\left[\frac{\partial U_{t+1}}{\partial B_{t+1}^S}\right] + \beta(1-h)\mathbf{E}_t\left[\frac{\partial U_{t+1}}{\partial B_{t+1}^H}\right] + \Theta_t^S \\
\Lambda_t^H &= \beta(1-s)\mathbf{E}_t\left[\frac{\partial U_{t+1}}{\partial B_{t+1}^S}\right] + \beta h\mathbf{E}_t\left[\frac{\partial U_{t+1}}{\partial B_{t+1}^H}\right] + \Theta_t^H \\
\Theta_t^S Z_{t+1}^S &= 0 \\
\Theta_t^H Z_{t+1}^H &= 0,
\end{aligned}$$

where Λ_t^S, Λ_t^H, Θ_t^S, and Θ_t^H are Lagrangian multipliers associated with budget constraints and borrowing constraints for each type. Additionally, the envelope conditions yields

$\frac{\partial U_t}{\partial B_t^S} = \Lambda_t^S \frac{R_{t-1}q_{t-1}}{\Pi_t}$, $\frac{\partial U_t}{\partial B_t^H} = \Lambda_t^H \frac{R_{t-1}q_{t-1}}{\Pi_t}$. Combining both first-order optimality and envelope conditions with stationary type distribution assumption, we obtain:

$$(C_t^S)^{-\sigma} = \beta \mathbf{E}_t \left[\frac{R_t q_t}{\Pi_{t+1}} \left(s(C_{t+1}^S)^{-\sigma} + (1-s)(C_{t+1}^H)^{-\sigma} \right) \right] + \frac{\Theta_t^S}{1-\lambda}$$
$$(C_t^H)^{-\sigma} = \beta \mathbf{E}_t \left[\frac{R_t q_t}{\Pi_{t+1}} \left((1-h)(C_{t+1}^S)^{-\sigma} + h(C_{t+1}^H)^{-\sigma} \right) \right] + \frac{\Theta_t^H}{\lambda}$$
$$\Theta_t^S Z_{t+1}^S = 0$$
$$\Theta_t^H Z_{t+1}^H = 0.$$

Since we focus on the zero liquidity equilibrium where $\Theta_t^S = 0, \Theta_t^H > 0$, household consumption saving behavior is characterized by the savers' consumption Euler equation and hand-to-mouth spenders' budget constraint:

$$(C_t^S)^{-\sigma} = \beta \mathbf{E}_t \left[\frac{R_t q_t}{\Pi_{t+1}} \left(s(C_{t+1}^S)^{-\sigma} + (1-s)(C_{t+1}^H)^{-\sigma} \right) \right]$$
$$C_t^H + TS_t + TG_t^H = W_t N_t + \tau_d D_t + \mathcal{T}^H.$$

Finally, a labor union sets wages on behalf of both households such that the labor-supply-like wage schedule is:

$$W_t \left((1-\lambda)(C_t^S)^{-\sigma} + \lambda (C_t^H)^{-\sigma} \right) = \chi N_t^{\phi}.$$

1.2 Firms

1.2.1 Final Good Producers

The final good producers use intermediate goods as inputs, and produce final goods according to CES technology:

$$Y_t = \left(\int Y_t(i)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} di \right)^{\frac{\varepsilon_p}{\varepsilon_p - 1}}.$$

They sell the final products to households in a competitive market. The profit maximization of final good producers yields a general relative demand function of intermediate good i and aggregate price level.

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} Y_t$$
$$P_t = \left(\int P_t(i)^{1-\varepsilon_p} di\right)^{\frac{1}{1-\varepsilon_p}}.$$

1.2.2 Intermediate Goods Producers

Firstly, the cost-minimization problem of intermediate goods firms reads:

$$\min_{N_t(i)} \ (1-\nu)W_t N_t(i)$$
$$s.t$$
$$Y_t(i) \leq A_t N_t(i),$$

where ν is a labor cost subsidy that eliminates steady state distortion arising from monopolistic competition. The first-order condition yields an expression for the real marginal cost:

$$MC_t = \frac{(1-\nu)W_t}{A_t}.$$

Secondly, the profit maximization problem of the intermediate goods firms is:

$$\max_{P_t(i)} E_t \left[\sum_{k=0}^{\infty} (\theta_p \beta)^k \left(\frac{C^S_{t+k}}{C^S_t}\right)^{-\sigma} \left(\frac{P_t(i)}{P_{t+k}} Y_{t+k}(i) - MC_{t+k} Y_{t+k}(i)\right)\right].$$

The optimality condition for reset price yields:

$$X_{1,t} = (C^S_t)^{-\sigma} MC_t Y_t + \theta_p \beta \mathrm{E}_t \left[\Pi^{\varepsilon_p}_{t+1} X_{1,t+1}\right]$$
$$X_{2,t} = (C^S_t)^{-\sigma} Y_t + \theta_p \beta \mathrm{E}_t \left[\Pi^{\varepsilon_p - 1}_{t+1} X_{2,t+1}\right]$$
$$\Pi^*_t = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{X_{1,t}}{X_{2,t}} u_t.$$

where u_t is the price markup shock.

1.3 Government

The central bank sets the gross policy rate R_t following the Taylor rule subject to a ZLB constraint:

$$R_t^* = (R_{t-1}^*)^{\phi_r} \left(\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{Y_t}{Y} \right)^{\phi_y} \right)^{1-\phi_r}$$

$$R_t = \begin{cases} R_t^* & \text{if } R_t^* \geq 1 \\ 1 & \text{if } R_t^* < 1, \end{cases}$$

where R_t^* is the shadow rate. ϕ_r governs the interest rate smoothing, while ϕ_π and ϕ_y measure the sensitivity of the policy rates with respect to the deviations of inflation and output from their respective steady state, respectively. The fiscal authority undertakes government spending following a feedback rule that responds to the deviation of output from its steady state:

$$G_t = G \left(\frac{Y_t}{Y} \right)^{\gamma_y^G},$$

where γ_y^G governs the cyclicality of government spending. We assume that government spending is funded according to:

$$\begin{aligned} G &= (1-\lambda) TG^S \\ (1-\alpha)(G_t - G) &= (1-\lambda)(TG_t^S - TG^S). \end{aligned}$$

The first equation indicates that, at steady state, government spending is funded fully by savers' taxes. The second equation shows that, away from the steady state, fraction $1-\alpha$ of government spending is financed by savers, while the remaining fraction is from taxes levied on spenders. If $\alpha = \lambda$, savers and borrowers are burdened equally, paying the same amount of taxes per capita. For the baseline, we assume $\alpha = 0$, meaning that all government spending is entirely from savers' taxes both at and away from the steady state.

1.4 Full Set of Equilibrium Conditions

$$W_t\left(\lambda(C_t^H)^{-\sigma}+(1-\lambda)(C_t^S)^{-\sigma}\right)=\chi N_t^{\phi} \tag{1}$$

$$(C_t^S)^{-\sigma}=\beta \mathrm{E}_t\left[\frac{R_t}{\Pi_{t+1}}q_t\left(s(C_{t+1}^S)^{-\sigma}+(1-s)(C_{t+1}^H)^{-\sigma}\right)\right] \tag{2}$$

$$C_t^H+TG_t^H=(1-\nu)W_tN_t+\tau_d D_t+\mathcal{T}^H \tag{3}$$

$$(1-\nu)W_t=A_tMC_t \tag{4}$$

$$X_{1,t}=(C_t^S)^{-\sigma}MC_tY_t+\theta_p\beta \mathrm{E}_t\left[\Pi_{t+1}^{\varepsilon_p}X_{1,t+1}\right] \tag{5}$$

$$X_{2,t}=(C_t^S)^{-\sigma}Y_t+\theta_p\beta \mathrm{E}_t\left[\Pi_{t+1}^{\varepsilon_p-1}X_{2,t+1}\right] \tag{6}$$

$$\Pi_t^*=\frac{\varepsilon_p}{\varepsilon_p-1}\frac{X_{1,t}}{X_{2,t}}u_t \tag{7}$$

$$1=(1-\theta_p)(\Pi_t^*)^{1-\varepsilon_p}+\theta_p\Pi_t^{\varepsilon_p-1} \tag{8}$$

$$Y_t=\frac{A_tN_t}{v_t^p} \tag{9}$$

$$(1-\lambda)C_t^S+\lambda C_t^H+G_t=Y_t \tag{10}$$

$$v_t^p=(1-\theta_p)(\Pi_t^*)^{-\varepsilon_p}+\theta_p\Pi_t^{\varepsilon_p}v_{t-1}^p \tag{11}$$

$$R_t^*=(R_{t-1}^*)^{\phi_r^1}\left(R_{ss}^*\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi}\left(\frac{Y_t}{Y}\right)^{\phi_y}\right)^{1-\phi_r^1} \tag{12}$$

$$R_t=max\left\{1,R_t^*\right\} \tag{13}$$

$$G=(1-\lambda)TG^S \tag{14}$$

$$(1-\alpha)(G_t-G)=(1-\lambda)(TG_t^S-TG^S) \tag{15}$$

$$G_t=G\left(\frac{Y_t}{Y}\right)^{\gamma_y^G} \tag{16}$$

$$\log(A_t)=\rho_a\log(A_{t-1})+\sigma_a\varepsilon_{a,t} \tag{17}$$

$$\log(u_t)=\rho_u\log(u_{t-1})+\sigma_u\varepsilon_{u,t} \tag{18}$$

$$\log(q_t)=\rho_q\log(q_{t-1})+\sigma_q\varepsilon_{q,t}, \tag{19}$$

where $v_t^p = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} di$. Equations (1), (2), and (3) characterize household consumption-saving and labor-supply behavior. Equations (4) $\sim$ (7) constitute the optimality conditions of intermediate goods firms. Equations (8) $\sim$ (10) come from the aggregation of intermediate goods and goods market clearing. Equation (11) describes the dynamics of price dispersion. Equations (12) $\sim$ (16) define the monetary and fiscal policy rules. Finally, equations (17) $\sim$ (19) specify exogenous TFP, price markup, and risk premium shock processes.

2 Second-Order Approximation of Household Welfare

In this section, we describe how to approximate the per-period welfare of households, assuming the log-utility function for consumption. The hatted variables in this appendix denote the log deviation from steady state (i.e. $\hat{x}_t = \log(X_t) - \log(X)$). Instantaneous household welfare for type $j \in \{S, H\}$ is

$$U_t^j \equiv \log(C_t^j) - \chi \frac{N_t^{1+\phi}}{1+\phi}.$$

Firstly, we will explain the method of approximating the price dispersion. Starting from the relationship between reset price inflation and inflation, we get the second-order approximation of inflation dynamics.

$$\begin{aligned} 1 &= (1-\theta_p)(\Pi_t^*)^{1-\varepsilon_p} + \theta_p(\Pi_t)^{\varepsilon_p - 1} \\ \rightarrow \hat{\pi}_t^* &= \frac{\overline{\theta}_p}{1-\overline{\theta}_p}\left(\hat{\pi}_t + \frac{1}{2}(\varepsilon_p - 1)\hat{\pi}_t^2\right) + \frac{1}{2}(\varepsilon_p - 1)\hat{\pi}_t^{*2}, \end{aligned} \tag{20}$$

where $\overline{\theta}_p = \theta_p \Pi^{\varepsilon_p - 1}$. Up to first order, (20) becomes $\hat{\pi}_t^* = \frac{\overline{\theta}_p}{1-\overline{\theta}_p}\hat{\pi}_t$. Substituting out $\hat{\pi}_t^{*2}$ in (20), we obtain the second order expression for reset price inflation:

$$\hat{\pi}_t^* = \frac{\overline{\theta}_p}{1-\overline{\theta}_p}\hat{\pi}_t + \frac{1}{2}(\varepsilon_p - 1)\frac{\overline{\theta}_p}{(1-\overline{\theta}_p)^2}\hat{\pi}_t^2.$$

Next, we use the price dispersion dynamics:

$$v_t^p = (1-\theta_p)(\Pi_t^*)^{-\varepsilon_p} + \theta_p \Pi_t^{\varepsilon_p} v_{t-1}^p.$$

The second-order approximation of the price dispersion dynamics is:

$$\hat{v}_t^p + \frac{1}{2}(\hat{v}_t^p)^2 = (1-\overline{\theta}_p\Pi)\left(-\varepsilon_p\hat{\pi}_t^* + \frac{1}{2}\varepsilon_p^2(\hat{\pi}_t^*)^2\right) + \overline{\theta}_p\Pi\left(\varepsilon_p\hat{\pi}_t + \hat{v}_{t-1}^p + \frac{1}{2}\varepsilon_p^2\hat{\pi}_t^2 + \frac{1}{2}(\hat{v}_{t-1}^p)^2 + \varepsilon_p\hat{\pi}_t\hat{v}_{t-1}^p\right) \tag{21}$$

Up to first order, (21) is:

$$\hat{v}_t^p = \frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{1-\overline{\theta}_p}\hat{\pi}_t + \overline{\theta}_p\Pi\hat{v}_{t-1}^p.$$

By replacing the reset price inflation and squared price dispersion term with their first-order expression, (21) can be rewritten as:

$$\begin{aligned}\hat{v}_t^p = &-\frac{1}{2}\left(\frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{1-\overline{\theta}_p}\hat{\pi}_t + \overline{\theta}_p\Pi\hat{v}_{t-1}^p\right)^2 \\ &+ (1-\overline{\theta}_p\Pi)\left(-\varepsilon_p\left(\frac{\overline{\theta}_p}{1-\overline{\theta}_p}\hat{\pi}_t + \frac{1}{2}(\varepsilon_p-1)\frac{\overline{\theta}_p}{(1-\overline{\theta}_p)^2}\hat{\pi}_t{}^2\right) + \frac{1}{2}\varepsilon_p^2\left(\frac{\overline{\theta}_p}{1-\overline{\theta}_p}\right)^2\hat{\pi}_t^2\right) \\ &+ \overline{\theta}_p\Pi\left(\varepsilon_p\hat{\pi}_t + \hat{v}_{t-1}^p + \frac{1}{2}\varepsilon_p^2\hat{\pi}_t^2 + \frac{1}{2}(\hat{v}_{t-1}^p)^2 + \varepsilon_p\hat{\pi}_t\hat{v}_{t-1}^p\right).\end{aligned}$$

With some mechanical algebra and collecting like terms, we obtain:

$$\begin{aligned}\hat{v}_t^p = &\frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{1-\overline{\theta}_p}\hat{\pi}_t + \overline{\theta}_p\Pi\hat{v}_{t-1}^p + \frac{1}{2}\frac{\varepsilon_p\overline{\theta}_p(\varepsilon_p\Pi-\varepsilon_p+1)(1-\overline{\theta}_p\Pi)}{(1-\overline{\theta}_p)^2}\hat{\pi}_t^2 \\ &+ \frac{\varepsilon_p(1-\overline{\theta}_p\Pi)}{1-\overline{\theta}_p}\hat{\pi}_t\hat{v}_{t-1}^p + \frac{1}{2}\overline{\theta}_p\Pi(1-\overline{\theta}_p\Pi)(\hat{v}_{t-1}^p)^2.\end{aligned}$$

As in *Alves* (2014), we assume that the price dispersion $\hat{v}_t^p$ is a second-order term, and, hence, will regard any terms taking the form of $\hat{v}_t^p\hat{x}_t$ as higher than the second order. Under this assumption, the second-order expression of price dispersion becomes:

$$\hat{v}_t^p = \overline{\theta}_p\Pi\hat{v}_{t-1}^p + \frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{1-\overline{\theta}_p}\hat{\pi}_t + \frac{1}{2}\frac{\varepsilon_p\overline{\theta}_p(\varepsilon_p\Pi-\varepsilon_p+1)(1-\overline{\theta}_p\Pi)}{(1-\overline{\theta}_p)^2}\hat{\pi}_t^2.$$

Taking the unconditional expectation, we have:

$$\mathrm{E}(\hat{v}_t^p) = \frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{(1-\overline{\theta}_p)(1-\overline{\theta}_p\Pi)}\mathrm{E}(\hat{\pi}_t) + \frac{1}{2}\frac{\varepsilon_p\overline{\theta}_p(\varepsilon_p\Pi-\varepsilon_p+1)}{(1-\overline{\theta}_p)^2}\mathrm{E}(\hat{\pi}_t^2). \tag{22}$$

Next, We write down the labor supply equation:

$$\begin{aligned}
&W_t\left((1-\lambda)\frac{1}{C_t^S}+\lambda\frac{1}{C_t^H}\right)=\chi N_t^{\phi}\\
\rightarrow\ &\frac{A_tMC_t}{1-\nu}\left((1-\lambda)\frac{1}{C_t^S}+\lambda\frac{1}{C_t^H}\right)=\chi N_t^{\phi}\\
\rightarrow\ &\frac{1-\lambda}{1-\nu}A_tMC_t\frac{1}{C_t^S}+\frac{\lambda}{1-\nu}A_tMC_t\frac{1}{C_t^H}=\chi\left(\frac{Y_tv_t^p}{A_t}\right)^{\phi}.
\end{aligned}$$

The second-order approximation of the labor supply schedule is

$$\begin{aligned}
&\frac{1-\lambda}{1-\nu}MC\frac{1}{C^S}\left(\hat{a}_t+\widehat{mc}_t-\hat{c}_t^S+\frac{1}{2}\hat{a}_t^2+\frac{1}{2}\widehat{mc}_t^2+\frac{1}{2}\left(\hat{c}_t^S\right)^2+\hat{a}_t\widehat{mc}_t-\widehat{mc}_t\hat{c}_t^S-\hat{a}_t\hat{c}_t^S\right)\\
&+\frac{\lambda}{1-\nu}MC\frac{1}{C^H}\left(\hat{a}_t+\widehat{mc}_t-\hat{c}_t^H+\frac{1}{2}\hat{a}_t^2+\frac{1}{2}\widehat{mc}_t^2+\frac{1}{2}\left(\hat{c}_t^H\right)^2+\hat{a}_t\widehat{mc}_t-\widehat{mc}_t\hat{c}_t^H-\hat{a}_t\hat{c}_t^H\right)\\
&=\chi\left(Yv^p\right)^{\phi}\left(\phi\hat{y}_t+\phi\hat{v}_t^p-\phi\hat{a}_t+\frac{1}{2}\phi^2\hat{y}_t^2+\frac{1}{2}\phi^2\hat{a}_t^2-\phi^2\hat{y}_t\hat{a}_t\right). \qquad (23)
\end{aligned}$$

Using the steady-state relationship of the labor-supply schedule, we obtain:

$$\frac{\frac{1-\lambda}{1-\nu}\frac{MC}{C^S}}{\chi\left(Yv^p\right)^{\phi}}=\frac{\frac{1-\lambda}{1-\nu}\frac{MC}{C^S}}{\frac{1-\lambda}{1-\nu}\frac{MC}{C^S}+\frac{\lambda}{1-\nu}\frac{MC}{C^H}}=\frac{(1-\lambda)\frac{Y}{C^S}}{(1-\lambda)\frac{Y}{C^S}+\lambda\frac{Y}{C^H}}.$$

From the steady state goods market clearing condition, we obtain:

$$(1-\lambda)C^S+\lambda C^H=(1-s_g)Y,$$

where $s_g=\frac{G}{Y}$. Using the fact $\frac{C^S}{C^H}=\mu_y$, steady-state household-specific consumption to output ratio can be represented as:

$$\frac{C^S}{Y}=\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda},\qquad \frac{C^H}{Y}=\frac{1-s_g}{(1-\lambda)\mu_y+\lambda}.$$

Finally, we can get the expressions for coefficients in (23) as:

$$\frac{\frac{1-\lambda}{1-\nu}\frac{MC}{C^S}}{\chi\left(Yv^p\right)^{\phi}}=\frac{(1-\lambda)\frac{Y}{C^S}}{(1-\lambda)\frac{Y}{C^S}+\lambda\frac{Y}{C^H}}=\frac{1-\lambda}{1-\lambda+\lambda\mu_y},\qquad \frac{\frac{\lambda}{1-\nu}\frac{MC}{C^H}}{\chi\left(Yv^p\right)^{\phi}}=\frac{\lambda\frac{Y}{C^H}}{(1-\lambda)\frac{Y}{C^S}+\lambda\frac{Y}{C^H}}=\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}.$$

Rewriting the second-order approximated labor-supply schedule:

$$
\begin{aligned}
&\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\left(\hat{a}_t+\widehat{mc}_t-\hat{c}_t^S+\frac{1}{2}\hat{a}_t^2+\frac{1}{2}\widehat{mc}_t^2+\frac{1}{2}\left(\hat{c}_t^S\right)^2+\hat{a}_t\widehat{mc}_t-\widehat{mc}_t\hat{c}_t^S-\hat{a}_t\hat{c}_t^S\right)\\
&+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\left(\hat{a}_t+\widehat{mc}_t-\hat{c}_t^H+\frac{1}{2}\hat{a}_t^2+\frac{1}{2}\widehat{mc}_t^2+\frac{1}{2}\left(\hat{c}_t^H\right)^2+\hat{a}_t\widehat{mc}_t-\widehat{mc}_t\hat{c}_t^H-\hat{a}_t\hat{c}_t^H\right)\\
&=\phi\hat{y}_t+\phi\hat{v}_t^p-\phi\hat{a}_t+\frac{1}{2}\phi^2\hat{y}_t^2+\frac{1}{2}\phi^2\hat{a}_t^2-\phi^2\hat{y}_t\hat{a}_t.
\end{aligned}
$$

Collecting the terms and with some algebra:

$$
\begin{aligned}
&\hat{a}_t+\widehat{mc}_t+\frac{1}{2}\hat{a}_t^2+\hat{a}_t\widehat{mc}_t+\frac{1}{2}\widehat{mc}_t\\
&+\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\left(-\hat{c}_t^S+\frac{1}{2}\left(\hat{c}_t^S\right)^2\right)+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\left(-\hat{c}_t^H+\frac{1}{2}\left(\hat{c}_t^H\right)^2\right)\\
&-\left(\hat{a}_t+\widehat{mc}_t\right)\left(\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H\right)\\
&=\phi\hat{y}_t+\phi\hat{v}_t^p-\phi\hat{a}_t+\frac{1}{2}\phi^2\hat{y}_t^2+\frac{1}{2}\phi^2\hat{a}_t^2-\phi^2\hat{y}_t\hat{a}_t.
\end{aligned}
\tag{24}
$$

Up to the first order, the labor-supply schedule equation is:

$$
\hat{a}_t+\widehat{mc}_t-\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S-\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H=\phi\hat{y}_t-\phi\hat{a}_t. \tag{25}
$$

We rewrite some second-order terms of equation (24):

$$
\begin{aligned}
&\frac{1}{2}\left(\hat{a}_t+\widehat{mc}_t\right)^2-\left(\hat{a}_t+\widehat{mc}_t\right)\left(\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H\right)\\
&=\left(\hat{a}_t+\widehat{mc}_t\right)\left(\frac{1}{2}\left(\hat{a}_t+\widehat{mc}_t\right)-\left(\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H\right)\right)\\
&=\frac{1}{2}\left(\left(\phi\hat{y}_t-\phi\hat{a}_t\right)^2-\left(\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H\right)^2\right),
\end{aligned}
$$

where the last equality makes use of (25). Using the above expression, we rewrite (24) as:

$$
\begin{aligned}
&\hat{a}_t+\widehat{mc}_t+\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\left(-\hat{c}_t^S+\frac{1}{2}\left(\hat{c}_t^S\right)^2\right)+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\left(-\hat{c}_t^H+\frac{1}{2}\left(\hat{c}_t^H\right)^2\right)\\
&+\frac{1}{2}\left(\left(\phi\hat{y}_t-\phi\hat{a}_t\right)^2-\left(\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H\right)^2\right)\\
&=\phi\hat{y}_t+\phi\hat{v}_t^p-\phi\hat{a}_t+\frac{1}{2}\phi^2\hat{y}_t^2+\frac{1}{2}\phi^2\hat{a}_t^2-\phi^2\hat{y}_t\hat{a}_t.
\end{aligned}
$$

With some algebra on $\hat{c}_t^S$ and $\hat{c}_t^H$, we can obtain the expression for the marginal cost:

$$\widehat{mc}_t = \frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S + \frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H - \frac{1}{2}\frac{\lambda(1-\lambda)\mu_y}{(1-\lambda+\lambda\mu_y)^2}\hat{\omega}_t^2 + \phi\hat{y}_t + \phi\hat{v}_t^p - (1+\phi)\hat{a}_t, \tag{26}$$

where $\hat{\omega}_t = \hat{c}_t^S - \hat{c}_t^H$. Next, we exploit the budget constraint of each household. The savers' budget constraint reads:

$$\begin{aligned} C_t^S + TG_t^S &= (1-\nu)W_tN_t + \frac{1-\tau_d\lambda}{1-\lambda}D_t + \mathcal{T}^S \\ &= MC_tv_t^pY_t + \frac{1-\tau_d\lambda}{1-\lambda}\left(1-MC_tv_t^p\right)Y_t + \mathcal{T}^S \\ &= \frac{\lambda(\tau_d-1)}{1-\lambda}MC_tv_t^pY_t + \frac{1-\tau_d\lambda}{1-\lambda}Y_t + \mathcal{T}^S. \end{aligned}$$

The second-order approximation of the savers' budget constraint is

$$\begin{aligned} &C^S\left(\hat{c}_t^S + \frac{1}{2}(\hat{c}_t^S)^2\right) + TG^S\left(\widehat{tg}_t^s + \frac{1}{2}(\widehat{tg}_t^s)^2\right) \\ &= \frac{\lambda(\tau_d-1)}{1-\lambda}MCv^pY\left(\widehat{mc}_t + \hat{v}_t^p + \hat{y}_t + \frac{1}{2}\widehat{mc}_t^2 + \frac{1}{2}\hat{y}_t^2 + \widehat{mc}_t\hat{y}_t\right) \\ &+ \frac{1-\tau_d\lambda}{1-\lambda}Y\left(\hat{y}_t + \frac{1}{2}\hat{y}_t^2\right). \end{aligned}$$

The second-order approximation of savers' tax burden (i.e., $TG_t^S = TG^S + \frac{1-\alpha}{1-\lambda}\left(G_t - G\right)$) is:

$$TG^S\left(\widehat{tg}_t^s + \frac{1}{2}(\widehat{tg}_t^s)^2\right) = \frac{1-\alpha}{1-\lambda}G\left(\hat{g}_t + \frac{1}{2}\hat{g}_t^2\right). \tag{27}$$

Substituting out the tax terms in the savers' budget constraint using (27):

$$\begin{aligned} &C^S\left(\hat{c}_t^S + \frac{1}{2}(\hat{c}_t^S)^2\right) + \frac{1-\alpha}{1-\lambda}G\left(\hat{g}_t + \frac{1}{2}\hat{g}_t^2\right) \\ &= \frac{\lambda(\tau_d-1)}{1-\lambda}MCv^pY\left(\widehat{mc}_t + \hat{v}_t^p + \hat{y}_t + \frac{1}{2}\widehat{mc}_t^2 + \frac{1}{2}\hat{y}_t^2 + \widehat{mc}_t\hat{y}_t\right) \\ &+ \frac{1-\tau_d\lambda}{1-\lambda}Y\left(\hat{y}_t + \frac{1}{2}\hat{y}_t^2\right). \end{aligned}$$

Applying the same procedure to the spenders' budget constraint, we obtain:

$$
\begin{aligned}
&C^H\left(\hat{c}_t^H+\frac{1}{2}(\hat{c}_t^H)^2\right)+\frac{\alpha}{\lambda}G\left(\hat{g}_t+\frac{1}{2}\hat{g}_t^2\right)\\
&=(1-\tau_d)MCv^pY\left(\widehat{mc}_t+\hat{v}_t^p+\hat{y}_t+\frac{1}{2}\widehat{mc}_t^2+\frac{1}{2}\hat{y}_t^2+\widehat{mc}_t\hat{y}_t\right)\\
&+\tau_dY\left(\hat{y}_t+\frac{1}{2}\hat{y}_t^2\right).
\end{aligned}
$$

Finally, approximating the government spending feedback rule (i.e., $G_t = G\left(\frac{Y_t}{Y}\right)^{\gamma_y^G}$) up to second order and then plugging the resulting approximation into the savers' and spenders' budget constraints:

$$
\begin{aligned}
&C^S\left(\hat{c}_t^S+\frac{1}{2}(\hat{c}_t^S)^2\right)+\frac{1-\alpha}{1-\lambda}G\left(\gamma_y^G\hat{y}_t+\frac{1}{2}(\gamma_y^G)^2\hat{y}_t^2\right) && (28)\\
&=\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^pY\left(\widehat{mc}_t+\hat{v}_t^p+\hat{y}_t+\frac{1}{2}\widehat{mc}_t^2+\frac{1}{2}\hat{y}_t^2+\widehat{mc}_t\hat{y}_t\right)+\frac{1-\tau_d\lambda}{1-\lambda}Y\left(\hat{y}_t+\frac{1}{2}\hat{y}_t^2\right)\\
&C^H\left(\hat{c}_t^H+\frac{1}{2}(\hat{c}_t^H)^2\right)+\frac{\alpha}{\lambda}G\left(\gamma_y^G\hat{y}_t+\frac{1}{2}(\gamma_y^G)^2\hat{y}_t^2\right) && (29)\\
&=(1-\tau_d)MCv^pY\left(\widehat{mc}_t+\hat{v}_t^p+\hat{y}_t+\frac{1}{2}\widehat{mc}_t^2+\frac{1}{2}\hat{y}_t^2+\widehat{mc}_t\hat{y}_t\right)+\tau_dY\left(\hat{y}_t+\frac{1}{2}\hat{y}_t^2\right).
\end{aligned}
$$

Using the steady state consumption ratios (i.e., $\frac{C^S}{Y}=\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}$, $\frac{C^H}{Y}=\frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}$), we can write the fist-order approximate of savers' and spenders' budget constraints:

$$
\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\hat{c}_t^S=\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\widehat{mc}_t+\left(\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p+\frac{1-\tau_d\lambda}{1-\lambda}-\frac{1-\alpha}{1-\lambda}s_g\gamma_y^G\right)\hat{y}_t \tag{30}
$$

$$
\frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}\hat{c}_t^H=(1-\tau_d)MCv^p\widehat{mc}_t+\left((1-\tau_d)MCv^p+\tau_d-\frac{\alpha}{\lambda}s_g\gamma_y^G\right)\hat{y}_t. \tag{31}
$$

Plugging the above two expressions into (25):

$$
\begin{aligned}
\widehat{mc}_t&=\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H+\phi\hat{y}_t-(1+\phi)\hat{a}_t\\
&=\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\frac{(1-\lambda)\mu_y+\lambda}{(1-s_g)\mu_y}\left(\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\widehat{mc}_t+\left(\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p+\frac{1-\tau_d\lambda}{1-\lambda}-\frac{1-\alpha}{1-\lambda}s_g\gamma_y^G\right)\hat{y}_t\right)\\
&+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\frac{(1-\lambda)\mu_y+\lambda}{(1-s_g)}\left((1-\tau_d)MCv^p\widehat{mc}_t+\left((1-\tau_d)MCv^p+\tau_d-\frac{\alpha}{\lambda}s_g\gamma_y^G\right)\hat{y}_t\right)\\
&+\phi\hat{y}_t-(1+\phi)\hat{a}_t.
\end{aligned}
$$

With some algebra, we can rewrite the above expression as:

$$\Big(1-\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\frac{(1-\lambda)\mu_y+\lambda}{(1-s_g)\mu_y}\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p-\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\frac{(1-\lambda)\mu_y+\lambda}{(1-s_g)}(1-\tau_d)MCv^p\Big)\widehat{mc}_t$$
$$=\Big(\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\frac{(1-\lambda)\mu_y+\lambda}{(1-s_g)\mu_y}\left(\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p+\frac{1-\tau_d\lambda}{1-\lambda}-\frac{1-\alpha}{1-\lambda}s_g\gamma_y^G\right)$$
$$+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\frac{(1-\lambda)\mu_y+\lambda}{(1-s_g)}\left((1-\tau_d)MCv^p+\tau_d-\frac{\alpha}{\lambda}s_g\gamma_y^G\right)+\phi\Big)\hat{y}_t-(1+\phi)\hat{a}_t.$$

To simplify notations, let us rewrite the above first-order approximation of marginal cost as:

$$\widehat{mc}_t=\theta_{mcy}\hat{y}_t+\theta_{mca}\hat{a}_t. \tag{32}$$

Plugging (32) into the budget constraints (30) and (31):

$$\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\hat{c}_t^S=\left(\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\theta_{mcy}+\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p+\frac{1-\tau_d\lambda}{1-\lambda}-\frac{1-\alpha}{1-\lambda}s_g\gamma_y^G\right)\hat{y}_t$$
$$+\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\theta_{mca}\hat{a}_t$$
$$\frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}\hat{c}_t^H=\left((1-\tau_d)MCv^p\theta_{mcy}+(1-\tau_d)MCv^p+\tau_d-\frac{\alpha}{\lambda}s_g\gamma_y^G\right)\hat{y}_t$$
$$+(1-\tau_d)MCv^p\theta_{mca}\hat{a}_t.$$

To simplify the notation, we rewrite the above two expressions as:

$$\hat{c}_t^S=\theta_{cSy}\hat{y}_t+\theta_{cSa}\hat{a}_t \tag{33}$$
$$\hat{c}_t^H=\theta_{cHy}\hat{y}_t+\theta_{cHa}\hat{a}_t. \tag{34}$$

Using (32), (33), and (26), the second-order approximated savers' budget constraint (28) can be rewritten as:

$$\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\left(\hat{c}_t^S+\frac{1}{2}(\hat{c}_t^S)^2\right)+\frac{1-\alpha}{1-\lambda}s_g\left(\gamma_y^G\hat{y}_t+\frac{1}{2}(\gamma_y^G)^2\hat{y}_t^2\right)$$
$$=\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\left(\widehat{mc}_t+\hat{v}_t^p+\hat{y}_t+\frac{1}{2}\widehat{mc}_t^2+\frac{1}{2}\hat{y}_t^2+\widehat{mc}_t\hat{y}_t\right)+\frac{1-\tau_d\lambda}{1-\lambda}\left(\hat{y}_t+\frac{1}{2}\hat{y}_t^2\right)$$

$$
\begin{aligned}
\rightarrow & \frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\left(\hat{c}_t^S+\frac{1}{2}(\theta_{cSy}\hat{y}_t+\theta_{cSa}\hat{a}_t)^2\right)+\frac{1-\alpha}{1-\lambda}s_g\left(\gamma_y^G\hat{y}_t+\frac{1}{2}(\gamma_y^G)^2\hat{y}_t^2\right) \\
& =\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\Big(\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H-\frac{1}{2}\frac{\lambda(1-\lambda)\mu_y}{(1-\lambda+\lambda\mu_y)^2}\hat{\omega}_t^2+\phi\hat{y}_t+\phi\hat{v}_t^p-(1+\phi)\hat{a}_t \\
& +\hat{v}_t^p+\hat{y}_t+\frac{1}{2}\left((1+\theta_{mcy})\hat{y}_t+\theta_{mca}\hat{a}_t)^2\right)+\frac{1-\tau_d\lambda}{1-\lambda}\left(\hat{y}_t+\frac{1}{2}\hat{y}_t^2\right).
\end{aligned}
$$

Likewise, the second-order approximated spenders' budget constraint (29) can be rewritten as:

$$
\begin{aligned}
& \frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}\left(\hat{c}_t^H+\frac{1}{2}(\hat{c}_t^H)^2\right)+\frac{\alpha}{\lambda}s_g\left(\gamma_y^G\hat{y}_t+\frac{1}{2}(\gamma_y^G)^2\hat{y}_t^2\right) \\
& =(1-\tau_d)MCv^p\left(\widehat{mc}_t+\hat{v}_t^p+\hat{y}_t+\frac{1}{2}\widehat{mc}_t^2+\frac{1}{2}\hat{y}_t^2+\widehat{mc}_t\hat{y}_t\right)+\tau_d\left(\hat{y}_t+\frac{1}{2}\hat{y}_t^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\rightarrow & \frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}\left(\hat{c}_t^H+\frac{1}{2}(\theta_{cHy}\hat{y}_t+\theta_{cHa}\hat{a}_t)^2\right)+\frac{\alpha}{\lambda}s_g\left(\gamma_y^G\hat{y}_t+\frac{1}{2}(\gamma_y^G)^2\hat{y}_t^2\right) \\
& =(1-\tau_d)MCv^p\Big(\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S+\frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H-\frac{1}{2}\frac{\lambda(1-\lambda)\mu_y}{(1-\lambda+\lambda\mu_y)^2}\hat{\omega}_t^2+\phi\hat{y}_t+\phi\hat{v}_t^p-(1+\phi)\hat{a}_t \\
& +\hat{v}_t^p+\hat{y}_t+\frac{1}{2}\left((1+\theta_{mcy})\hat{y}_t+\theta_{mca}\hat{a}_t)^2\right)+\tau_d\left(\hat{y}_t+\frac{1}{2}\hat{y}_t^2\right).
\end{aligned}
$$

With some algebraic manipulation and collecting like terms, we can rewrite the above two expressions as:

$$
\begin{aligned}
&\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\hat{c}_t^S - \frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\left(\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S + \frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H\right)\\
&= \left(\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p(1+\phi) + \frac{1-\tau_d\lambda}{1-\lambda} - \frac{1-\alpha}{1-\lambda}s_g\gamma_y^G\right)\hat{y}_t\\
&+ \frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\left(-(1+\phi)\right)\hat{a}_t\\
&+ \left(\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\frac{1}{2}\left(1+\theta_{mcy}\right)^2 + \frac{1}{2}\frac{1-\tau_d\lambda}{1-\lambda} - \frac{1}{2}\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\theta_{cSy}^2 - \frac{1-\alpha}{1-\lambda}s_g\frac{1}{2}(\gamma_y^G)^2\right)\hat{y}_t^2\\
&+ \left(\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p(1+\theta_{mcy})\theta_{mca} - \frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\theta_{cSy}\theta_{cSa}\right)\hat{y}_t\hat{a}_t\\
&+ \left(\frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\frac{1}{2}\theta_{mca}^2 - \frac{1}{2}\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\theta_{cSa}^2\right)\hat{a}_t^2\\
&+ \frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p\left(-\frac{1}{2}\frac{\lambda(1-\lambda)\mu_y}{(1-\lambda+\lambda\mu_y)^2}\right)\hat{\omega}_t^2\\
&+ \frac{\lambda(\tau_d-1)}{1-\lambda}MCv^p(1+\phi)\hat{v}_t^p
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
&\frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}\hat{c}_t^H - (1-\tau_d)MCv^p\left(\frac{1-\lambda}{1-\lambda+\lambda\mu_y}\hat{c}_t^S + \frac{\lambda\mu_y}{1-\lambda+\lambda\mu_y}\hat{c}_t^H\right)\\
&= \left((1-\tau_d)MCv^p(1+\phi) + \tau_d - \frac{\alpha}{\lambda}s_g\gamma_y^G\right)\hat{y}_t\\
&+ (1-\tau_d)MCv^p(-(1+\phi))\hat{a}_t\\
&+ \left((1-\tau_d)MCv^p\frac{1}{2}(1+\theta_{mcy})^2 + \frac{1}{2}\tau_d - \frac{1}{2}\frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}\theta_{cHy}^2 - \frac{\alpha}{\lambda}s_g\frac{1}{2}(\gamma_y^G)^2\right)\hat{y}_t^2\\
&+ \left((1-\tau_d)MCv^p(1+\theta_{mcy})\theta_{mca} - \frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}\theta_{cHy}\theta_{cHa}\right)\hat{y}_t\hat{a}_t\\
&+ \left((1-\tau_d)MCv^p\frac{1}{2}\theta_{mca}^2 - \frac{1}{2}\frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}\theta_{cHa}^2\right)\hat{a}_t^2\\
&+ (1-\tau_d)MCv^p\left(-\frac{1}{2}\frac{\lambda(1-\lambda)\mu_y}{(1-\lambda+\lambda\mu_y)^2}\right)\hat{\omega}_t^2\\
&+ (1-\tau_d)MCv^p(1+\phi)\hat{v}_t^p.
\end{aligned}
\tag{36}
$$

Rewriting (35) and (36) in a matrix form:

$$A_0 \begin{pmatrix} \hat{c}_t^S \\ \hat{c}_t^H \end{pmatrix} = A_1 \begin{pmatrix} \hat{y}_t \\ \hat{a}_t \\ \hat{y}_t^2 \\ \hat{y}_t \hat{a}_t \\ \hat{a}_t^2 \\ \hat{\omega}_t^2 \\ \hat{v}_t^p \end{pmatrix}.$$

Taking the inverse of matrix A_0 yields:

$$\hat{c}_t^S = \gamma_{cSy}\hat{y}_t + \gamma_{cSa}\hat{a}_t + \gamma_{cSy2}\hat{y}_t^2 + \gamma_{cSya}\hat{y}_t\hat{a}_t + \gamma_{cSa2}\hat{a}_t^2 + \gamma_{cS\omega 2}\hat{\omega}_t^2 + \gamma_{cSvp}\hat{v}_t^p \tag{37}$$

$$\hat{c}_t^H = \gamma_{cHy}\hat{y}_t + \gamma_{cHa}\hat{a}_t + \gamma_{cHy2}\hat{y}_t^2 + \gamma_{cHya}\hat{y}_t\hat{a}_t + \gamma_{cHa2}\hat{a}_t^2 + \gamma_{cH\omega 2}\hat{\omega}_t^2 + \gamma_{cHvp}\hat{v}_t^p. \tag{38}$$

Finally, per-period welfare of household type $j \in \{S, H\}$ can be approximated as:

$$\begin{aligned} U_t^j &\equiv log(C_t^j) - \chi\frac{N_t^{1+\phi}}{1+\phi} \\ &\approx log(C^j) - \chi\frac{N^{1+\phi}}{1+\phi} \\ &+ \hat{c}_t^j - \Omega_y\left(\hat{y}_t + \hat{v}_t^p - \hat{a}_t + \frac{1}{2}(1+\phi)\hat{y}_t^2 + \frac{1}{2}(1+\phi)\hat{a}_t^2 - (1+\phi)\hat{y}_t\hat{a}_t\right), \end{aligned}$$

where $\Omega_y \equiv \chi N^{1+\phi} = \frac{MC}{1-\nu}v^p\left((1-\lambda)\frac{Y}{C^S} + \lambda\frac{Y}{C^H}\right)$. Substituting out $\hat{c}_t^j$ using (37) and (38):

$$\begin{aligned} U_t^j &\approx log(C^j) - \chi\frac{N^{1+\phi}}{1+\phi} \\ &+ (\gamma_{cjy} - \Omega_y)\hat{y}_t + (\gamma_{cja} + \Omega_y)\hat{a}_t + \left(\gamma_{cjy2} - \frac{1}{2}(1+\phi)\Omega_y\right)\hat{y}_t^2 + (\gamma_{cjya} + (1+\phi)\Omega_y)\hat{y}_t\hat{a}_t \end{aligned} \tag{39}$$

$$+ \left(\gamma_{cja2} - \frac{1}{2}(1+\phi)\Omega_y\right)\hat{a}_t^2 + \gamma_{cj\omega 2}\hat{\omega}_t^2 + (\gamma_{cjvp} - \Omega_y)\hat{v}_t^p. \tag{40}$$

Using (37) and (38) and removing terms higher than second-order, we can express the squared consumption dispersion as $\omega_t^2 = (\gamma_{cSy} - \gamma_{cHy})^2\hat{y}_t^2 + 2(\gamma_{cSy} - \gamma_{cHy})(\gamma_{cSa} - \gamma_{cHa})\hat{y}_t\hat{a}_t +$

$(\gamma_{cSa}-\gamma_{cHa})^2\hat{a}_t^2$. Plugging this into (40):

$$\begin{aligned}
U_t^j \approx\ & log(C^j) - \chi\frac{N^{1+\phi}}{1+\phi} \\
& + (\gamma_{cjy}-\Omega_y)\,\hat{y}_t + (\gamma_{cja}+\Omega_y)\,\hat{a}_t + \left(\gamma_{cjy2}-\frac{1}{2}(1+\phi)\Omega_y+\gamma_{cj\omega 2}(\gamma_{cSy}-\gamma_{cHy})^2\right)\hat{y}_t^2 \\
& + (\gamma_{cjya}+(1+\phi)\Omega_y+2\gamma_{cj\omega 2}(\gamma_{cSy}-\gamma_{cHy})(\gamma_{cSa}-\gamma_{cHa}))\,\hat{y}_t\hat{a}_t \\
& + \left(\gamma_{cja2}-\frac{1}{2}(1+\phi)\Omega_y+\gamma_{cj\omega 2}(\gamma_{cSa}-\gamma_{cHa})^2\right)\hat{a}_t^2 + (\gamma_{cjvp}-\Omega_y)\,\hat{v}_t^p.
\end{aligned}$$

Taking the unconditional expectations and substituting out the price dispersion using (22), we write the expected utility of type j:

$$\begin{aligned}
EU^j \approx\ & log(C^j) - \chi\frac{N^{1+\phi}}{1+\phi} + \left(\gamma_{cja2}-\frac{1}{2}(1+\phi)\Omega_y+\gamma_{cj\omega 2}(\gamma_{cSa}-\gamma_{cHa})^2\right)\frac{\sigma_a^2}{1-\rho_a^2} \\
& + (\gamma_{cjy}-\Omega_y)\,\mathbf{E}(\hat{y}_t) \\
& + \left(\gamma_{cjy2}-\frac{1}{2}(1+\phi)\Omega_y+\gamma_{cj\omega 2}(\gamma_{cSy}-\gamma_{cHy})^2\right)\mathbf{E}(\hat{y}_t^2) \\
& + (\gamma_{cjya}+(1+\phi)\Omega_y+2\gamma_{cj\omega 2}(\gamma_{cSy}-\gamma_{cHy})(\gamma_{cSa}-\gamma_{cHa}))\,\mathbf{E}(\hat{y}_t\hat{a}_t) \\
& + (\gamma_{cjvp}-\Omega_y)\left(\frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{(1-\overline{\theta}_p)(1-\overline{\theta}_p\Pi)}\mathbf{E}(\hat{\pi}_t)+\frac{1}{2}\frac{\varepsilon_p\overline{\theta}_p(\varepsilon_p\Pi-\varepsilon_p+1)}{(1-\overline{\theta}_p)^2}\mathbf{E}(\hat{\pi}_t^2)\right)
\end{aligned}$$

The expected value under steady-state inflation $\overline{\Pi}$ for each type is given as follows. When there is no type switching ($s=1$),

$$\begin{aligned}
EV^S(\Pi=\overline{\Pi}) &= \frac{EU^S(\Pi=\overline{\Pi})}{1-\beta} \\
EV^H(\Pi=\overline{\Pi}) &= \frac{EU^H(\Pi=\overline{\Pi})}{1-\beta}.
\end{aligned}$$

When type switching occurs, the expected value for each household is determined by solving:

$$\begin{aligned}
EV^S(\Pi=\overline{\Pi}) &= EU^S(\Pi=\overline{\Pi}) + s\beta EV^S(\Pi=\overline{\Pi}) + (1-s)\beta EV^H(\Pi=\overline{\Pi}) \\
EV^H(\Pi=\overline{\Pi}) &= EU^H(\Pi=\overline{\Pi}) + (1-h)\beta EV^S(\Pi=\overline{\Pi}) + h\beta EV^H(\Pi=\overline{\Pi}),
\end{aligned}$$

where $\lambda=\frac{1-s}{2-s-h}$. To calculate the per-period welfare, we multiply $EV^S(\Pi=\overline{\Pi})$ and $EV^H(\Pi=\overline{\Pi})$ by $(1-\beta)$.

References

Alves, Sergio Afonso Lago, "Lack of divine coincidence in New Keynesian models," *Journal of Monetary Economics*, 2014, *67*, 33–46.