Oneline Appendix to Growth through Learning

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1. Single Agent Case $N = 1$.

This section reports the proofs of Lemma 1 and Proposition 2. Both pertain to the single agent case.

Proof of Lemma 1

Strictly concave "reduced form" utility.—Since maximization of $E_t(U)$ is dynamically optimal, (9) via Lemma 1 gives the optimal action for a forward looking agent as long as the objective is concave in x.

Then we have

$$\begin{aligned} \frac{\partial U}{\partial x} &= \rho\left[(\theta - x_t + \varepsilon_t)^{-\rho} - n_t\right]^{-\gamma}(\theta - x_t + \varepsilon_t)^{-\rho-1} \\ \text{and} \quad \frac{\partial^2 U}{\partial x^2} &= -\gamma\rho^2 c_t^{-\gamma-1}\left[(\theta - x_t + \varepsilon_t)^{-\rho-1}\right]^2 \\ &\quad +\rho(\rho+1)[c_t]^{-\gamma}(\theta - x_t + \varepsilon_t)^{-\rho-2} \end{aligned}$$

or

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} &= -\gamma\rho^2 c_t^{-\gamma-1}(\theta - x_t + \varepsilon_t)^{-2\rho-2} + \rho(\rho+1)[c_t]^{-\gamma}(\theta - x_t + \varepsilon_t)^{-\rho-2} \\ &= \left[c_t^{-\gamma}(\theta - x_t + \varepsilon_t)^{-2}\right]\left\{-\gamma\rho^2 c_t^{-1}(\theta - x_t + \varepsilon_t)^{-2\rho} + \rho(\rho+1)(\theta - x_t + \varepsilon_t)^{-\rho}\right\} \\ &= \left[c_t^{-\gamma}(\theta - x_t + \varepsilon_t)^{-2}\right]\left\{-\gamma\rho^2 c_t^{-1} y_t^2 + \rho(\rho+1) y_t\right\} \\ &= \left[c_t^{-\gamma}(\theta - x_t + \varepsilon_t)^{-2}\right] y_t\rho\left\{-\gamma\rho\frac{y_t}{c_t} + \rho + 1\right\} \end{aligned}$$

Then if

$$\rho(\gamma - 1) > 1, \tag{A.1}$$

We have that

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} &= \left[c_t^{-\gamma}(\theta - x_t + \varepsilon_t)^{-2}\right] y_t\rho\left\{-\gamma\rho\frac{y_t}{c_t} + \rho + 1\right\} \\ &\leq \left[c_t^{-\gamma}(\theta - x_t + \varepsilon_t)^{-2}\right] y_t\rho\left\{-\gamma\rho + \rho + 1\right\} \quad \because \frac{y_t}{c_t} \geq 1 \\ &= \left[c_t^{-\gamma}(\theta - x_t + \varepsilon_t)^{-2}\right] y_t\rho\left[-\rho(\gamma - 1) + 1\right] \\ &< 0 \quad \text{because } \rho(\gamma - 1) > 1 \end{aligned}$$

because $\rho(\gamma - 1)$ is an even integer that is larger than 2.

CLAIM: Let $r > 0$ be an even integer and a be integer. Then

$$\arg\min_x \int ((z - x)^r - n)^a \, d\Phi\left(\frac{z - \mu}{\sigma_z}\right) = \mu. \tag{A.2}$$

PROOF OF CLAIM: The FOC to the problem in (A.2) reads

$$a\int\left[(z-x)^{r}-n\right]^{a-1}(z-x)^{r-1}\,d\Phi\left(\frac{z-\mu}{\sigma_z}\right)=0 \tag{A.3}$$

Using Newton's generalized binomial theorem,[1] then for any non-zero constant a, we have

$$\left[(z-x)^{r}-n\right]^{a}=\sum_{k=0}^{\infty}\binom{a-1}{k}(z-x)^{r(a-k-1)}\,n^{k}.$$

Since r is even integer, this implies that $r\,(a-k)$ is even for any a and k. This implies that (A.3) reads

$$\begin{aligned}0 &= a\int\sum_{k=0}^{\infty}\binom{a-1}{k}(z-x)^{r(a-k-1)+(r-1)}\,n^{k}d\Phi\left(\frac{z-\mu}{\sigma_z}\right)\\ &= \sum_{k=0}^{\infty}\int(z-x)^{r(a-k-1)+(r-1)}\,d\Phi\left(\frac{z-\mu}{\sigma_z}\right)\end{aligned}$$

where the second equality follows because $\binom{a-1}{k}$ and n do not depend on z. For any a and k, we always have that $r\,(a-k-1)+(r-1)$ is a odd number because r is even. The objective is strictly convex in x, so that (A.3) has exactly one solution for x, which is the globally optimal action. For the normal distribution, all odd moments around the mean are zero and (A.3) therefore holds at $x=\mu$. Therefore, (9) is the unique solution to (A.3) and this proves Lemma 1.

2. N agents: Derivation of planner's problem and proof of Claim 1 and Proposition 5

We guess and verify that the value function and policy function have the following form

$$V(y)=\frac{B}{\sum y_i}\ \text{ and }\ \Phi\left(y_i,\mathbf{y}_{-\mathbf{i}}\right)=ay+b\sum_{j\neq i}y_j$$

where the coefficients B,a,b are to be determined.

Now plug in the guessed solution into the value function formulation, we have

$$\frac{B}{\sum y_i}=\max_{\mathbf{n}}\left\{-\sum_j\left(\frac{1}{y_j-n_j}\right)+\beta Bl^2\int\frac{\zeta^2}{N}\frac{e^{-\zeta^2/2}}{\sqrt{2\pi}}d\zeta\right\}$$

where

$$l^2=\left[\left(\frac{1}{\sigma_\eta^2}\sum n\right)^{-1/2}\right]^2=\sigma_\eta^2\left(\sum n\right)^{-1}$$

[1] https://en.wikipedia.org/wiki/Binomial_theorem#Newton.27s_generalised_binomial_theorem

Note that because $\int \zeta^2 \frac{e^{-\zeta^2/2}}{\sqrt{2\pi}} d\zeta = 1$, we have

$$\int \frac{\zeta^2}{N} \frac{e^{-\zeta^2/2}}{\sqrt{2\pi}} d\zeta = \frac{1}{N}$$

therefore, the value function now becomes

$$\frac{B}{\sum y_i} = \max_{\mathbf{n}} \left\{ -\sum_j \left(\frac{1}{y_j - n_j} \right) + \beta B \sigma_\eta^2 \left(\sum n \right)^{-1} \frac{1}{N} \right\}$$

Now we take first order condition w.r.t. to n_i,

$$-\frac{1}{(y_i - n_i)^2} - \frac{\beta B \frac{1}{N} \sigma_\eta^2}{\left(\sum n\right)^2} = 0$$

for all $i \in \{1, 2 ... N\}$

We first notice that

$$\begin{aligned} \sum_i n &= \sum_i \left(a y_i + b \sum_{j \neq i} y_j \right) \\ &= (a + b(N-1)) \sum_i y_i \end{aligned}$$

therefore, FOC becomes

$$\frac{1}{\left(y_i - a y_i - b \sum_{j \neq i} y_j\right)^2} = -\frac{\beta B \frac{1}{N} \sigma_\eta^2}{\left[(a + b(N-1)) \sum_j y_j\right]^2}$$

or

$$(a + b(N-1))^2 \left(y_i + \sum_{j \neq i} y_j \right)^2 = -\left(\beta B \frac{1}{N} \sigma_\eta^2 \right) \left((1-a) y_i - b \sum_{j \neq i} y_j \right)^2$$

Matching the coefficients from LHS and RHS gives 3 equations for term y^2, $y_i \sum_{j \neq i} y_j$ and $\left(\sum_{j \neq i} y_j\right)^2$

$$\begin{aligned} (a + b(N-1))^2 &= -\left(\beta B \frac{1}{N} \sigma_\eta^2 \right) (1-a)^2 \\ (a + b(N-1))^2 &= \left(\beta B \frac{1}{N} \sigma_\eta^2 \right) (1-a) b \\ (a + b(N-1))^2 &= -\left(\beta B \frac{1}{N} \sigma_\eta^2 \right) b^2 \end{aligned}$$

This implies that

$$(1-a)^2 = b^2 = -(1-a) b$$

or

$$a = 1 + b$$

Plug in this relationship into $\frac{(a+b(N-1))^2}{b^2} = -\left(\beta B \frac{1}{N}\sigma_\eta^2\right)$, we have

$$\frac{(1+bN)^2}{b^2} = -\left(\beta B \frac{1}{N}\sigma_\eta^2\right)$$

Now from the value function,

$$\begin{aligned}
\frac{B}{\sum y_i} &= -\sum_i \left(\frac{1}{y_i - n_i}\right) + \beta B \sigma_\eta^2 \left(\sum n\right)^{-1} \frac{1}{N} \\
&= -\sum_i \frac{1}{(1-a)\, y_i - b \sum_{j \neq i} y_j} + \frac{\beta B \sigma_\eta^2 \frac{1}{N}}{(a + b(N-1)) \sum_i y_i}
\end{aligned}$$

and now using the identity that $\frac{(1+bN)^2}{b^2} = -\left(\beta B \frac{1}{N}\sigma_\eta^2\right)$, we have

$$\frac{(1+bN)^2}{-\beta \frac{1}{N}\sigma_\eta^2 b^2 \sum y_i} = \frac{N}{b \sum_i y_i} - \frac{\frac{(1+bN)^2}{b^2}}{(1+bN) \sum_i y_i}$$

This solves b (notice that $\sum y_i$ is canceled out in LHS and RHS),

$$b = \frac{\sqrt{\beta \frac{1}{N}\sigma_\eta^2} - 1}{N}$$

Using identity $a = 1 + b$, we have

$$a = \frac{\sqrt{\beta \frac{1}{N}\sigma_\eta^2} - 1 + N}{N}$$

As in the two-player case, we define

$$\alpha(N) = \sqrt{\frac{1}{N}\beta\sigma_\eta^2}$$

and thus the policy function is

$$\Phi(y_i, \mathbf{y_{-i}}) = \frac{N - 1 + \alpha(N)}{N} y - \frac{1 - \alpha(N)}{N} \sum_{j \neq i} y_j$$

Lastly since we have that

$$\frac{(1+bN)^2}{b^2} = -\left(\beta B \frac{1}{N}\sigma_\eta^2\right)$$

and this solves B

$$B = \frac{N \frac{(1+bN)^2}{b^2}}{-\beta\sigma_\eta^2} = -\frac{N^2}{(\alpha(N) - 1)^2}$$

Thus we have that,

$$V(\mathbf{y}) = -\frac{N^2}{(\alpha(N)-1)^2}\left(\sum_j y_j\right)^{-1} \quad \text{and} \quad \Phi(y_i,\mathbf{y_{-i}}) = \frac{N-1+\alpha(N)}{N}y - \frac{1-\alpha(N)}{N}\sum_{j\neq i} y_j$$

where

$$\alpha(N) = \sqrt{\frac{1}{N}\beta\sigma_\eta^2}$$

3. Proof of Lemmas 2 and 3

In this appendix we shall drop the "+1" subscript on ε, so as to shorten the notation. Conditional on $\left(\bar{\xi}_n, \bar{\boldsymbol{\xi}}_{n_j}\right)$, the posterior mean of ε is

$$E\left(\varepsilon \mid \bar{\xi}_n, \bar{\boldsymbol{\xi}}_{n_j}\right) \equiv \frac{\left[\frac{n_i}{\sigma_\eta^2} + \sum_{j\neq i}\frac{1}{2\sigma_\omega^2+\sigma_\eta^2/n_j}\right]\frac{\left(\frac{\sigma_\eta^2}{n}\right)^{-1}\bar{\xi}_n + \sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}\bar{\xi}_{n_j}}{\left(\frac{\sigma_\eta^2}{n}\right)^{-1} + \sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}}}{\frac{1}{\sigma_u^2+\sigma_\omega^2} + \frac{n_i}{\sigma_\eta^2} + \sum_{j\neq i}\frac{1}{2\sigma_\omega^2+\sigma_\eta^2/n_j}}$$

where w_i is defined in Eq. (37). It is the Bayes weight on the evidence. Since the prior mean on ε is zero,

$$\varepsilon - x = \varepsilon - E\left(\varepsilon \mid \bar{\xi}_n, \bar{\boldsymbol{\xi}}_{n_j}\right) = -w_i\frac{\bar{\eta}_n\left(\frac{\sigma_\eta^2}{n}\right)^{-1} + \sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}(\omega_j-\omega_i+\eta_j)}{\left(\frac{\sigma_\eta^2}{n}\right)^{-1} + \sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}}$$

Similarly,

$$\varepsilon_j - x_j = -w_j\frac{\bar{\eta}_{n_j}\left(\frac{\sigma_\eta^2}{n_j}\right)^{-1} + \sum_{k\neq j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}(\omega_k-\omega_j+\eta_k)}{\left(\frac{\sigma_\eta^2}{n_j}\right)^{-1} + \sum_{k\neq j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}}$$

Since they are zero mean, $\mathrm{Cov}(\varepsilon-x)(\varepsilon_j-x_j) = E(\varepsilon-x)(\varepsilon_j-x_j)$.

For $i\neq j$, and using $E\left(\eta\eta_j\right) = E(\eta\omega) = 0$, we multiply the two expressions above to get

$$E(\varepsilon-x)(\varepsilon_j-x_j) = \\ w_iw_jE\left(\begin{array}{c}\frac{\bar{\eta}_n\left(\frac{\sigma_\eta^2}{n}\right)^{-1} + \sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}(\omega_j-\omega_i+\eta_j)}{\left(\frac{\sigma_\eta^2}{n}\right)^{-1} + \sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}}\times \\ \times\frac{\bar{\eta}_{n_j}\left(\frac{\sigma_\eta^2}{n_j}\right)^{-1} + \sum_{k\neq j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}(\omega_k-\omega_j+\eta_k)}{\left(\frac{\sigma_\eta^2}{n_j}\right)^{-1} + \sum_{k\neq j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}}\end{array}\right). \tag{A.4}$$

Now we deal first with the numerator in (A.4) and then with the denominator. The numerator reads,

$$
\begin{aligned}
& E\left\{\begin{array}{c}\left[\bar{\eta}_n\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n'}\right)^{-1}(\omega_j-\omega_i+\eta_j)\right]\times \\ \times\left[\bar{\eta}_{n_j}\left(\frac{\sigma_\eta^2}{n_j}\right)^{-1}+\sum_{k\neq j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}(\omega_k-\omega_j+\eta_k)\right]\end{array}\right\} \\
= \; & E\left[\sum_{k\neq j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}(\omega_k-\omega_j+\eta_k)\right]\left[\bar{\eta}_n\left(\frac{\sigma_\eta^2}{n}\right)^{-1}\right]+ \\
& +E\left[\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n'}\right)^{-1}(\omega_j-\omega_i+\eta_j)\right]\left[\bar{\eta}_{n_j}\left(\frac{\sigma_\eta^2}{n_j}\right)^{-1}\right] \\
& +E\left[\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}(\omega_j-\omega_i+\eta_j)\right]\left[\sum_{k\neq j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}(\omega_k-\omega_j+\eta_k)\right] \\
= \; & \left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_i}\right)^{-1}\left(\frac{\sigma_\eta^2}{n_i}\right)^{-1}\left(\frac{\sigma_\eta^2}{n_i}\right)+ \\
= \; & \left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}\left(\frac{\sigma_\eta^2}{n_j}\right)^{-1}\left(\frac{\sigma_\eta^2}{n_j}\right)+\sum_{k\neq i,j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}\frac{\sigma_\eta^2}{n_k} \\
= \; & \left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_i}\right)^{-1}+\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}+\sum_{k\neq i,j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-2}\left[\frac{\sigma_\eta^2}{n_k}+\sigma_\omega^2\right].
\end{aligned}
$$

Next, the denominator in (A.4) reads

$$
\left[\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n'}\right)^{-1}\right]\left[\left(\frac{\sigma_\eta^2}{n_j}\right)^{-1}+\sum_{k\neq j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}\right].
$$

Therefore,

$$
C_{ij}=E(\varepsilon-x)(\varepsilon_j-x_j)=w_iw_j\frac{\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_i}\right)^{-1}+\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}+\sum_{k\neq i,j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-2}\left[\frac{\sigma_\eta^2}{n_k}+\sigma_\omega^2\right]}{\left[\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n'}\right)^{-1}\right]\left[\left(\frac{\sigma_\eta^2}{n_j}\right)^{-1}+\sum_{k\neq j}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}\right]}
$$

which proves Eq. (36) in Lemma 2

When $i=j$, we have an extra common term ω_i to consider. Therefore,

$$
E(\varepsilon-x)^2 \quad = \tag{A.5}
$$

$$
=w_i^2E\frac{\bar{\eta}_n\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}(\omega_j-\omega_i+\eta_j)}{\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}}\frac{\bar{\eta}_n\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}(\omega_j-\omega_i+\eta_j)}{\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}}
$$

The numerator in Eq. (A.5) reads

$$E\left\{\begin{array}{c}\left[\bar{\eta}_n\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n'}\right)^{-1}(\omega_j-\omega_i+\eta_j)\right]\times\\ \left[\bar{\eta}_n\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}(\omega_j-\omega_i+\eta_j)\right]\end{array}\right\}$$

$$\begin{aligned}
&= \left(\frac{\sigma_\eta^2}{n}\right)\left(\frac{\sigma_\eta^2}{n}\right)^{-2}+E\left[\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}(\omega_j-\omega_i+\eta_j)\right]\left[\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}(\omega_j-\omega_i+\eta_j)\right]\\
&= \left(\frac{\sigma_\eta^2}{n}\right)\left(\frac{\sigma_\eta^2}{n}\right)^{-2}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-2}\left[2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right]+\sum_{j\neq i}\sum_{k\neq i}\sigma_\omega^2\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}\\
&= \left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}+\sum_{j\neq i}\sum_{k\neq i}\sigma_\omega^2\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}
\end{aligned}$$

so that

$$C_{ii}=w_i^2\frac{\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}+\sum_{j\neq i}\sum_{k\neq i}\sigma_\omega^2\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_k}\right)^{-1}}{\left[\left(\frac{\sigma_\eta^2}{n}\right)^{-1}+\sum_{j\neq i}\left(2\sigma_\omega^2+\frac{\sigma_\eta^2}{n_j}\right)^{-1}\right]^2}.$$

This proves Eq. (40) just following Lemma 2

Since the N-player vector $(\varepsilon_i)_{i=1}^{\mathrm{N}}$ is i.i.d. over time, ε is independent of $\theta-E(\theta)$. Therefore the variance of the sum of $\theta-E(\theta)$ and $\varepsilon_i-E(\varepsilon_i)$ is the sum of their variances. Therefore correlation coefficient between i and j is

$$\begin{aligned}
r_{ij} &= \frac{E(\theta+\varepsilon_i-x_i)(\theta+\varepsilon_j-x_j)}{\sqrt{E(\theta+\varepsilon_i-x_i)^2}\sqrt{E(\theta+\varepsilon_j-x_j)^2}}\\
&= \frac{E(\theta-E[\theta])^2+E(\varepsilon_i-x_i)(\varepsilon_j-x_j)}{\sqrt{E(\theta-E[\theta])^2+C_{ii}}\sqrt{E(\theta-E[\theta])^2+C_{ij}}}\\
&= \frac{\tau+C_{ij}}{\sqrt{\tau+C_{ii}}\sqrt{\tau+C_{ij}}}.
\end{aligned}$$

which is Eq. (44) in Lemma 3.

Checking that as the idiosyncrasy of ε's goes away the correlations converge to 1: As $\sigma_\omega\to 0$,

$$r_{ij}\to\frac{\left[\tau+w_iw_j\sum_i\left[\frac{\sigma_\eta^2}{n_k}\right]^{-1}\right]}{\sqrt{\tau+w_i^2\sum_i\left[\frac{\sigma_\eta^2}{n_k}\right]^{-1}}\sqrt{\tau+w_j^2\sum_i\left[\frac{\sigma_\eta^2}{n_k}\right]^{-1}}}=1.$$

The equality follows because $\lim_{\sigma_\omega\to 0}w_i=\lim_{\sigma_\omega\to 0}w_j$ for all i,j.

As for $\hat{\sigma}$, we have

$$\hat{\sigma} = Var\left(\frac{1}{N}\sum_{i=1}^{N}(\varepsilon_i - x_i)\right) = \frac{1}{N^2}\left(\sum_{i,j} C_{ij}\right)$$

which is Eq. (39) of Lemma 2.

4. Proof of Propositions 6 and 7

Since $\zeta \sim N(0,1) \Rightarrow \int \zeta^2 \frac{e^{-\zeta^2/2}}{\sqrt{2\pi}} d\zeta = 1$, (45) reads

$$\frac{a}{y+y'} + \frac{b}{y+y'} = \max_n \left\{-\frac{1}{y-n} + \beta l^2 (a+b)\right\}. \tag{A.6}$$

where Matching coefficients, we obtain

$$a = \frac{1+2\lambda}{2\lambda} \quad \text{and} \quad b = \frac{a}{\lambda}$$

Let

$$B = \beta\sigma_\eta^2 (a+b) < 0. \tag{A.7}$$

Then (A.6) reads

$$\frac{a}{y+y'} + \frac{b}{y+y'} = \max_n \left\{-\frac{1}{y-n} + \frac{B}{n+\varphi(y',y)}\right\} \tag{A.8}$$

The FOC is

$$-\frac{1}{(y-n)^2} - \frac{B}{(n+\varphi(y',y))^2} = 0,$$

i.e.,

$$\frac{1}{(y-n)^2} = \frac{-B}{(n+\varphi(y',y))^2}$$

Then

$$y - n = \frac{n+\varphi(y',y)}{\sqrt{-B}} = \lambda(n+\varphi(y',y))$$

where $-B > 0$,

$$\lambda = (-B)^{-1/2} > 0. \tag{A.9}$$

This gives

$$\varphi(y,y') = \frac{1}{1+\lambda}(y - \lambda\varphi(y',y))$$

Using symmetry of φ,

$$\varphi(y',y) = \frac{1}{1+\lambda}(y' - \lambda\varphi(y,y'))$$

This gives two equations in two unknowns. Eliminating $\varphi(y', y)$ leaves us with

$$\varphi(y, y') = \frac{1}{1+\lambda}\left(y - \lambda\frac{1}{1+\lambda}(y' - \lambda\varphi(y, y'))\right)$$

i.e.,

$$(1+\lambda)\varphi(y, y') = y - \frac{\lambda}{1+\lambda}(y' - \lambda\varphi(y, y')),$$

i.e.,

$$\left(1 + \lambda - \frac{\lambda^2}{1+\lambda}\right)\varphi(y, y') = y - \frac{\lambda}{1+\lambda}y'.$$

Now $1 + \lambda - \frac{\lambda^2}{1+\lambda} = \frac{1+2\lambda}{1+\lambda}$, and therefore

$$\frac{1+2\lambda}{1+\lambda}\varphi(y, y') = y - \frac{\lambda}{1+\lambda}y',$$

i.e.,

$$(1+2\lambda)\varphi(y, y') = (1+\lambda)y - \lambda y',$$

since $\lambda > 0$ shows the free-riding effect.

It remains to be seen whether (A.8) holds, i.e., whether when we substitute the policy φ for n on the RHS of (A.6), the Bellman equation holds for all (y, y').

$$\frac{a}{y+y'} + \frac{b}{y+y'} = -\frac{1}{y - \varphi(y, y')} - \frac{\lambda^{-2}}{\varphi(y, y') + \varphi(y', y)}. \tag{A.10}$$

The RHS of (A.10) reads

$$\begin{aligned}
\frac{a}{y+y'} + \frac{b}{y+y'} &= -\frac{1}{y - \frac{(1+\lambda)y - \lambda y'}{1+2\lambda}} - \frac{1+2\lambda}{\lambda^2(y+y')} \\
&= -(1+2\lambda)\left(\frac{1}{(1+2\lambda)y - (1+\lambda)y + \lambda y'} + \frac{1}{\lambda^2(y+y')}\right) \\
&= -(1+2\lambda)\left(\frac{1}{\lambda(y+y')} + \frac{1}{\lambda^2(y+y')}\right)
\end{aligned}$$

But λ itself is defined in terms of (a, b). From (A.7) and (A.9) we have

$$B = -\beta\sigma_\eta^2\left(\frac{1+2\lambda}{2\lambda} + \frac{1+2\lambda}{2\lambda^2}\right) = \frac{-\beta\sigma_\eta^2}{2\lambda^2}(1+2\lambda)(1+\lambda) \tag{A.11}$$

But from (A.9), $B = -\lambda^{-2}$. Substituting for B, we have

$$-\lambda^{-2} = -\beta\sigma_\eta^2\frac{(1+2\lambda)(1+\lambda)}{2\lambda^2}$$

or

$$2=\beta\sigma_\eta^2\left(1+2\lambda\right)\left(1+\lambda\right)$$

or

$$\beta\sigma_\eta^2 2\lambda^2+3\beta\sigma_\eta^2\lambda+\left(\beta\sigma_\eta^2-2\right)=0$$

or

$$\begin{aligned}
\lambda &= \frac{-3\beta\sigma_\eta^2\pm\sqrt{\left(3\beta\sigma_\eta^2\right)^2-4\left(2\beta\sigma_\eta^2\right)\left(\beta\sigma_\eta^2-2\right)}}{2\left(\beta\sigma_\eta^2 2\right)} \\
&= \frac{-3\beta\sigma_\eta^2\pm\sqrt{9\left(\beta\sigma_\eta^2\right)^2-8\beta\sigma_\eta^2\left(\beta\sigma_\eta^2-2\right)}}{4\beta\sigma_\eta^2} \\
&= \frac{-3\beta\sigma_\eta^2\pm\sqrt{\left(\beta\sigma_\eta^2\right)^2+16\beta\sigma_\eta^2}}{4\beta\sigma_\eta^2} \\
&= \frac{-3\pm\sqrt{1+\frac{16}{\beta\sigma_\eta^2}}}{4} \\
&= \frac{1}{4}\sqrt{1+\frac{16}{\beta\sigma_\eta^2}}-\frac{3}{4} \qquad \text{(A.12)}
\end{aligned}$$

From (A.9) we know that $\lambda>0$ which means that only the larger root is a valid solution and only when

$$1+\frac{16}{\beta\sigma_\eta^2}>9\Longleftrightarrow 2>\beta\sigma_\eta^2 \qquad \text{(A.13)}$$

which is always met when the individual problem has an interior solution for which (22) implies we need $\beta\sigma_\eta^2<1$. If the signals are poor, σ_η^2 is large and the inequality is violated. Investment is

$$\varphi\left(y,y'\right)=\frac{\left(1+\lambda\right)y-\lambda y'}{1+2\lambda}.$$

Proof of proposition 7

Re-write (68) as $\left(y_{+1}^{-1/2},y_{+1}'^{-1/2}\right)=\left(\delta\left|\zeta\right|,\delta'\left|\zeta'\right|\right)$, with $\left(\delta,\delta'\right)$ defined in (70). Then

$$\begin{aligned}
\Pr\left[\begin{pmatrix} y_{+1}^{-1/2} \\ y_{+1}'^{-1/2}\end{pmatrix}\leq\begin{pmatrix} a \\ a'\end{pmatrix}\right] &= \Pr\left[\begin{pmatrix} \left|\zeta\right| \\ \left|\zeta'\right|\end{pmatrix}\leq\begin{pmatrix} a/\delta \\ a'/\delta'\end{pmatrix}\right] \\
&= \Pr\left[-\begin{pmatrix} a/\delta \\ a'/\delta'\end{pmatrix}\leq\begin{pmatrix} \zeta \\ \zeta'\end{pmatrix}\leq\begin{pmatrix} a/\delta \\ a'/\delta'\end{pmatrix}\right] \\
&= \Phi\left(\frac{a}{\delta},\frac{a'}{\delta'}\right)-\Phi\left(-\frac{a}{\delta},-\frac{a'}{\delta'}\right)
\end{aligned}$$

where Φ is the bivariate standard normal distribution with correlation coefficient r which is the correlation between $\varepsilon - x$ and $\varepsilon' - x'$ given in (44). The density of (a, a') is

$$\frac{\partial^2}{\partial a \partial a'}\left[\Phi\left(\frac{a}{\delta}, \frac{a'}{\delta'}\right) - \Phi\left(-\frac{a}{\delta}, -\frac{a'}{\delta'}\right)\right] = \frac{2}{\delta\delta'}p\left(\frac{a}{\delta}, \frac{a'}{\delta'}\right)$$

Multiplying by the Jacobian of the transformation from $\left(y_{+1}^{-1/2}, y_{+1}'^{-1/2}\right)$ to $\left(y_{+1}, y_{+1}'\right)$ then leads to (71).

5. Numerical Algorithm for Fig. 13

1. Using basis functions guess $v^{(I)}(y, y')$ for each (y, y')
   2. (a) For each guess of $v^{(I)}(y, y')$, guess a function $\varphi^{(k,I)}(y, y')$ such that

$$\varphi^{(k+1,I)}(y, y') = \arg\max_{n \geq 0}\left\{\frac{(y-n)^{1-\gamma}}{1-\gamma} + \beta\int v\left((s\zeta)^{-2}, (l'\zeta')^{-2}\right)p(\zeta, \zeta')\, d\zeta d\zeta'\right\}$$
$$s.t. \qquad n' = \varphi^{(k)}(y', y)$$

Note the non-negativity restriction on n.

2. (b) the loop stops if

$$\sup\left\|\varphi^{(k+1,I)}(y, y') - \varphi^{(k,I)}(y, y')\right\| < \epsilon$$

denote the stopped policy function as $\varphi^{(I)}$

3. Using $\varphi^{(I)}(y, y')$, solve the value function

$$v^{(I+1)}(y, y') = \frac{\left(y - \varphi^{(I)}(y, y')\right)^{1-\gamma}}{1-\gamma} + \beta\int v^{(I)}\left((s\zeta)^{-2}, (l'\zeta')^{-2}\right)p(\zeta, \zeta')\, d\zeta d\zeta'$$

4. Stop the algorithm if

$$\sup\left\|v^{(I+1)}(y, y') - v^{(I)}(y, y')\right\| < \epsilon.$$

for some tolerance level $\epsilon > 0$

If not, go back to step 1 with the new guess of $v^{(I+1)}(y, y')$.

The second-order conditions check for the planner.—Following Fig. 12 we verified that the planner indeed wants to deviate upwards from equilibrium investment (Panel 1 in Fig. A.1) and that the second-order conditions to the planner's problem hold (Panel 2).

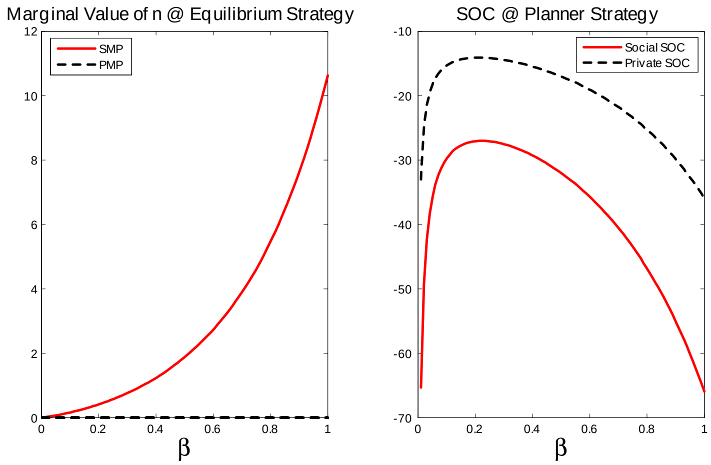


Figure A.1: Marginal value of raising investment from its equilibrium value, and at Second-order conditions check for the planner

6. Calibration based on the investment ratio relative to business-sector output

One interpretation of the model is that it really has two sectors: information and all other goods/services. In the starkest case we may think of two identities: $y_i = c_i + n$ and $y_0 = c_0 + x_0$, where i denotes the business sector and 0 denotes non-business services. GDP is $p_i y_i + p_0 y_0$. Utility is defined over both final goods. For an example of this type of accounting, see pp. 101-2 of McGrattan and Prescott (2010). So, if we measure investment relative not to total GDP but to business-sector output only – see McGrattan and Prescott (2010) for a justification – we obtain a targeted A of 0.31 compared to only .24 in Eq. (29). This gives rise to a slightly higher discount factor of $\beta = (.95)^{27} = 0.25$.

The details are as follows: The BLS (2015) reports that the business sector accounted for about 78 percent of the value of gross domestic product (GDP) in 2000. if n is financed from final goods only, whereas both business and non-business output is consumed, then we should equate A not to 0.24, but to

$$A = \frac{n_t}{y_{i,t} - y_{0,t}} = \frac{n_t}{(0.78)\, GDP_t} = \frac{.24}{.78} = .31$$

With the same two percent target for growth, we now have

$$1.02^T = 2.13 \frac{\beta^{1/2}}{\sigma}, \quad A = \sqrt{\beta \sigma_\eta^2} = 0.31, \text{ and } \beta = 0.95^T \tag{A.14}$$

This yields the parameter values

$$\beta = 0.25, \quad \sigma = 0.62 \quad \text{and} \quad T = 27.1$$

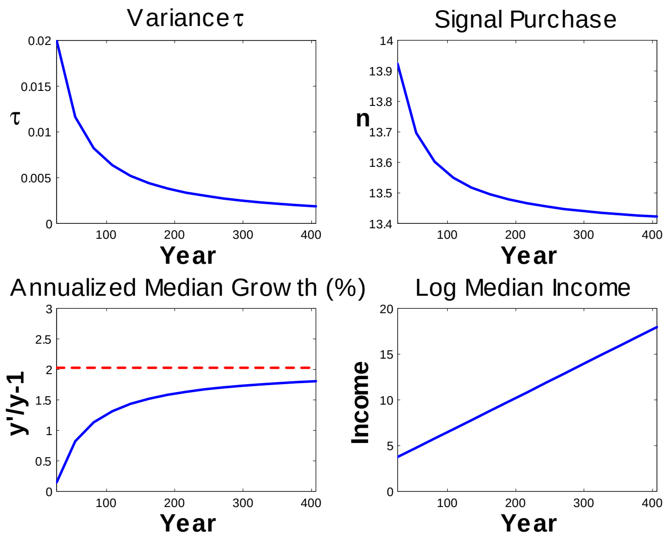


Figure A.2: Convergence in real time in the alternative calibration

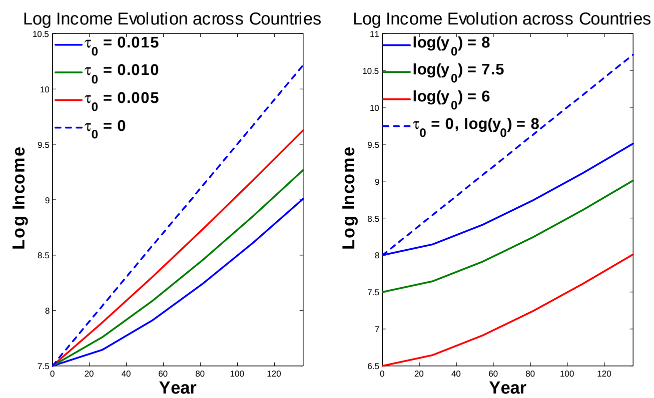


Figure A.3: Second calibration: Panel 1: Equal y_0. Panel 2: Equal τ_0

The outcome is depicted in the Appendix Figures A.2, A.3, and A.4. They are quite similar compared to Figs 2, 5 and 7. Although T is smaller, there is little qualitative difference between the model's implications.

7. Allowing for non-integer ρ

This appendix is in two parts. The first provides proof for the allowing for non-integer ρ. The second was written by Michal Fabinger. They use different reasoning to arrive at the conclusion that ρ and γ need not be integer valued. The idea is to use a production function of the form $y_t = |\theta + \varepsilon_t - x_t|^{-\rho}$ so that output remains positive. Both proofs assume that $c = y$.

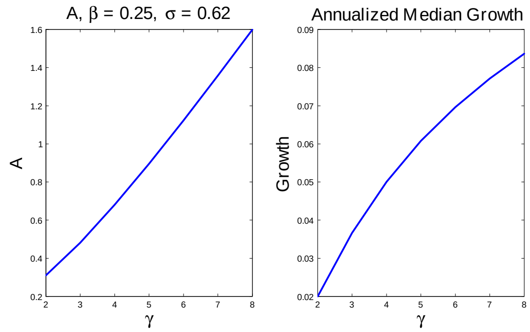


Figure A.4: THE EFFECT OF γ IN THE ALTERNATIVE CALIBRATION

7.1 Generalizing to non-integer values of the parameters

The output function can be expressed in terms of absolute values

$$y_t = |\theta + \epsilon_t - x_t|^{-\rho}$$

The following generalized Lemma 1 states that when $|z-x|^r$ is strictly convex in x for $r > 0$, then we have that

$$\arg\min_x \int |z-x|^r \, d\Phi\left(\frac{z-\mu}{\sigma_z}\right) = \mu$$

The first order condition with respect to x reads,

$$\begin{aligned} & -\frac{r}{2}\int \left[2\left(z-x\right)\right]\left|\left(z-x\right)^2\right|^{\frac{r}{2}-1} d\Phi\left(\frac{z-\mu}{\sigma_z}\right) \\ = & \quad -r\int \left(z-x\right)\left|\left(z-x\right)^2\right|^{\frac{r}{2}-1} d\Phi\left(\frac{z-\mu}{\sigma_z}\right) \\ = & \quad -r\int \frac{|z-x|^r}{z-x} d\Phi\left(\frac{z-\mu}{\sigma_z}\right) \end{aligned}$$

In order to show it is zero at $x = \mu$, we can transform this equation using $\xi = \frac{z-\mu}{\sigma_z}$, thus we have that

$$\begin{aligned} & r\int \frac{|z-x|^r}{z-x} d\Phi\left(\frac{z-\mu}{\sigma_z}\right) \\ = & \quad r\int \left(\sigma_z\xi + \mu - x\right)\left|\left(\sigma_z\xi + \mu - x\right)^2\right|^{\frac{r}{2}-1} d\Phi\left(\xi\right) \end{aligned}$$

Since the objective function is strictly convex, we guess that $x = \mu$ is the only solution, then we have

$$\begin{aligned}
& \int (\sigma_z \xi + \mu - x) \left|(\sigma_z \xi + \mu - x)^2\right|^{\frac{r}{2}-1} d\Phi(\xi) \\
= & \int \frac{|\sigma_z \xi|^r}{\sigma_z \xi} d\Phi(\xi) \\
= & \sigma_z^{r-1} \int \frac{|\xi|^r}{\xi} d\Phi(\xi) \\
= & \sigma_z^{r-1} \int \frac{|\xi|^r}{\xi} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi = 0
\end{aligned}$$

The last equality follows from the fact that $\frac{|\xi|^r}{\xi} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}$ is an odd function. Therefore, the first order condition is satisfied at $x = \mu$ as $r \int \frac{|z-x|^r}{z-x} d\Phi\left(\frac{z-\mu}{\sigma_z}\right) = 0$.

The generalized version of Lemma 1 gives the optimal action for a forward looking agent as long as the objective is concave in x.

$$U(x) = \frac{1}{1-\gamma} |\theta - x_t + \varepsilon_t|^{\rho(\gamma-1)}.$$

We can write the objective function in an alternative form as follows,

$$U(x) = \frac{1}{1-\gamma} \left|(\theta - x_t + \varepsilon_t)^2\right|^{\frac{\rho(\gamma-1)}{2}}$$

Thus we have that

$$\begin{aligned}
\frac{\partial U}{\partial x} &= \frac{\rho(\gamma-1)}{2} \frac{1}{1-\gamma} \left|(\theta - x_t + \varepsilon_t)^2\right|^{\frac{\rho(\gamma-1)}{2}-1} \left[-2(\theta - x_t + \varepsilon_t)\right] \\
&= \rho(\theta - x_t + \varepsilon_t) \left|(\theta - x_t + \varepsilon_t)^2\right|^{\frac{\rho(\gamma-1)}{2}-1}
\end{aligned}$$

Now the second derivative is

$$\begin{aligned}
\frac{\partial^2 U}{\partial x^2} &= \rho(\theta - x_t + \varepsilon_t) \left|(\theta - x_t + \varepsilon_t)^2\right|^{\frac{\rho(\gamma-1)}{2}-1} \\
&= -\rho \left|(\theta - x_t + \varepsilon_t)^2\right|^{\frac{\rho(\gamma-1)}{2}-1} - 2\rho\left(\frac{\rho(\gamma-1)}{2} - 1\right)(\theta - x_t + \varepsilon_t)^2 \left|(\theta - x_t + \varepsilon_t)^2\right|^{\frac{\rho(\gamma-1)}{2}-2} \\
&= -\left[\rho + 2\rho\left(\frac{\rho(\gamma-1)}{2} - 1\right)\right] \left|(\theta - x_t + \varepsilon_t)^2\right|^{\frac{\rho(\gamma-1)}{2}-1} \\
&= -\left[\rho^2(\gamma-1) - \rho\right] \left|(\theta - x_t + \varepsilon_t)^2\right|^{\frac{\rho(\gamma-1)}{2}-1}
\end{aligned}$$

i.e., this is concave if

$$\rho^2(\gamma-1) - \rho > 0$$

Since x maximizes a strictly concave function and, in light of the independence of θ and ε, since $\theta + \varepsilon \sim N\left(m + \bar{\xi}_{\mathrm{n}}, \tau + \sigma_\eta^2/n\right)$, we can set $z = \theta + \varepsilon$ and apply the generalized Lemma 1 to conclude that

$$x = m + E\left(\varepsilon \mid \bar{\xi}_{\mathrm{n}}\right).$$

Then,

$$\theta + \varepsilon - x \sim N\left(0, \tau + \frac{\sigma_\eta^2}{n}\right),$$

so that

$$\zeta = \left(\tau + \frac{\sigma_\eta^2}{n}\right)^{-1/2} (\theta + \varepsilon - x) \sim N(0,1).$$

This implies that

$$(\theta + \varepsilon - x) = \zeta \left(\tau + \frac{\sigma_\eta^2}{n}\right)^{1/2}$$

Then, we have that we can now re-write the output

$$\begin{aligned} y &= \left|\zeta \left(\tau + \frac{\sigma_\eta^2}{n}\right)^{1/2}\right|^{-\rho} \\ &= \left(\tau + \frac{\sigma_\eta^2}{n}\right)^{-\rho/2} |\zeta|^{-\rho} \end{aligned}$$

Then the Bellman equation can be written

$$v(y,\tau) = \max_{n \geq 0} \left\{ \frac{(y-n)^{1-\gamma}}{1-\gamma} + \beta \int v\left(y', b(\tau, n)\right) dP\left(y' \mid \tau, n\right) \right\}, \tag{A.15}$$

We need to know the distribution of $|\xi|$, which follows the Folded Normal distribution

$$v(y,\tau) = \max_{n} \left\{ \frac{(y-n)^{1-\gamma}}{1-\gamma} + \beta \int_0^\infty v\left(\left(\tau + \frac{\sigma_\eta^2}{n}\right)^{-\rho/2} |\zeta|^{-\rho}, b(\tau,n) \right) \left(\sqrt{\frac{2}{\pi}} e^{-\zeta^2/2} \right) d|\zeta| \right\}.$$

on the support $|\xi| \in [0, \infty)$. The folded normal distribution has pdf

$$f = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\xi-\mu)^2}{2\sigma_\eta^2}} + \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\xi+\mu)^2}{2\sigma_\eta^2}}.$$

7.2 Generalizing to non-integer values of the parameters, Alternative Proof by Michal Fabinger

Generalized Lemma 1. Let r be a positive real number. Then

$$\arg\min_x \int_{-\infty}^{\infty} |z-x|^r d\Phi\left(\frac{z-\mu}{\sigma_z}\right) = \mu.$$

Here z, x, μ are σ_z are understood to be real and σ_z positive.

Proof. The integral on the left-hand-side may be evaluated explicitly. The result is

$$\frac{2^{r/2}\sigma_z^r}{\sqrt{\pi}}\Gamma\left(\frac{r+1}{2}\right) \times {}_1F_1\left(-\frac{r}{2};\frac{1}{2};-\frac{(x-\mu)^2}{2\sigma_z{}^2}\right)$$

where $\Gamma(.)$ is the gamma function and ${}_1F_1(\ .\ ;\ .\ ;\ .\ )$ is the Kummer confluent hypergeometric function of the first kind, as in

https://en.wikipedia.org/wiki/Confluent_hypergeometric_function

The first factor $\frac{2^{r/2}\sigma_z^r}{\sqrt{\pi}}\Gamma\left(\frac{r+1}{2}\right)$ is a positive constant. We just need to minimize the second factor, which may be written as

$${}_1F_1\left(-\frac{r}{2};\frac{1}{2};-y\right), \quad y \equiv \frac{(x-\mu)^2}{2\sigma_z{}^2} \geq 0.$$

Its derivative with respect to y

$$\frac{d}{dy}\,{}_1F_1\left(-\frac{r}{2};\frac{1}{2};-y\right) = r\ \ {}_1F_1\left(1-\frac{r}{2};\frac{3}{2};-y\right)$$

is positive for any $r > 0$ and $y \geq 0$, which means that the minimum of the original function is achieved for $y = 0$, i.e. $x = \mu$.

8. N agents: Proof of Proposition 4, and feasible growth

Let $\chi = \left(\frac{1}{2} - \frac{1}{2N}\right) + \sqrt{\left(\frac{1+N}{2N}\right)^2 + \frac{(N-1)}{N\beta\sigma_\eta^2}}$ so that in (56) $B = \frac{\chi^{-2}(N-1)}{\beta\sigma_\eta^2}$ or, equivalently, $\chi = \left(\frac{\beta\sigma_\eta^2 B}{N-1}\right)^{-\frac{1}{2}}$. Note that when $N = 2$, $\chi = \lambda$, and therefore the results in Appendix 4 in fact follow from the (more complicated) development and results in this Appendix – by plugging in $N = 2$ into the results of Appendix 8, we have the results in Appendix 4. .

Proof of Proposition 4

From the first order conditions,

$$y - n = \frac{n + \sum_{j\neq i}\varphi\left(y_j, \mathbf{y}_{-j}\right)}{\sqrt{\frac{\beta\sigma_\eta^2 B}{N-1}}}.$$

Then we have

$$y - n = \chi\left(n + \sum_{j\neq i}\varphi\left(y_j, \mathbf{y}_{-j}\right)\right),$$

or

$$\varphi\left(y,\mathbf{y}'\right)=\frac{1}{1+\chi}\left(y-\chi\sum_{j\neq i}\varphi\left(y_j,\mathbf{y}_{-j}\right)\right).$$

Now substitute from (52) to obtain

$$\begin{aligned}\varphi\left(y,\mathbf{y}'\right) &= \frac{1}{1+\chi}\left(y-\chi\sum_{j\neq i}\left(ay_j+b\sum_{s\neq j}y_s\right)\right)\\ &= \frac{1}{1+\chi}y-\frac{\chi}{1+\chi}\sum_{j\neq i}\left(ay_j+b\sum_{s\neq j}y_s\right)\end{aligned}$$

Now note that

$$\sum_{j\neq i}\sum_{s\neq j}y_s=\left(N-1\right)y_i+\left(N-2\right)\sum_{j\neq i}y_j$$

Therefore,

$$\begin{aligned}\varphi\left(y,\mathbf{y}'\right) &= \frac{1}{1+\chi}y-\frac{\chi}{1+\chi}\sum_{j\neq i}\left(ay_j+b\sum_{s\neq j}y_s\right)\\ &= \frac{1}{1+\chi}y-\frac{a\chi}{1+\chi}\sum_{j\neq i}y_j-\frac{\chi b}{1+\chi}\sum_{j\neq i}\sum_{s\neq j}y_s\\ &= \frac{1}{1+\chi}y-\frac{a\chi}{1+\chi}\sum_{j\neq i}y_j-\frac{\chi b}{1+\chi}\left(\left(N-1\right)y_i+\left(N-2\right)\sum_{j\neq i}y_j\right)\end{aligned}$$

or

$$\varphi\left(y,\mathbf{y}'\right)=\left[\frac{1}{1+\chi}-\frac{\chi b}{1+\chi}\left(N-1\right)\right]y_i-\left[\frac{a\chi}{1+\chi}+\frac{\chi b\left(N-2\right)}{1+\chi}\right]\sum_{j\neq i}y_j$$

Matching the coefficients, we have

$$a=\left[\frac{1}{1+\chi}-\frac{\chi b}{1+\chi}\left(N-1\right)\right]\quad\text{and}\quad b=-\left[\frac{a\chi}{1+\chi}+\frac{\chi b\left(N-2\right)}{1+\chi}\right]$$

The latter reads

$$\begin{aligned}b &= -\left(\frac{\left[\frac{1}{1+\chi}-\frac{\chi b}{1+\chi}\left(N-1\right)\right]\chi}{1+\chi}+\frac{\chi b\left(N-2\right)}{1+\chi}\right)\\ &= -\frac{\left[1-\chi b\left(N-1\right)\right]\chi}{\left(1+\chi\right)^2}-\frac{\chi b\left(N-2\right)}{1+\chi}\\ &= -\frac{\chi}{\left(1+\chi\right)^2}+\frac{\chi^2 b\left(N-1\right)}{\left(1+\chi\right)^2}-\frac{\chi b\left(N-2\right)}{1+\chi},\end{aligned}$$

i.e.,

$$\left(1-\frac{\chi^2\left(N-1\right)}{\left(1+\chi\right)^2}+\frac{\chi\left(N-2\right)}{1+\chi}\right)b=-\frac{\chi}{\left(1+\chi\right)^2}$$

Notice that

$$\begin{aligned}
&\frac{(1+\chi)^2-\chi^2(N-1)+(1+\chi)\chi(N-2)}{(1+\chi)^2}\\
=\;&\frac{1+2\chi+\chi^2-\chi^2N+\chi^2+\chi(N-2)+\chi^2(N-2)}{(1+\chi)^2}\\
=\;&\frac{1+\chi N}{(1+\chi)^2},
\end{aligned}$$

so that

$$\frac{1+\chi N}{(1+\chi)^2}b=-\frac{\chi}{(1+\chi)^2},$$

or,

$$b=-\frac{\chi}{(1+\chi)^2}\frac{(1+\chi)^2}{1+\chi N}=-\frac{\chi}{1+\chi N}.$$

This implies that,

$$a=\frac{1}{1+\chi}-\frac{\chi b}{1+\chi}(N-1)=\frac{(1+\chi N)+\chi^2(N-1)}{(1+\chi)(1+\chi N)}=\frac{1+\chi N+\chi^2(N-1)}{(1+\chi)(1+\chi N)}.$$

We thus have that

$$\varphi(y,\mathbf{y}')=\frac{1+\chi N+\chi^2(N-1)}{(1+\chi)(1+\chi N)}y-\frac{\chi}{1+\chi N}\sum_{i\neq j}y_j,$$

where

$$\chi=\left(\frac{\beta\sigma_\eta^2B}{N-1}\right)^{-\frac{1}{2}} \tag{A.16}$$

As $\chi>0$, the free riding effect exists, and also as N increases, $\frac{\chi}{1+\chi N}$ decreases.

Derivation of (54).—We have

$$\sum_{i=1}^{N}\varphi(y_i,\mathbf{y}_{-\mathbf{i}})=(a+b(N-1))\sum_{s=1}^{N}y_s,$$

where

$$\begin{aligned}
a+b(N-1)\;&=\;\frac{1+\chi N+\chi^2(N-1)}{(1+\chi)(1+\chi N)}-\left(\frac{\chi(N-1)}{1+\chi N}\right)\\
&=\;\frac{1+\chi N+\chi^2(N-1)-\chi(1+\chi)(N-1)}{(1+\chi)(1+\chi N)}\\
&=\;\frac{1}{1+\chi N},
\end{aligned}$$

which implies (54).

Lastly, notice that Note that

$$\begin{aligned}
& 1-a \\
= \;& 1-\frac{1+\chi N+\chi^2(N-1)}{(1+\chi)(1+\chi N)} \\
= \;& \frac{(1+\chi)(1+\chi N)-\left[1+\chi N+\chi^2(N-1)\right]}{(1+\chi)(1+\chi N)} \\
= \;& \frac{1+\chi+\chi N+\chi^2 N-1-\chi N-\chi^2(N-1)}{(1+\chi)(1+\chi N)} \\
= \;& \frac{\chi+\chi^2}{(1+\chi)(1+\chi N)}=\frac{\chi}{1+\chi N}=-b
\end{aligned}$$

It remains to see if the value function (55) holds at the allocations. Using the form in (55), we have

$$-B\left(\sum_{s=1}^{N} y_s\right)^{-1}=-\frac{1}{y_i-n_i}-\beta\frac{\sigma_\eta^2 B}{N-1}\left(n+\sum_{j\neq i}\varphi\left(y_j,\mathbf{y}_{-j}\right)\right)^{-1}.$$

Since $n=\varphi\left(y_i,\mathbf{y}_{-\mathbf{i}}\right)=ay_i+b\sum_{i\neq j}y_j$, we have

$$\begin{aligned}
& n+\sum_{j\neq i}\varphi\left(y_j,\mathbf{y}_{-j}\right)=ay_i+b\sum_{i\neq j}y_j+\sum_{j\neq i}\left(ay_j+b\sum_{s\neq j}y_s\right) \\
= \;& \left(a\sum_{s=1}^{N}y_s\right)+b\sum_{i\neq j}y_j+b\sum_{j\neq i}\sum_{s\neq j}y_s \\
= \;& \left(a\sum_{s=1}^{N}y_s\right)+b\sum_{i\neq j}y_j+b\left((N-1)\,y_i+(N-2)\sum_{j\neq i}y_j\right) \\
= \;& \left(a\sum_{s=1}^{N}y_s\right)+b\,(N-1)\,y_i+b\,(N-1)\sum_{j\neq i}y_j=\left[a+b\,(N-1)\right]\sum_{s=1}^{N}y_s
\end{aligned}$$

In addition,

$$y-n=y-ay_i-b\sum_{i\neq j}y_j=(1-a)\,y_i-b\sum_{i\neq j}y_j=-b\sum_{s=1}^{N}y_s,$$

because $1-a=-b$. Therefore,

$$\begin{aligned}
B\left(\sum_{s=1}^{N}y_s\right)^{-1} \;&=\; -\frac{1}{b}\left(\sum_{s=1}^{N}y_s\right)^{-1}+\frac{\beta\sigma_\eta^2 B}{N-1}\left(\left[a+b\,(N-1)\right]\sum_{s=1}^{N}y_s\right)^{-1} \\
&=\; \left(-\frac{1}{b}+\frac{\beta\sigma_\eta^2 B}{(N-1)\left[a+b\,(N-1)\right]}\right)\left(\sum_{s=1}^{N}y_s\right)^{-1}
\end{aligned}$$

Matching coefficient we thus have,

$$B = -\frac{1}{b} + \frac{\beta\sigma_\eta^2 B}{(N-1)\left[a + b(N-1)\right]}.$$

Using (A.16), we then have $B = \frac{\chi^{-2}(N-1)}{\beta\sigma_\eta^2}$. We then have

$$\frac{(N-1)}{\beta\sigma_\eta^2} = \frac{\chi^2}{\frac{\chi}{1+\chi N}} + \frac{1}{\frac{1+\chi N+\chi^2(N-1)}{(1+\chi)(1+\chi N)} - \frac{\chi}{1+\chi N}(N-1)}$$

Simplifying,

$$\begin{aligned}
\frac{(N-1)}{\beta\sigma_\eta^2} &= \chi(1+\chi N) + \frac{(1+\chi)(1+\chi N)}{1+\chi N+\chi^2(N-1) - \chi(1+\chi)(N-1)} \\
&= \chi(1+\chi N) + (1+\chi N) = \chi + \chi^2 N + 1 + \chi N,
\end{aligned}$$

This implies that

$$\begin{aligned}
B &= \frac{\chi^{-2}(N-1)}{\beta\sigma_\eta^2} \\
&= \chi^{-2}\left[\chi(1+\chi N) + (1+\chi N)\right] \\
&= \frac{1+\chi N}{\chi N}\left(1 + \frac{1}{\chi}\right)
\end{aligned}$$

This delivers (56)

Plugging $B = \frac{\chi^{-2}(N-1)}{\beta\sigma_\eta^2}$ into the above equation we get the quadratic equation in χ:

$$N\chi^2 + (1+N)\chi + 1 - \frac{(N-1)}{\beta\sigma_\eta^2} = 0.$$

The solution to this equation is

$$\begin{aligned}
\chi &= \frac{-(1+N) \pm \sqrt{(1+N)^2 - 4N\left(1 - \frac{N}{\beta\sigma_\eta^2}\right)}}{2N} \\
&= \frac{-N - 1 \pm \sqrt{1 - 2N + N^2 + \frac{4N^2}{\beta\sigma_\eta^2}}}{2N} \\
&= \left(-\frac{1}{2} - \frac{1}{2N}\right) + \sqrt{\left(\frac{1-N}{2N}\right)^2 + \frac{1}{\beta\sigma_\eta^2}} \\
&= \frac{1}{2}\left(\sqrt{\left(\frac{N-1}{N}\right)^2 + \frac{4}{\beta\sigma_\eta^2}} - 1 - \frac{1}{N}\right),
\end{aligned}$$

and $B = \frac{\chi^{-2}(N-1)}{\beta\sigma_\eta^2}$.

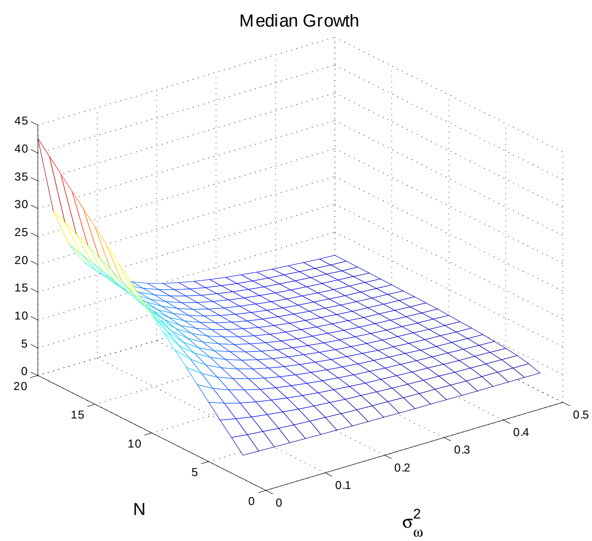


Figure A.5: FEASIBLE GROWTH FOR $\sigma_\omega^2 \in [0,1]$ AND VARIOUS $N \geq 2$

Then

$$\chi > 0 \Longleftrightarrow \left(1 - \frac{1}{N}\right)^2 + \frac{4}{\beta\sigma_\eta^2} > \left(1 + \frac{1}{N}\right)^2 \Longleftrightarrow \beta\sigma_\eta^2 < N,$$

i.e., (51).

When $N = 2$, we have

$$\begin{aligned} \chi &= -\frac{3}{4} + \sqrt{\frac{1}{16} + \frac{1}{\beta\sigma_\eta^2}} \\ &= -\frac{3}{4} + \frac{1}{4}\sqrt{1 + \frac{16}{\beta\sigma_\eta^2}} \\ &= \lambda \end{aligned}$$

and

$$B = \frac{1 + 2\lambda}{2\lambda}\left(1 + \frac{1}{\lambda}\right)$$

which satisfies the two-agent solution.

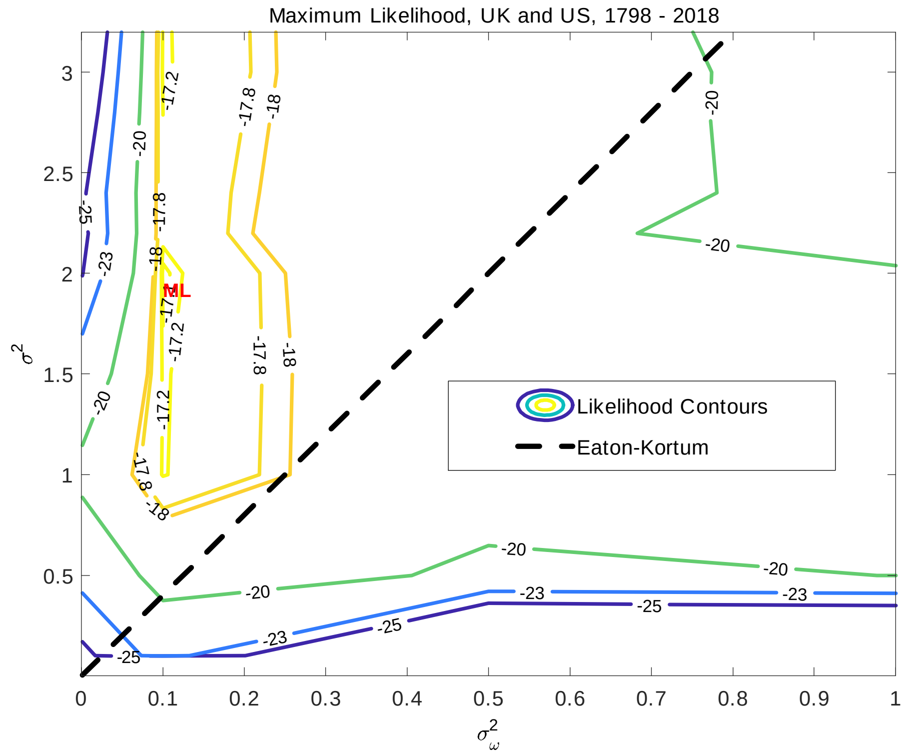
Feasible median growth

Figures A.5 show that with more agents, they all can grow faster – a statement about feasibility, not equilibrium.

Estimated MLs based on the US and UK data

Figure A.6 reports the contour plot of the ML estimates using the US and UK data, described in section 3.

Figure A.6: ML ESTIMATE $\left(\sigma_\eta^2 = 1.8, \sigma_\omega^2 = 0.12\right)$ AND LIKELIHOOD CONTOURS



9. Model with Capital

In this section, we incorporate physical capital k in the single-agent model.

Production.—Let θ, ε_t and x_t be as in Eq. (31). Output is

$$y_t = Z_t^{1-\alpha} k_t^\alpha = (\theta + \varepsilon_t - x_t)^{-\rho(1-\alpha)} k_t^\alpha,$$

where $\rho(1-\alpha) > 0$ is an even integer and k is physical capital. Fig. 1 now illustrates Z.

Investment.—Output can be invested in physical capital k_t and in information relevant for *next* period's production decisions. The income identity is

$$y_t = c_t + n_t + \underbrace{k_{t+1} - (1-\delta) k_t}_{i_t},$$

where as before n_t is the number of independent signals on ε_{t+1}, call these signals, $\left(\xi_1, ..., \xi_{n,t}\right)$. Each ξ_i is an independent signal on next period's ε as follows:

$$\xi_i = \varepsilon_{t+1} + \eta_i,$$

where $\eta_i \sim N\left(0, \sigma_\eta^2\right)$ are i.i.d. random variables.

Choice of x.—There are no adjustment costs to changing x. Therefore the history of x is irrelevant once beliefs are specified. Since the choice of x does not affect the information about θ or ε, the agent chooses it so as to maximize his current period expected utility conditional on the beginning-of-period priors over θ, which is $N(m, \tau)$, and over ε which is $N\left(\bar{\xi}_{\mathrm{n}}, \sigma_\eta^2/n\right)$. Let $U = \frac{1}{1-\gamma}\left[(\theta - x_t + \varepsilon_t)^{-\rho(1-\alpha)} k_t^\alpha - n_t - i_t\right]^{1-\gamma}$.

From Lemma 1, we have $\theta+\varepsilon-x \sim N\left(0, \tau + \frac{\sigma_\eta^2}{n}\right)$, so that $\zeta = \left(\tau + \frac{\sigma_\eta^2}{n}\right)^{-1/2}(\theta + \varepsilon - x) \sim N(0,1)$. This fact allows us to express next-period output as a function of τ, n and ζ alone, and

$$y' = \left(\tau + \frac{\sigma_\eta^2}{n}\right)^{-\rho(1-\alpha)/2} \zeta^{-\rho(1-\alpha)} k'^\alpha,$$

where ζ is a standard normal variable and y' now depends on k'.

Choice of n.—Let us evaluate the agent's lifetime utility just after the realization of y, but before he chooses n. The agent's belief over θ is normal $N(m, \tau)$, but m is not a state because, in light of (9), the agent offsets it by x. Moreover, ε is i.i.d.. The state is the pair (y, τ, k). Lifetime expected utility, $v(y, \tau, k)$, and optimal investment, $n(y, \tau)$ satisfy

$$v(y, \tau, k) = \max_{n \geq 0, i}\left\{\frac{(y - n - i)^{1-\gamma}}{1-\gamma} + \beta \int v(y', b(\tau, n), (1-\delta)k + i)\, dP(y' \mid \tau, n)\right\},$$

with $b(\cdot)$ defined in (7). Using (10), this becomes

$$v(y, \tau, k) = \max_{n,i}\left\{\begin{array}{c} \frac{(y-n-i)^{1-\gamma}}{1-\gamma} + \\ \beta \int v\left(\left(\tau + \frac{\sigma_\eta^2}{n}\right)^{-\rho(1-\alpha)/2} \zeta^{-\rho(1-\alpha)} k'^\alpha, b(\tau, n), (1-\delta)k + i\right) \frac{1}{\sqrt{2\pi}} e^{-\zeta^2/2} d\zeta \end{array}\right\}.$$

The following proposition provides the analytical solution with $\delta = 1$. It also proves Proposition 2 which assumes away k so that $\alpha = 0$. Let

$$h_2 = (1-\alpha)\, r(\alpha)\, \sigma_\eta^{2(\gamma-1)(1-\alpha)} \left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}$$

where $r(\alpha) = \int \zeta^{2(\gamma-1)(1-\alpha)} \phi(\zeta)\, d\zeta$. The following proposition shows the policy and value functions.

Proposition 1 *When $\rho = 2$ and $\tau = 0$, and when $h_2 < 1$ holds , the policy and value functions are*

$$\begin{aligned} n &= Ay, \quad \textit{and} \\ k' &= \chi y, \quad \textit{and} \\ v(y) &= v y^{1-\gamma}, \end{aligned}$$

for scalars A, χ and v given by

$$A = \left(\beta\left(1-\alpha\right)r\left(\alpha\right)\sigma_\eta^{2(\gamma-1)(1-\alpha)}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}\right)^{\frac{1}{\gamma}}, \tag{A.17}$$

$$\chi = \frac{\alpha}{1-\alpha}A, \quad \textit{and} \tag{A.18}$$

$$v = \frac{1}{1-\gamma}\left(1-\frac{1}{1-\alpha}A\right)^{-\gamma}. \tag{A.19}$$

Similar to the one player case, when $h_2 \geq 1$, consumption is zero or negative and the solutions are not valid.

Proof. When $\delta = 1$ and $\tau = 0$, the only state variable is y

$$v\left(y\right) = \max_{n,k'}\left\{\frac{\left(y-n-k'\right)^{1-\gamma}}{1-\gamma} + \beta\int v\left(\left(\frac{\sigma_\eta^2}{n}\right)^{-(1-\alpha)}\zeta^{-2(1-\alpha)}k\prime^\alpha\right)\frac{1}{\sqrt{2\pi}}e^{-\zeta^2/2}d\zeta\right\}$$

Using the guess, we have

$$vy^{1-\gamma} = \max_{n,k'}\left\{\frac{\left(y-n-k'\right)^{1-\gamma}}{1-\gamma} + \beta v\int\left(\left(\sigma_\eta^2\right)^{-(1-\alpha)}\zeta^{-2(1-\alpha)}k\prime^\alpha\right)^{1-\gamma}n^{(1-\alpha)(1-\gamma)}\frac{1}{\sqrt{2\pi}}e^{-\zeta^2/2}d\zeta\right\}$$

Now we denote

$$r\left(\alpha\right) = \int\zeta^{2(\gamma-1)(1-\alpha)}\phi\left(\zeta\right)d\zeta \rightarrow_{\alpha\rightarrow 0}\left(2\gamma-3\right)!!.$$

Next, verify that $n = Ay$ will satisfy the FOC. The FOC w.r.t. n then reads

$$-\left(y-n-k'\right)^{-\gamma} + \left(\alpha-1\right)\left(1-\gamma\right)vr\left(\alpha\right)\beta\sigma_\eta^{2(\gamma-1)(1-\alpha)}n^{(1-\alpha)(1-\gamma)-1}\left(k'\right)^{\alpha(1-\gamma)} = 0.$$

Substituting $n = Ay$ and $k' = \chi y$, it reads

$$\left(1-A-\chi\right)^{-\gamma}y^{-\gamma} = \left(\alpha-1\right)\left(1-\gamma\right)vr\left(\alpha\right)\beta\sigma_\eta^{2(\gamma-1)(1-\alpha)}\left(Ay\right)^{(1-\alpha)(1-\gamma)-1}\left(\chi y\right)^{\alpha(1-\gamma)}.$$

Then we have

$$\left(1-A-\chi\right)^{-\gamma}y^{-\gamma} = \left(1-\alpha\right)\left(1-\gamma\right)vr\left(\alpha\right)\beta\sigma_\eta^{2(\gamma-1)(1-\alpha)}\left(A\right)^{(1-\alpha)(1-\gamma)-1}\left(\chi\right)^{\alpha(1-\gamma)}y^{(1-\alpha)(1-\gamma)-1+\alpha(1-\gamma)}.$$

Now we note that

$$\left(1-\alpha\right)\left(1-\gamma\right)-1+\alpha\left(1-\gamma\right) = -\gamma.$$

Our guess on $v\left(y\right)$ is verified.

$$\left(1-A-\chi\right)^{-\gamma} = \beta\left(1-\alpha\right)\left(1-\gamma\right)vr\left(\alpha\right)\sigma_\eta^{2(\gamma-1)(1-\alpha)}\left(A\right)^{(1-\alpha)(1-\gamma)-1}\left(\chi\right)^{\alpha(1-\gamma)}. \tag{A.20}$$

For the FOC w.r.t. k', we have

$$-\left(y-n-k'\right)^{-\gamma} + \beta\alpha\left(1-\gamma\right)vr\left(\alpha\right)\sigma_\eta^{2(\gamma-1)(1-\alpha)}n^{(1-\alpha)(1-\gamma)}\left(k'\right)^{(1-\gamma)\alpha-1} = 0.$$

Substituting $n = Ay$ and $k' = \chi y$, it reads

$$(1-A-\chi)^{-\gamma} = \beta\alpha\,(1-\gamma)\,vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\,(A)^{(1-\alpha)(1-\gamma)}\,(\chi)^{\alpha(1-\gamma)-1}\,, \tag{A.21}$$

The last equation can be used to solve for v,

$$vy^{1-\gamma} = \left\{\frac{(1-A-\chi)^{1-\gamma}}{1-\gamma}y^{1-\gamma} + \beta v\int\left(\left(\frac{\sigma_\eta^2}{n}\right)^{-(1-\alpha)}\zeta^{-2(1-\alpha)}k\prime^{\alpha}\right)^{1-\gamma}\frac{1}{\sqrt{2\pi}}e^{-\zeta^2/2}d\zeta\right\},$$

or equivalently

$$\begin{aligned} vy^{1-\gamma} &= \frac{(1-A-\chi)^{1-\gamma}}{1-\gamma}y^{1-\gamma} + \beta vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\left(n^{(1-\alpha)}k\prime^{\alpha}\right)^{1-\gamma} \\ &= \frac{(1-A-\chi)^{1-\gamma}}{1-\gamma}y^{1-\gamma} + \beta vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\,(A)^{(1-\alpha)(1-\gamma)}\,(\chi)^{\alpha(1-\gamma)}\,y^{1-\gamma}, \end{aligned}$$

or equivalently

$$v = \frac{(1-A-\chi)^{1-\gamma}}{1-\gamma} + \beta vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\,(A)^{(1-\alpha)(1-\gamma)}\,(\chi)^{\alpha(1-\gamma)}\,. \tag{A.22}$$

We now have three equations to solve the 3 unknowns A, χ, v

$$\begin{aligned} (1-A-\chi)^{-\gamma} &= \beta\,(1-\alpha)\,(1-\gamma)\,vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\,(A)^{(1-\alpha)(1-\gamma)-1}\,(\chi)^{\alpha(1-\gamma)} \\ (1-A-\chi)^{-\gamma} &= \beta\alpha\,(1-\gamma)\,vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\,(A)^{(1-\alpha)(1-\gamma)}\,(\chi)^{\alpha(1-\gamma)-1} \\ v &= \frac{(1-A-\chi)^{1-\gamma}}{1-\gamma} + \beta vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\,(A)^{(1-\alpha)(1-\gamma)}\,(\chi)^{\alpha(1-\gamma)}\,. \end{aligned}$$

From the first two equations, we have

$$\begin{aligned} &\beta\,(1-\alpha)\,(1-\gamma)\,vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\,(A)^{(1-\alpha)(1-\gamma)-1}\,(\chi)^{\alpha(1-\gamma)} \\ =\;& \beta\alpha\,(1-\gamma)\,vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\,(A)^{(1-\alpha)(1-\gamma)}\,(\chi)^{\alpha(1-\gamma)-1}\,, \end{aligned}$$

or equivalently

$$(1-\alpha)\,\chi = \alpha A.$$

From the first equation, we have that

$$\left(\frac{1}{A} - 1 - \frac{\alpha}{1-\alpha}\right)^{-\gamma} = \beta\,(1-\alpha)\,(1-\gamma)\,vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}.$$

This implies that

$$\frac{1}{A} - 1 - \frac{\alpha}{1-\alpha} = \left[\beta\,(1-\alpha)\,(1-\gamma)\,vr\,(\alpha)\,\sigma_\eta^{2(\gamma-1)(1-\alpha)}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}\right]^{-\frac{1}{\gamma}},$$

or equivalently

$$A=\left\{\left[\beta\left(1-\alpha\right)\left(1-\gamma\right)r\left(\alpha\right)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}\right]^{-\frac{1}{\gamma}}+\frac{1}{1-\alpha}\right\}^{-1}.$$

Now from the third equation, we have

$$v=\frac{\left(\frac{1}{A}-1-\frac{\alpha}{1-\alpha}\right)^{1-\gamma}}{1-\gamma}A^{1-\gamma}+\beta vr\left(\alpha\right)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}\left(A\right)^{1-\gamma}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}.$$

This implies that

$$v=\frac{\frac{\left(\frac{1}{A}-1-\frac{\alpha}{1-\alpha}\right)^{1-\gamma}}{1-\gamma}A^{1-\gamma}}{1-\beta r\left(\alpha\right)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}\left(A\right)^{1-\gamma}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}}.$$

Form the first equation again, we have

$$\begin{aligned}\left(\frac{1}{A}-1-\frac{\alpha}{1-\alpha}\right)^{-\gamma} &= \beta\left(1-\alpha\right)\left(1-\gamma\right)r\left(\alpha\right)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)} \\ &\quad \frac{\frac{\left(\frac{1}{A}-1-\frac{\alpha}{1-\alpha}\right)^{1-\gamma}}{1-\gamma}A^{1-\gamma}}{1-\beta r\left(\alpha\right)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}\left(A\right)^{1-\gamma}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}}\end{aligned}$$

or equivalently

$$\frac{1}{\frac{1}{A}-1-\frac{\alpha}{1-\alpha}}=\beta\left(1-\alpha\right)r\left(\alpha\right)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}\frac{A^{1-\gamma}}{1-\beta r\left(\alpha\right)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}\left(A\right)^{1-\gamma}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}}$$

or equivalently

$$\frac{A}{1-\frac{1}{1-\alpha}A}=\beta\left(1-\alpha\right)r\left(\alpha\right)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}\frac{A^{1-\gamma}}{1-\frac{(1-\alpha)\beta r(\alpha)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}}{(1-\alpha)}\left(A\right)^{1-\gamma}}$$

This equality will hold if

$$1=\beta\left(1-\alpha\right)r\left(\alpha\right)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}\left(A\right)^{-\gamma}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}.$$

This yields (A.17). And since $\chi=\frac{\alpha}{1-\alpha}A$, this yields (A.18).

Using the equation $1=\beta\left(1-\alpha\right)r\left(\alpha\right)\sigma_{\eta}^{2(\gamma-1)(1-\alpha)}A^{-\gamma}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\gamma)}$, we have that

$$v=\frac{1}{1-\gamma}\frac{\left(1-\frac{1}{1-\alpha}A\right)^{1-\gamma}}{1-\frac{1}{1-\alpha}A}=\frac{1}{1-\gamma}\left(1-\frac{1}{1-\alpha}A\right)^{-\gamma},$$

i.e., (A.19) ■

10. Miracles *vs.* disasters

The production function $y_t = (\theta + \varepsilon_t - x_t)^{-\rho}$ in Eq. (2) does not let an agent stabilize his income by repeatedly choosing the same value of x. This is because ε_t fluctuates. By contrast, in Jovanovic and Ma (2022), an agent *can* repeatedly choose the same technology and repeatedly get the same output. Output is

$$y = \exp\left\{A - \frac{\lambda}{2}\left(s_A - h'\right)^2\right\}, \tag{A.23}$$

where h' is the agent's skill level and s_A is a random variable the value of which depends on the level of technology denoted by A. The agent can raise A by setting $x > 0$ in the law of motion for A:

$$A' = A + x. \tag{A.24}$$

Raising A, however, is risky because it may shift s_A further away from where the agent's skill level is:

$$s_{A'} = s_A + x\varepsilon, \tag{A.25}$$

where $\varepsilon \sim F(\varepsilon)$.

The agent can change h to $h' \equiv h + \Delta$ at an adjustment cost of

$$C(y, \Delta) \equiv \left[1 - \exp\left\{-\frac{\theta}{2}\Delta^2\right\}\right] y.$$

Finally, the agent's net output is

$$y - C(y, \Delta) = \exp\left\{A - \frac{\lambda}{2}(u - \Delta)^2 - \frac{\theta}{2}\Delta^2\right\}, \tag{A.26}$$

where

$$u \equiv s_A - h' = \text{ "skill gap."} \tag{A.27}$$

In the left panel of Fig. A.7, the *convexity* of the production function in $(\theta + \varepsilon - x)$, as in this paper, means that a large mistake is unlikely to cause a disaster in that output is bounded from below whereas reducing the mistake to zero raises output without bound. By contrast, in the right panel of Fig. A.7, we have *concavity,* as in Jovanovic and Ma (2022), and then the opposite occurs, a large mistake creates a disaster whereas a small mistake does not cause a miracle in that its effect on output is bounded.

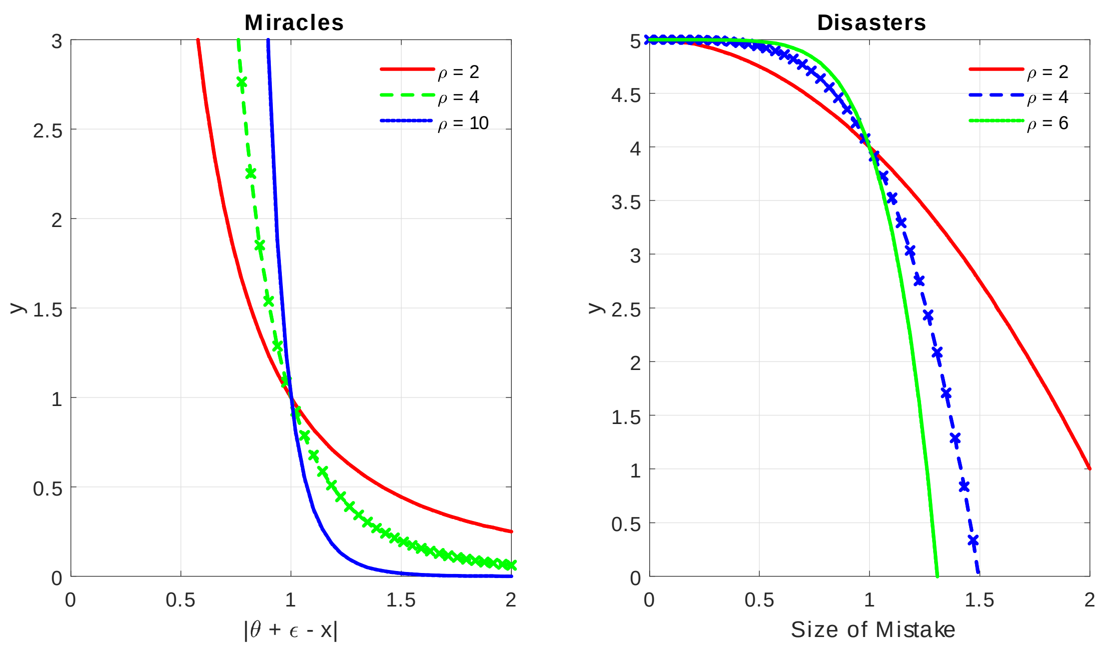


Figure A.7: Thicker right tail (left panel), thicker left tail (right panel)