Online Appendix for “Rescue Policies for Small Businesses in the COVID-19 Recession”

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A PPP Loan Terms

Eligibility The following requirements apply to first draw PPP loans:

- The business was operational before February 15, 2020, and is still open and operational.
- There are no more than 500 employees. If a business has multiple locations, there must be no more than 500 employees per location.

Loan Amount The maximum amount of a PPP loan is determined based on the average monthly payroll up to a threshold. The maximum amount of first draw PPP loans is 2.5 times the average monthly payroll up to \$10 million. In the Accommodation and Food services sector, it is 3.5 times the average monthly payroll up to \$10 million. For non-employers, the maximum amount is based on the business net profit.

Interest Rate PPP loans have an interest rate of 1%.

Maturity PPP loans issued after June 5, 2020, have a maturity of five years. PPP loans issued prior to June 5, 2020, have a maturity of two years. However, by mutual agreement, the maturity can be extended to 5 years.

Covered Period The covered period is the 8- or 24-week after loan disbursement. The 24-week period applies to all borrowers that received forgiveness prior to December 27, 2020, but borrowers that received an SBA loan number before June 5, 2020, have the option to use an eight-week period.

Conditions for Loan Forgiveness Full loan forgiveness is possible if, during the 8- or 24-week covered period following loan disbursement, all of the following are achieved:

- Employee and compensation levels are maintained (temporary layoff is allowed if the workers are rehired),

- all loan proceeds are spent on payroll costs and other eligible expenses, and
- at least 60% of the proceeds are spent on payroll costs.

Partial Forgiveness "If a borrower uses less than 60% of the loan amount for payroll costs during the forgiveness covered period, the borrower will continue to be eligible for partial loan forgiveness, subject to at least 60 percent of the loan forgiveness amount having been used for payroll costs."[1]

B Targeted Grant

In addition to the baseline grant and the targeted grant described in Section 5.2, we consider two additional targeted grant policies with different grant amounts. In "targeted grant small", we set $X_p^{small} = 0.5X_p$ and in "targeted grant large", we set $X_p^{large} = 6.32X_p$, where X_p is the amount of the grant as a fraction of the quarterly payroll in the baseline grant and the targeted grant scenarios in Section 5.2. The grant amount in "targeted grant large" is chosen such that the fiscal cost of the grant is the same as that in the baseline grant.

Table B.1 compares the cost-effectiveness of the baseline grant to the three targeted grants. As the grant amount increases, the targeted grant becomes less cost-effective. In particular, a targeted grant as costly as the untargeted baseline grant turns out to be less cost effective, even in comparison to the baseline grant. Intuitively, while a larger targeted grant saves more impacted firms and positively affects investment and labor in those firms, it does so with a diminishing impact at the margin.

	Baseline grant	Targeted grant small	Targeted grant	Targeted grant large
Cost (Frac. GDP)	4.99%	0.39%	0.78%	4.99%
Emp. save (Frac. Emp)	0.76%	0.32%	0.39%	0.64%
Cost per perc. jobs saved	6.54%	1.23%	2.02%	7.75%

Notes: The fiscal cost is computed as a fraction of GDP in the steady state. Employment saved is computed as the per-quarter difference in small-firm employment relative to the laissez-faire economy over a 10 year period from the onset of the pandemic.

Table B.1: Cost per Job Saved in Small Firms

Table B.2 shows that the large targeted grant is the most effective in rescuing firms that would had exited without any grant, but it also generates the most zombie firms.

[1] Source: https://home.treasury.gov/news/press-releases/sm1026

Comparing the effect of the two policies on small firm investment, we find that the large targeted grant is less effective in preventing capital liquidation at the exit margin than the equally-costly baseline grant in the first two quarters of the pandemic. It is also not as effective in preventing active firms from under-investing as the baseline grant.

	Baseline grant	Targeted grant	Targeted grant large
Fraction of saved firms	47.1%	68.5%	89.8%
Fraction of zombie firms	1.35%	0.36%	1.80%

Notes: Fraction of saved firms is the measure of saved firms in period $t = 0$ divided by the measure of firms that exit under the laissez-faire environment, conditional on receiving the grant. Fraction of zombie firms is the measure of zombie firms divided by the measure of saved firms. See Section 5.3 for definitions of saved firms and zombie firms.

Table B.2: Zombie Firms

C Conditional Grant

As a robustness check, we assume that the grant comes with a minimum employment requirement in period $t = 0$ that resembles the condition for forgiveness in the PPP program (see Online Appendix A).[2] That is, firms that choose to take up the grant must maintain employment at 60% of the pre-pandemic level. The grant is only given in period $t = 0$, as in our benchmark model, and the minimum employment condition only applies in period $t = 0$. This is consistent with the length of the covered period in the PPP program that ranges from 8 to 24 weeks. Other than the minimum employment requirement, all policy assumptions are the same as in Section 3.3.

Specifically, an η fraction of small firms have the opportunity to take up the grant. They choose to do so if $v_0(x, b, k, \iota, s = 1) \geq v_0(x, b, k, \iota, s = 0)$, where $s \in \{0, 1\}$ indicates grant take-up and $\iota \in \{0, 1\}$ is the impact status. The profit of a firm that takes up the grant in $t = 0$ changes from equation (27) to

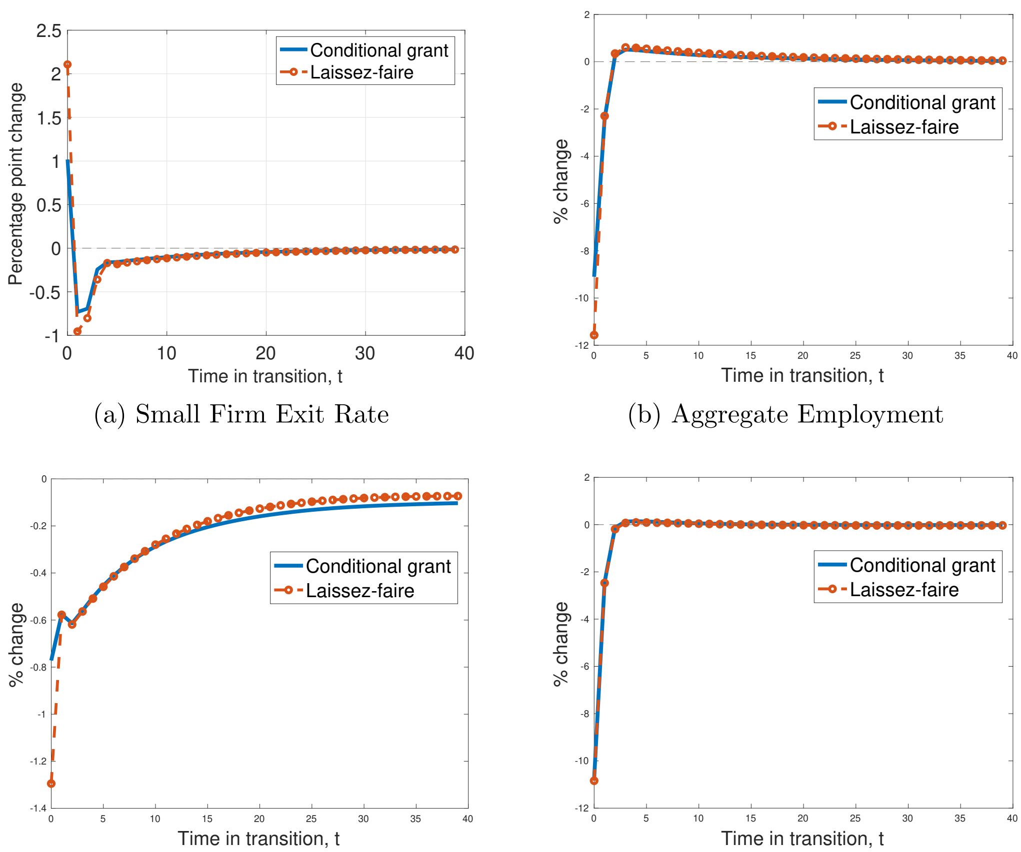
$$\begin{aligned} \pi_0(x, b, k, \iota, s = 1) = \max_{\ell} (1 + \nu_0^n \iota) x f(k, \ell) + b_p(x, k) - w_0 \ell - c^f(k) \\ \text{such that} \quad \ell \geq 0.6 \times \ell(x, k) \end{aligned}$$

where $b_p(x, k)$ is the grant amount defined in equation (26), $\ell(x, k)$ is the labor demand in the steady state, and $0.6 \times \ell(x, k)$ is the minimum employment requirement.

[2] For simplicity, we do not model the option of not fulfilling the condition, thus abandoning forgiveness.

We simulate the economy with the conditional grant using the same calibrated parameter values reported in Section 4.2. The results indicate that, although the minimum employment condition stimulates short-run employment in small firms, the conditional grant delivers similar medium- and long-run impacts on aggregate output, employment, and capital as the baseline, unconditional grant. Thus, the main conclusions of the paper carry over. The conditional grant modeled here can be viewed as an extreme case because, in practice, the conditionality of the PPP loan forgiveness is not strictly enforced and partial forgiveness is possible.

Figures C.1 and C.2 show the impulse responses to a number of variables of interest, comparing the laissez-faire to the conditional grant. Figures C.3 and C.4 show the short- and long-run impact of the conditional grant policy. In comparison to the impulse responses under the unconditional grant, the conditional grant dampens the initial employment decline while mitigating the initial decline in firm exits. This is because employment in staying impacted firms remains higher while other impacted firms prefer to exit rather than accepting the conditional grant. From period $t = 1$ onward, however, the observed reallocation effects are similar to those under the unconditional grant scenario.



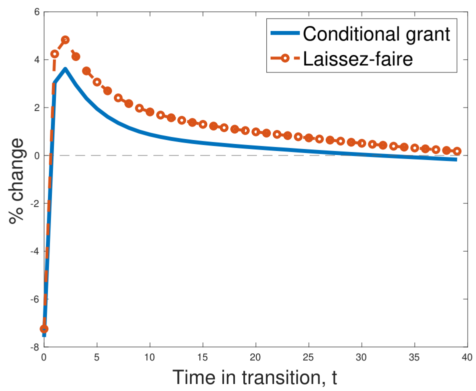
(a) Small Firm Exit Rate

(b) Aggregate Employment

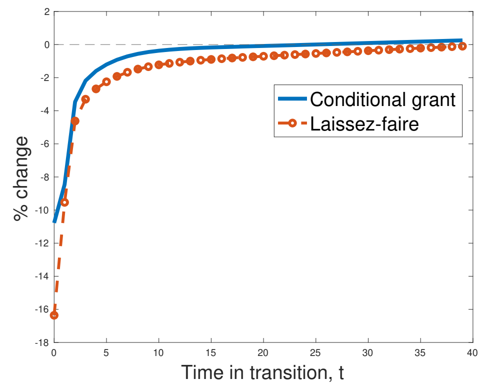
(c) Aggregate Capital

(d) Aggregate Output

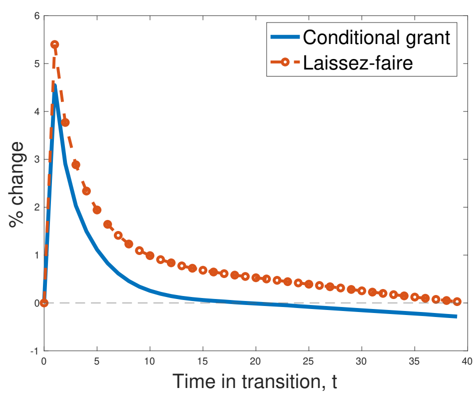
Figure C.1: Impulse Response to the Pandemic Shock (Conditional Grant)



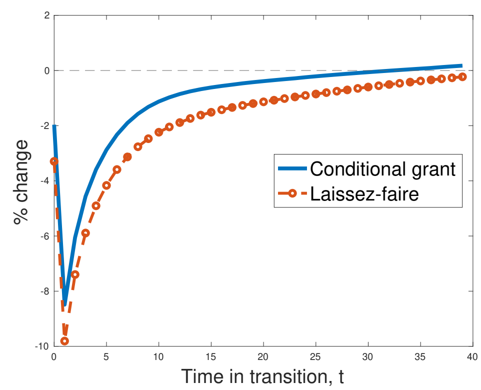
(a) Employment in the Corporate Sector



(b) Employment in Small Firms

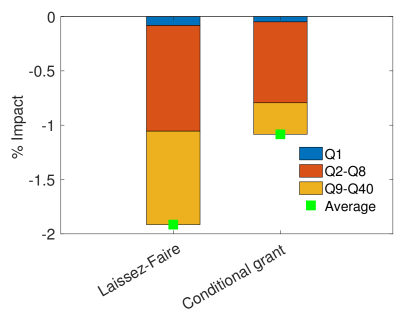


(c) Capital in the Corporate Sector

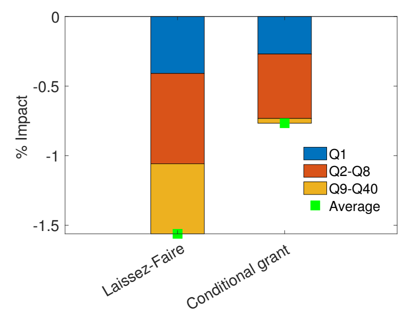


(d) Capital in Small Firms

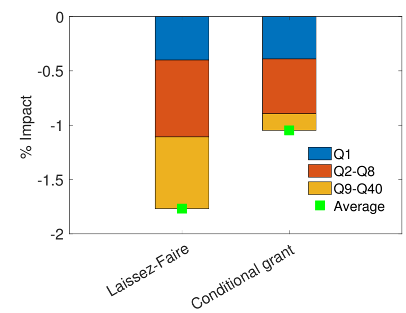
Figure C.2: Impulse Response to the Pandemic Shock by Sector (Conditional Grant)



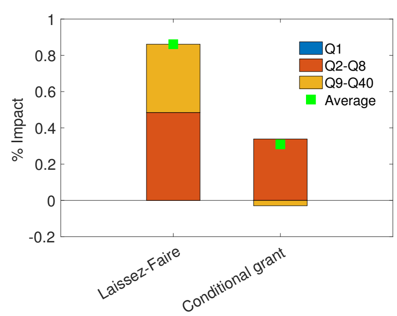
(a) Small Firm Capital



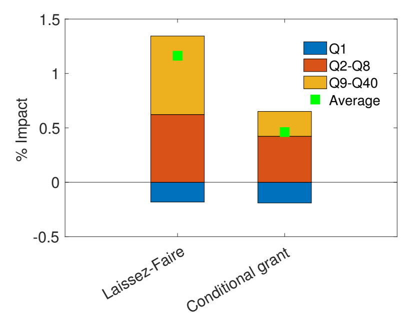
(b) Small Firm Employment



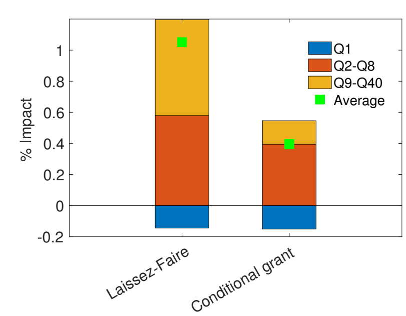
(c) Small Firm Output



(d) Corporate Capital



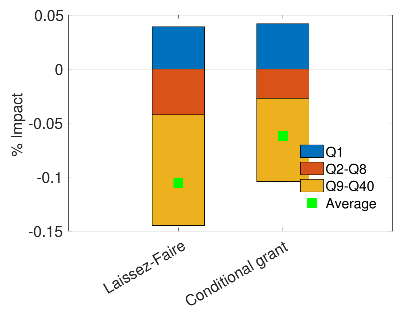
(e) Corporate Employment



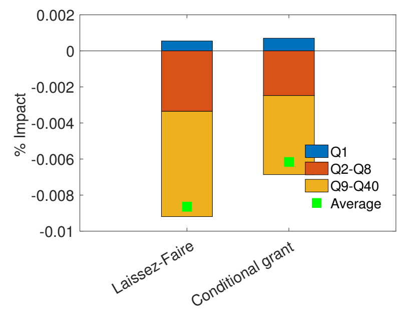
(f) Corporate Output

Notes: The net height of the bars represents the average quarterly effect of the pandemic over 10 years. Short-run: first two quarters; medium-run = quarter 3 to 8; long-run = quarter 9 to 40.

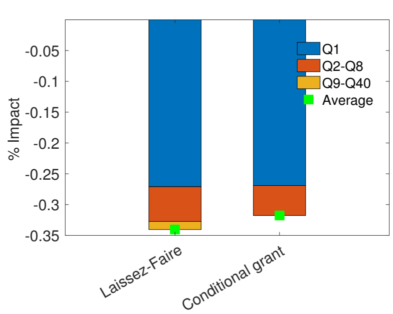
Figure C.3: Short- and Long-Run Effects (Conditional Grant)



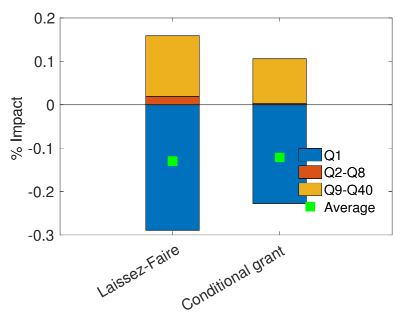
(a) Wage, w_t



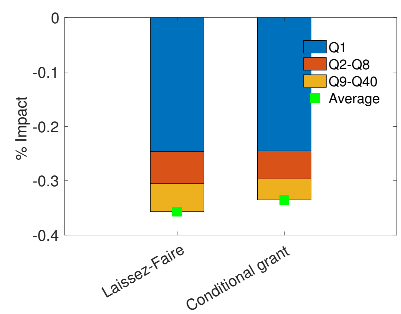
(b) Price of Financial Asset, q_t



(c) Aggregate Output



(d) Aggregate Employment

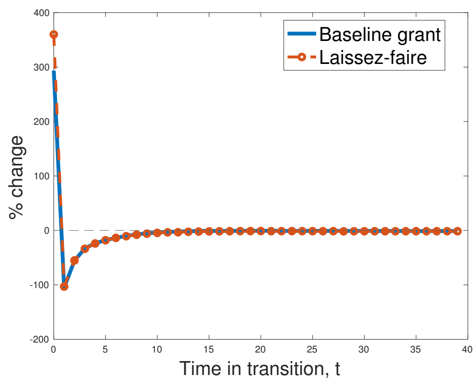


(e) Aggregate Consumption

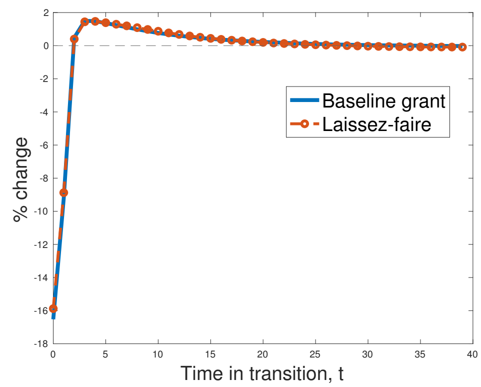
Figure C.4: Short- and Long-Run Effects (Conditional Grant, Other Variables)

D Further Details

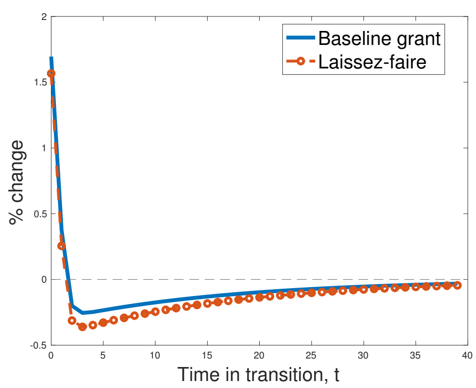
D.1 Additional Figures



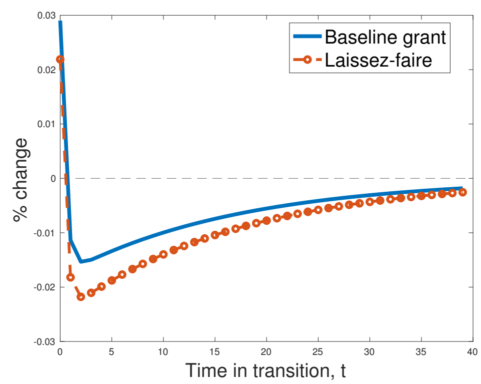
(a) Corporate Sector Investment, I_t^c



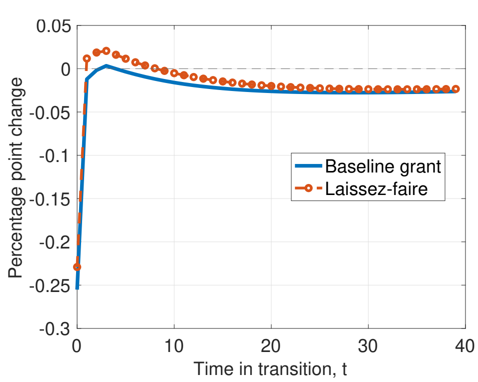
(b) Aggregate Investment, I_t



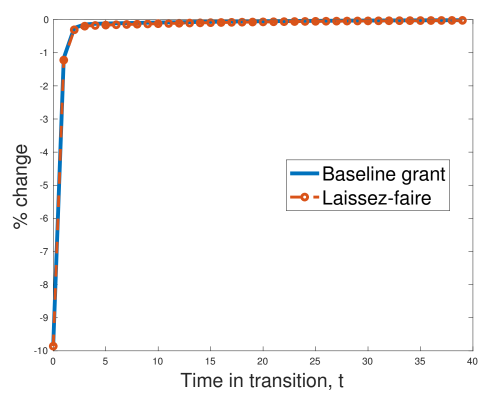
(c) Wage, w_t



(d) Price of Financial Asset, q_t



(e) Entry rate



(f) Aggregate Consumption, C_t

Figure D.1: IRFs: Other variables of interest

D.2 Short- and Long-Run Decomposition

We decompose the total cumulative effect of the pandemic shock in a 10-year period into short-run, medium-run, and long-run effects. Specifically, let $irf_t = \frac{y_t - y_{ss}}{y_{ss}}$ be the percentage effect of the pandemic on variable y in period t, and let $\overline{irf}$ be the average effect of the pandemic on a variable over a period of T_{LR} periods, i.e.

$$\overline{irf} = \frac{1}{T_{LR}} \sum_{t=0}^{T_{LR}-1} irf_t \ .$$

We compute $\overline{irf}_{SR}, \overline{irf}_{MR}$ and $\overline{irf}_{LR}$ such that

$$\overline{irf} = \overline{irf}_{SR} + \overline{irf}_{MR} + \overline{irf}_{LR},$$

where

$$\overline{irf}_{SR} = \frac{1}{T_{LR}} \sum_{t=0}^{T_{SR}-1} irf_t \ ,$$

$$\overline{irf}_{MR} = \frac{1}{T_{LR}} \sum_{t=T_{SR}}^{T_{MR}-1} irf_t \ ,$$

$$\overline{irf}_{LR} = \frac{1}{T_{LR}} \sum_{t=T_{MR}}^{T_{LR}-1} irf_t \ .$$

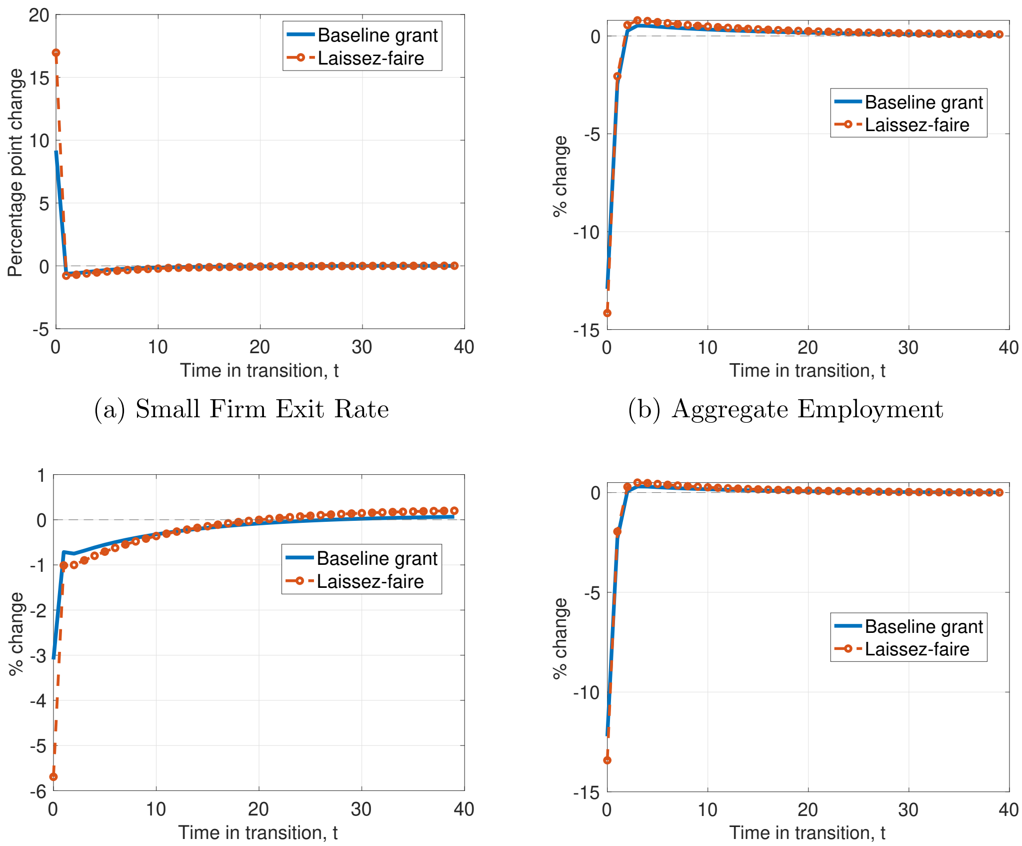
We consider the short-run to be the first quarter post-pandemic, the medium-run to be quarter 2 to year 3, and the long-run to be year 4 to year 10.

E Robustness: Alternative Calibration of the Productivity Process

We vary ε_x and $\bar{x}$ to match the job creation and job destruction rates, while holding other parameter values the same. The new ε_x is 0.025, smaller that in the original calibration. We change the value of $\bar{x}$ to 1.40 so that the small-firm share of employment is consistent with the data. Table E.1 shows the fit of this alternative calibration. With a smaller ε_x, we are not able to generate enough dispersion in the firm size distribution such that the average firm size is significantly smaller than in the benchmark model. Nevertheless, the main conclusions of the model remain. Figure E.1 shows that the baseline grant reduces the sharp increase in firm exit at the onset of the pandemic, leading to a smaller drop in aggregate capital. But the grant has only modest effects on aggregate employment and output at the onset of the pandemic, and small effects after the initial impact of the pandemic. As in the benchmark calibration, the grant also significantly dampens the medium- and long-run reallocation of production factors from the small-firm sector to the corporate sector (Figure E.2).

Moment	Data	Model
Job creation rate	0.0250	0.0268
Job destruction rate	0.0215	0.0298
Average employment in small firms	9.2516	4.1666
Small firm share of employment	0.4895	0.4504
Small firm exit rate	0.0198	0.0126
Average employment, age 0	5.2935	3.4815
Fixed expense to revenue ratio	0.2448	0.3994
Autocorr. employment	0.9667	0.9882
Time spent in market work	0.3300	0.3302
Exit rate, emp. size 0 to 9	0.0246	0.0127
serial corr. investment rate	0.0580	0.1588

Table E.1: Model Fit (Alternative Calibration)



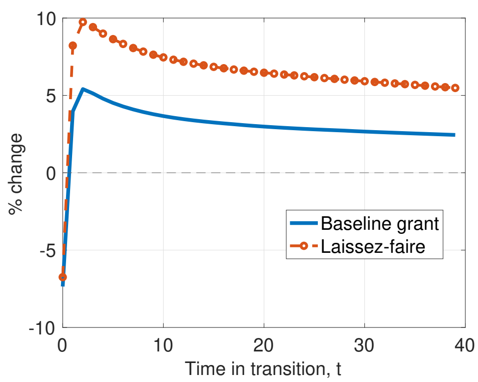
(a) Small Firm Exit Rate

(b) Aggregate Employment

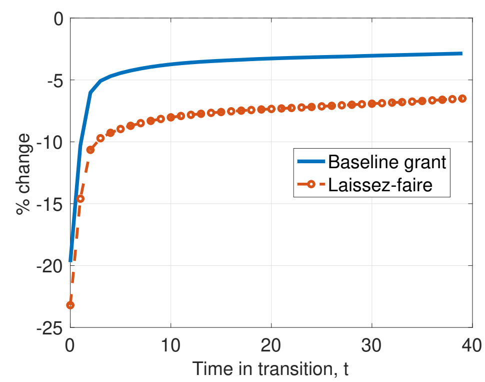
(c) Aggregate Capital

(d) Aggregate Output

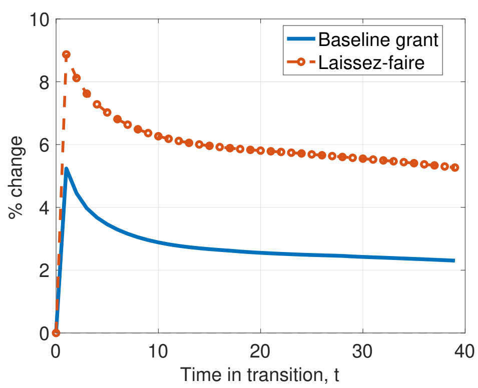
Figure E.1: Impulse Response to the Pandemic Shock (Alternative Calibration)



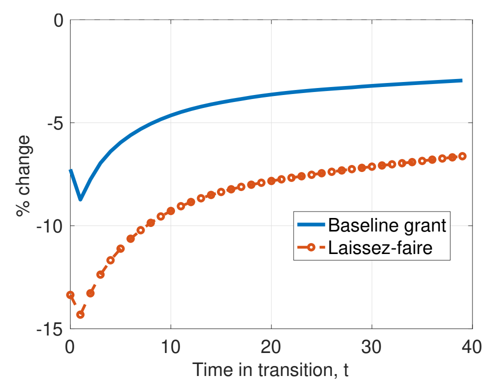
(a) Employment in the Corporate Sector



(b) Employment in Small Firms



(c) Capital in the Corporate Sector



(d) Capital in Small Firms

Figure E.2: Impulse Response to the Pandemic Shock by Sector (Alternative Calibration)

F Computation

F.1 Computation of the Stationary Equilibrium

The financial discount factor is $q = \beta$. The capital-labor ratio in the corporate sector is pinned down from $1 = \beta[1 - \delta + F_K]$. Using the functional form $F(K^c, L^c) = A(K^c)^\alpha (L^c)^{1-\alpha}$, we obtain

$$1 = q\left[1 - \delta + A\alpha\left(\frac{K^c}{L^c}\right)^{\alpha-1}\right] ,$$

from which we obtain the capital-labor ratio in corporate firms

$$\kappa^* = \frac{K^c}{L^c} = \left(\frac{A\alpha}{\frac{1}{q} - 1 + \delta}\right)^{\frac{1}{1-\alpha}} .$$

The real wage follows from $w = F_L$, i.e.

$$w = A\left(1 - \alpha\right)\left(\kappa^*\right)^\alpha .$$

Consumption follows from the first order condition $u_C\left(C, 1 - L\right) w = u_{1-L}\left(C, 1 - L\right)$, i.e.

$$C^\sigma = \frac{w}{\zeta} .$$

This permits the calculation of $\pi\left(x, k\right) = \max_{\ell \geq 0} xf(k, \ell) - w\ell - c^f(x)$. The optimal labor demand can be derived as

$$\ell\left(x, k\right) = \left(\frac{w}{xA\left(1 - \gamma_1\right)\gamma_2}\right)^{\frac{1}{(1-\gamma_1)\gamma_2 - 1}} k^{-\frac{\gamma_1\gamma_2}{(1-\gamma_1)\gamma_2 - 1}} .$$

Given $\pi\left(x, k\right)$, the value of an unconstrained firm net of debt $V(x, k)$ can be obtained from (18), which allows us to compute $\tilde{x}(k)$ from (19), the investment policy $k'(x, k)$ from (20), and $\hat{B}(x, k)$ as the fixed point of (21) and (22). We discretize the state space for capital $k_i \in [0, \bar{k}]$ and define a flexible grid for debt which depends on the firm's capital stock such that $b \in [\tilde{B}(k_i), \lambda k_i]$ where $\tilde{B}(k) = \min_{x \geq \tilde{x}(k)} \hat{B}(x, k)$ is the maximum debt level at which the firm is unconstrained. As described in the main text, unconstrained firms keep the debt level at $\tilde{B}(k)$. That is, in a given period they invest $k'(x, k)$ and set $b' = \tilde{B}(k'(x, k))$. Value and policy functions are then defined on a finite grid G which is a subset of $\mathbb{X} \times \mathbb{R} \times \mathbb{R}_+$. Value functions for unconstrained firms are $v_u(x, b, k) = V(x, k) - b$. For constrained firms they follow from (23)–(25). Liquidation and entry policy functions follow from (12) and (13), and the investment and borrowing policy functions $k'(x,b,k)$ and $b'(x,b,k)$ follow from (25).

This permits calculation of the stationary measure of small firms. Combining (5) and (8) yields one equation for the stationary measure μ^0:

$$\mu^0(A) = \int \mathbb{I}_{(x',b'(x,b,k),k'(x,b,k))\in A}\, g\left(x'|x\right)(1-\psi)\left(1-d^l\left(x,b,k\right)\right)\ d\mu^0(x,b,k) \\ + M\int \mathbb{I}_{(x',b'(x,b,k),k'(x,b,k))\in A}\, g\left(x'|x\right) d^e\left(x,b,k\right) d\Phi(x,b,k)\ ,$$

for all Borel subsets $A \subset \mathbb{X}\times\mathbb{R}\times\mathbb{R}_+$. After discretization, μ^0 is a vector with dimension equal to the cardinality of the grid G, and the above equation can be solved by matrix inversion. This also defines the measure of active firms $\mu(x,b,k) = (1-\psi)(1-d^l(x,b,k))\mu^0(x,b,k) + Md^e(x,b,k)\Phi(x,b,k)$ for $(x,b,k) \in G$. Note that both μ^0 and μ are linear in M; hence parameter M linearly scales the size of the non-corporate sector.

The capital stock in the corporate sector K^c can be backed out from the goods-market equilibrium condition

$$C - \int \left\{xf(k,\ell(x,k)) - c^f(k) - \xi[k'(x,b,k)-(1-\delta)k]\right\}\ d\mu(x,b,k) \\ + M\int (k+c^e)d^e(x,b,k)d\Phi(x,b,k) - \theta(1-\delta)\int k\left(\psi+(1-\psi)d^l(x,b,k)\right)\ d\mu^0(x,b,k) \\ = K^c[F(1,1/\kappa^*)-\delta]\ ,$$

with capital-labor ratio in the corporate sector κ^* obtained before. Finally, employment in the corporate sector is $L^c = K^c/\kappa^*$ and labor supply is $L = L^c + \int \ell(x,k)\ d\mu(x,b,k)$.

F.2 Computation of the Transition Path

We describe how the transition path after the unexpected pandemic shock at $t=0$ back to the original steady state can be calculated numerically. Choose large enough T and suppose that the economy has approximately reached the steady state from period $T+1$ onward. That is, all endogenous variables attain their steady state values for $t \geq T+1$.

1. Start with a guess for initial consumption C_0. Then use the following first-order condi-

tions of the household and corporate sector decision problems:

$$q_t = \beta \frac{D_{t+1}}{D_t} \left(\frac{C_t}{C_{t+1}} \right)^{\sigma} , \tag{F.1}$$

$$w_t = \zeta_t C_t^{\sigma} , \tag{F.2}$$

$$w_t = (1-\alpha) A_t \kappa_t^{\alpha} , \tag{F.3}$$

$$1 = q_t [1 - \delta + \alpha A_{t+1} \kappa_{t+1}^{\alpha-1}] , \tag{F.4}$$

which hold for all $t = 0, \ldots, T$, where $A_t = A\,(1 + \nu_t^c)$, $D_t = \left(1 + \nu_t^d\right)$, and $\zeta_t = \left(1 + \nu_t^{\ell}\right)$ are time-varying multipliers of corporate productivity, utility of consumption, and utility of leisure, respectively. κ_t is the capital-labor ratio in the corporate sector. The guess for C_0 directly yields w_0 and κ_0. Then use all four equations to obtain a dynamic equation for κ_t:

$$\beta \kappa_t^{\alpha} = \frac{A_{t+1} D_t \zeta_t}{A_t D_{t+1} \zeta_{t+1}} \cdot \frac{\kappa_{t+1}^{\alpha}}{1 - \delta + \alpha A_{t+1} \kappa_{t+1}^{\alpha-1}} .$$

This equation can be inverted numerically to obtain iteratively $\kappa_1, \ldots, \kappa_T$.

Given this solution, the above equations can be used to back out C_t, w_t and q_t for all $t = 0, 1, \ldots, T$. This obtains operating profits of small firms $\pi_t(x, k, \iota)$ for all periods $t = 1, \ldots, T$ where dummy variable ι indicates whether the firm is impacted by the shock. In period 0, write $\pi_0(x, k, \iota, s)$, where index $s = 1$ indicates that these firms receive the grant in $t = 0$. We further write $\ell_t(x, k, \iota)$ for the employment decision of firm (x, k) in impact status ι which depends on t through the real wage.

2. Iterate (10) and (11) (adjusted for the pandemic shock) backwards from steady state value functions yields value functions $v_t(x, b, k, \iota)$ and $v_t^0(x, b, k, \iota)$ for $t = 0, \ldots, T$ and $\iota \in \{0, 1\}$. See below for a description how this can be done without optimization over assets. Regarding period $t = 0$, calculate $v_0(x, b, k, \iota, s)$ and $v_0^0(x, b, k, \iota, s)$ separately for firms with and without grant ($s = 0, 1$) as these have different profits in $t = 0$.

   Calculate entry, exit and investment policies $d_0^e(x, b, k, \iota)$, $d_0^l(x, b, k, \iota, s)$ and $k_0'(x, b, k, \iota, s)$ in period $t = 0$, and $d_t^e(x, b, k, \iota)$, $d_t^l(x, b, k, \iota)$ and $k_t'(x, b, k, \iota)$ for $t = 1, \ldots, T$. Note that the grant is only paid in period $t = 0$ to incumbent firms which is why s enters only the exit policy function d_0^l and the investment policy k_0', but not the entry policy d_0^e.

   Calculate the distribution of active firms in $t = 0$ after entry and exit, $\mu_0(x, b, k, \iota, s)$ from (5) and the given steady-state distribution of incumbent firms at the beginning of the period $\mu^0(x,b,k)$. Here we impose that fraction η_i of incumbent firms and potential entrants are hit by the pandemic and that fraction η of continuing firms in period $t=0$ receive the grant, while entrants in period $t=0$ are not eligible for the grant. Hence,

$$\mu_{t=0}(x,b,k,\iota,s) = \begin{cases} \eta_i \eta \mu^0(x,b,k)(1-\psi)(1-d_0^l(x,b,k,1,1)) & , \ \iota=1, \ s=1, \\ \eta_i \Big[(1-\eta)\mu^0(x,b,k)(1-\psi)(1-d_0^l(x,b,k,1,0)) & \\ +Md_0^e(x,b,k,1)\Phi(x,b,k)\Big] & , \ \iota=1, \ s=0, \\ (1-\eta_i)\eta\mu^0(x,b,k)(1-\psi)(1-d_0^l(x,b,k,0,1)) & , \ \iota=0, \ s=1, \\ (1-\eta_i)\Big[(1-\eta)\mu^0(x,b,k)(1-\psi)(1-d_0^l(x,b,k,0,0)) & \\ +Md_0^e(x,b,k,0)\Phi(x,b,k)\Big] & , \ \iota=0, \ s=0. \end{cases}$$

Continue to iterate over (5) and (8), adjusted for the additional state variable ι, to obtain firm distributions before and after entry/exit in all subsequent periods $t=1,\dots,T$, $\mu_t^0(x,b,k,\iota)$ and $\mu_t(x,b,k,\iota)$ (note that state variable s only matters in $t=0$). Calculate the output of small firms net of investment expenditures:

$$\begin{aligned} Y_0^n = & \int \left\{(1-\nu_0^n\iota)xf(k,\ell_0(x,k,\iota)) - c^f(k) - \xi[k_0'(x,b,k,\iota,s)-(1-\delta)k]\right\} \, d\mu_0(x,b,k,\iota,s) \\ & + \theta(1-\delta)\int k\Big[\psi + (1-\psi)[\eta_i\big(\eta d_0^l(x,b,k,1,1) + (1-\eta)d_0^l(x,b,k,1,0)\big) \\ & \qquad + (1-\eta_i)\big(\eta d_0^l(x,b,k,0,1) + (1-\eta)d_0^l(x,b,k,0,0)\big)]\Big] \, d\mu^0(x,b,k) \\ & - M\int \big[\eta_i d_0^e(x,b,k,1) + (1-\eta_i)d_0^e(x,b,k,0)\big](k+c^e)d\Phi(x,b,k) \ , \end{aligned}$$

$$\begin{aligned} Y_t^n = & \int \left\{(1-\nu_t^n\iota)xf(k,\ell_t(x,k,\iota)) - c^f(k) - \xi[k_t'(x,b,k,\iota)-(1-\delta)k]\right\} \, d\mu_t(x,b,k,\iota) \\ & + \theta(1-\delta)\int k\left[\psi + (1-\psi)d_t^l(x,b,k,\iota)\right] \, d\mu_t^0(x,b,k,\iota) \\ & - M\int [\eta_i d_t^e(x,b,k,1) + (1-\eta_i)d_t^e(x,b,k,0)](k+c^e)d\Phi(x,b,k) \ , \ t=1,\dots,T. \end{aligned}$$

Starting at $t=0$, given the initial steady-state capital stock in the corporate sector $K_0^c = K^{c*}$ and the previously obtained path for κ_t, calculate L_t^c, output Y_t^c, investment I_t^c, and K_{t+1}^c iteratively for all $t=0,\dots,T$:

$$L_t^c = K_t^c/\kappa_t \ \Rightarrow \ Y_t^c = A_tF(K_t^c, L_t^c)$$

$$\Rightarrow I_t^c = Y_t^n + Y_t^c - C_t \ \Rightarrow K_{t+1}^c = (1-\delta)K_t^c + I_t^c \ .$$

If $K_{T+1}^c > K^{c*}$, increase C_0; if $K_{T+1}^c < K^{c*}$, decrease C_0. Repeat until K_{T+1}^c is reasonably close to K^{c*} (saddle path).

Regarding value function iteration (step 2), investment and borrowing/savings policies can be obtained using a similar logic as explained in the main text for the stationary equilibrium. Firms become unconstrained when their financial savings exceed a certain threshold level after which they are able to pay positive dividends. The value function of these unconstrained firms takes the form $v_{ut}(x, b, k, \iota) = V_t(x, k, \iota) - b$ where

$$\begin{aligned} V_t(x, k, \iota) = & \max_{k' \geq 0} \pi_t(x, k, \iota) - \xi[k' - (1-\delta)k] \\ & + q_t \left[\psi\theta(1-\theta)k' + (1-\psi)\mathbb{E}_{x'|x} \max[\theta(1-\delta)k', V_{t+1}(x', k', \iota)] \right] \ . \end{aligned} \tag{F.5}$$

This equation can be solved backwards starting from $V_{T+1} = V$ (steady state). Define cutoff productivity levels $\tilde{x}_t(k, \iota) = \min\{x \in \mathbb{X} | V_t(x, k, \iota) \geq \theta(1-\delta)k\}$ such that unconstrained firms with productivity below $\tilde{x}_t(k, \iota)$ liquidate the firm in period t, while all others stay. Investment policies of unconstrained firms are

$$k_t'(x, k, \iota) = \begin{cases} k_t^{**}(x, \iota) \ , & \text{if } (1-\delta)k > k_t^{**}(x, \iota) \ , \\ (1-\delta)k \ , & \text{if } (1-\delta)k \in [k_t^*(x, \iota), k_t^{**}(x, \iota)] \ , \\ k_t^*(x, \iota) \ , & \text{if } (1-\delta)k < k_t^*(x, \iota) \ , \end{cases}$$

where the downward and upward investment levels are

$$\begin{aligned} k_t^{**}(x, \iota) = & \arg\max_{k' \geq 0} -(1 - q_t\psi(1-\delta))\theta k' + q_t(1-\psi)\mathbb{E}_{x'|x} \max[\theta(1-\delta)k', V_{t+1}(x', k', \iota)] \ , \\ k_t^*(x, \iota) = & \arg\max_{k' \geq 0} -(1 - q_t\psi\theta(1-\delta))k' + q_t(1-\psi)\mathbb{E}_{x'|x} \max[\theta(1-\delta)k', V_{t+1}(x', k', \iota)] \ . \end{aligned}$$

Borrowing policies of unconstrained firms and maximum debt levels consistent with non-negative dividends follow similarly as in the stationary equilibrium. The borrowing policy in period t is

$$b_t'(x, k, \iota) = \min \left[\lambda k_t'(x, k, \iota), \min_{x' \geq \tilde{x}_{t+1}(k_t'(x, k, \iota), \iota)} \hat{B}_{t+1}(x', k_t'(x, k, \iota), \iota) \right] \ ,$$

with maximum debt levels defined by

$$\hat{B}_t(x,k,\iota) = \pi_t(x,k,\iota) + q_t b'_t(x,k,\iota) - \xi[k'_t(x,k,\iota) - (1-\delta)k] \ .$$

These equations can be solved recursively, starting from the steady-state in period $T+1$, i.e. $\hat{B}_{T+1}(x,k,\iota) = \hat{B}(x,k)$. Note that in period $t=0$, $\hat{B}_0$ also depends on the receipt of the grant which impacts the profit π_0.

In any period t, a firm is unconstrained if $b \leq \hat{B}_t(x,k,\iota)$ in which case it chooses $k'_t(x,k,\iota)$ and $b'_t(x,k,\iota)$ as specified above, conditional on survival which necessitates $x \geq \tilde{x}_t(k,\iota)$. Otherwise, if $x < \tilde{x}_t(k,\iota)$, the firm is liquidated, $d^l_t(x,b,k,\iota) = 1$. Because $\hat{B}_0(x,k,\iota,s)$ also depends on the receipt status of the grant s in the pandemic period $t=0$, the debt cutoff that differentiates constrained from unconstrained firms depends on the receipt of the grant.

Any firm with $b > \hat{B}_t(x,k,\iota)$ is constrained. The value function of a firm at the beginning of period t is then

$$v^0_t(x,b,k,\iota) = \begin{cases} v^0_{ut}(x,b,k,\iota) \ , & \text{if } b \leq \hat{B}_t(x,k,\iota) \ , \\ v^0_{ct}(x,b,k,\iota) \ , & \text{otherwise} \ , \end{cases}$$

where the value of a constrained firm before exit is

$$v^0_{ct}(x,b,k,\iota) = \begin{cases} \theta(1-\delta)k - b & \text{if } \pi_t(x,k,\iota) - b + \theta(1-\delta)k < 0 \text{ or } v_{ct}(x,b,k,\iota) < \theta(1-\delta)k - b \ , \\ v_{ct}(x,b,k,\iota) & \text{else} \ , \end{cases}$$

with $v_{ct}(x,b,k,\iota)$ denoting the value of a continuing constrained firm in period t. This firm's Bellman equation is

$$\begin{aligned} v_{ct}(x,b,k,\iota) = & \max_{k' \geq 0} q_t \left[\psi(\theta(1-\delta)k' - b') + (1-\psi)\mathbb{E}_{x'|x} v^0_{t+1}(x',b',k',\iota)\right] \ , \\ & \text{s.t. } b' = \frac{1}{q_t}\left\{b - \pi_t(x,k,\iota) + \xi[k' - (1-\delta)k]\right\} \leq \lambda k' \ . \end{aligned}$$

The borrowing constraint implies an upper bound for investment in period t:

$$k' \leq \bar{k}'_t(x,b,k,\iota) = \begin{cases} \frac{\pi_t(x,k,\iota) + (1-\delta)k - b}{1 - q_t\lambda} & \text{if } \pi_t(x,k,\iota) + q_t\lambda(1-\delta)k - b \geq 0 \ , \\ \frac{\pi_t(x,k,\iota) + \theta(1-\delta)k - b}{\theta - q_t\lambda} & \text{else} \ . \end{cases}$$

Again, these value functions can be solved by backward induction starting at the steady-state value function of constrained firms v_c in period $T+1$. Note again that in period $t=0$, the value and policy functions of constrained firms differs between firms with grant and firms without grant via the profit function $\pi_0(x, k, \iota, s)$ in equation (27).

G Moment Calculation

Firm exit rate. The firm exit rate is the quarterly rate at which small firms permanently exit. In the model, we calculate the firm exit rate as the measure of firms that exit in a period divided by the measure of firms at the beginning of the period. In the steady state, the exit rate is

$$r_{exit} = \frac{\int_{x,b,k} d^l(x, b, k) \;\, d\mu^0(x, b, k)}{\int_{x,b,k} d\mu^0(x, b, k)}.$$

The data counterpart is computed based on data from the BDS. Since the BDS data is annual, we first compute the annual exit rate by dividing the number of establishment death in year t, and then convert them into the quarterly rate. The relationship between annual exit rate (r^y_{exit}) and quarterly exit rate (r^q_{exit}) is as follows.

$$r^y_{exit} = 1 - \prod_{q=1}^{4} (1 - r^q_{exit}).$$

We use the establishment exit rate to proxy for the firm exit rate since we do not have high-quality micro data on firm exits.

Fixed expense to revenue ratio. Fixed expenses are firms' non-payroll overhead expense. We compute the data target based on a table prepared by Sageworks published in the Washington Post, which is constructed based on the 2007 U.S. Economic Census.[3] We weight the expense-to-revenue ratios of the nine types of small businesses by their yearly revenues.

The model counterpart of fixed expenses of small firms include the maintenance cost of capital and the additional fixed cost, and the model counterpart of small firm revenue is their output. We compute fixed expense to revenue ratio in the steady state as

$$\frac{\int_{x,b,k} \left[\delta k + c^f(x, k)\right] \;\, d\mu(x, b, k)}{\int_{x,b,k} x f(k, \ell(x, k)) \; d\mu(x, b, k)}.$$

[3]Link: https://www.washingtonpost.com/wp-srv/special/business/costofrunningabusiness.html, accessed on May 28, 2021.

Debt to Asset Ratio. The data target for the debt-to-asset ratio is computed based on the KFS. It is computed as the ratio between the sum of positive net debt of small firms divided by the sum of their real assets. The net debt of a firm is its total debt minus financial assets; we treat firms with a negative net debt (positive financial asset) as having zero debt. The model counterpart of the debt to asset ratio is

$$\frac{\int_{x,b,k} \max\{b,0\}\ d\mu(x,b,k)}{\int_{x,b,k} k\ d\mu(x,b,k)}.$$

H Consumption-Equivalent Variation (CEV)

We measure the welfare effect of government policies by a one-year consumption-equivalent variation. Specifically, CEV^G is defined as the percentage increase in consumption for the first four quarters in the pandemic in the laissez-faire (LF) environment that would make the representative household indifferent between the laissez-faire economy and the economy with grant G. That is,

$$\sum_{t=0}^{3} \beta^t U_t \left((1+CEV^G)C_t^{LF}, L_t^{LF}\right) + \sum_{t=4}^{\infty} \beta^t U_t \left(C_t^{LF}, L_t^{LF}\right) = V_0^G \ , \tag{H.1}$$

where $\{C_t^{LF}, L_t^{LF}\}$ are consumption and labor supply in period t in the laissez-faire economy, and V_0^G is the value of the household in period 0 under the economy with grant G, which can be either the baseline grant or the targeted grant.

Since the utility function is additively separable (see Section 4.1.1), we can rewrite the left-hand side of (H.1) as

$$\sum_{t=0}^{3} \beta^t (1+CEV)^{1-\sigma} \left(1+\nu_t^d\right) \frac{(C_t^{LF})^{1-\sigma}}{1-\sigma} + \sum_{t=4}^{\infty} \beta^t \left(1+\nu_t^d\right) \frac{(C_t^{LF})^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} (1+\nu_t^\ell)\zeta(1-L_t^{LF}) \ , \tag{H.2}$$

which can be further simplified as

$$[(1+CEV^G)^{1-\sigma} - 1]\tilde{V}^C + V_0^{LF} \ , \tag{H.3}$$

where V_0^{LF} is the value in period 0 in the laissez-faire economy and

$$\tilde{V}^C \equiv \sum_{t=0}^{3} \beta^t \left(1+\nu_t^d\right) \frac{(C_t^{LF})^{1-\sigma}}{1-\sigma} \ .$$

Substituting the left-hand side of (H.1) with (H.3) and rearranging, we obtain

$$CEV^G = \left(\frac{V_0^G - V_0^{LF}}{\tilde{V}^C} + 1 \right)^{\frac{1}{1-\sigma}} - 1.$$