Online Appendix: Does my model predict a forward guidance puzzle?*

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The following contains further analysis, examples, and robustness exercises to supplement the analysis in the main text of, "Does my model predict a forward guidance puzzle?"

1 The Fisher model

Sections 2 and 3 use a Fisher model of inflation determination. This appendix includes more discussion of the equilibrium properties of the model when there is active fiscal policy.

The Fisher model is an endowment economy (see Leeper and Leith, 2016 for the micro-foundations of the model). Inflation is determined in the economy by combining the household's Euler equation with rules for fiscal and monetary policy, which gives rise to the following linearized system of equations:

$$
\begin{aligned}
i_t &= \phi\pi_t + \bar{i} \\
i_t &= \mathbb{E}_t\pi_{t+1} \\
b_t &= \delta b_{t-1} + i_t - \beta^{-1}\pi_t
\end{aligned}
$$

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For the simple example in section 3, we assume that $0 < \phi = \phi^a < 1$, $\delta = \delta^a > 1$ for $T^a \leq t < T^*$, $\bar{i} \neq 0$ in $t = T^* - 1$ and otherwise $\bar{i} = 0$. The system for $T^a \leq t < T^*$ can be expressed as:

$$y_t = \Gamma_a + A_a y_{t-1} + B_a \mathbb{E}_t y_{t+1}$$

where $y = (\pi, b)'$ and

$$B_a = \begin{pmatrix} \frac{1}{\phi^a} & 0 \\ 1 - \frac{1}{\beta\phi^a} & 0 \end{pmatrix} \qquad A_a = \begin{pmatrix} 0 & 0 \\ 0 & \delta^a \end{pmatrix}$$

and $\Gamma_a = (-\frac{\bar{i}}{\phi^a}, \frac{\bar{i}}{\beta\phi^a})'$ if $t = T^* - 1$, otherwise $\Gamma_a = 0_{2\times 1}$. Because $\bar{i} = 0$ when $t = T^a$, the θ_{T^a} regime admits two MSV solutions of the form: $y_t = \bar{b}(\theta_{T^a}) y_{t-1}$, where, $\bar{a}(\theta_{T^a}) = 0_{2\times 1}$ and

$$\bar{b}(\theta_{T^a}) = \begin{pmatrix} 0 & \frac{\phi^a - \delta^a}{\phi^a - \beta^{-1}} \\ 0 & \phi^a \end{pmatrix} \text{ and } \tilde{b}(\theta_{T^a}) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^a \end{pmatrix}.$$

One can verify through explicit computation that the largest eigenvalues of the first solution for $DT_a(\bar{a}, \bar{b})$ and $DT_b(\bar{b})$ are $1/\delta^a < 1$ and $\phi^a/\delta^a < 1$, respectively. Hence, $\bar{\phi}(\theta_{T^a}) = (0_{2\times 1}, \bar{b}(\theta_{T^a}))$ is IE-stable.[1] Now we discuss the implications of two alternative assumptions about the θ_{T^*} regime.

Case. Suppose $\phi = \phi^a$, $\delta = \delta^a$ for $t \geq T^*$. Then the unique dynamically stable MSV solution for $t \geq T^*$ is given by[2]

$$y_t = \begin{pmatrix} 0 & \frac{\phi^a - \delta^a}{\phi^a - \beta^{-1}} \\ 0 & \phi^a \end{pmatrix} y_{t-1} = \bar{b}(\theta_{T^*}) y_{t-1} = \bar{b}(\theta_{T^a}) y_{t-1}$$

Thus the unique stable terminal solution is given by $\Phi_0(\theta_{T^*}) = (0_{2\times 1}, \bar{b}(\theta_{T^*}))$. We now obtain the FGA solution. We first obtain the solution for $\bar{b}_j$. Because $\bar{b}(\theta_{T^*}) = (I - B_a \bar{b}(\theta_{T^*}))^{-1} A_a =$

[1]The θ_{T^a} regime admits two MSV solutions. The second is given by $\pi_t = 0$, and $b_t = \delta^a b_{t-1}$. It is easy to show that the IE-stability eigenvalue in this case is $1/\phi^a > 1$, and therefore this equilibrium is not IE-stable.

[2]By Proposition 1, this MSV solution is the unique dynamically stable rational expectations solution. Note that the θ_{T^*} regime admits two MSV solutions. The second is given by $\pi_t = 0$, and $b_t = \delta^a b_{t-1}$. Because $\delta^a > 1$ the second MSV solution is not a dynamically stable equilibrium. Further note that for simplicity we assume $\phi = \phi^a$, $\delta = \delta^a$ for $t \geq T^*$, but qualitatively similar results emerge if $\delta > 1 > \phi > 0$ but $\phi \neq \phi^a$, $\delta \neq \delta^a$ for $t \geq T^*$.

$\bar{b}(\theta_{T^a})$, we have $\bar{b}_1 = (I - B_a\bar{b}(\theta_{T^*}))^{-1}A_a = \bar{b}(\theta_{T^a})$ which implies $\bar{b}_j = \bar{b}(\theta_{T^a})$ for all $j \geq 0$. Now consider the recursion for $\bar{a}_j$. Since $\bar{b}_j = \bar{b}(\theta_{T^*})$ for all j we have:

$$\begin{aligned}\bar{a}_j &= (I - B_a\bar{b}(\theta_{T^*}))^{-1}B_a\bar{a}_{j-1} + (I - B_a\bar{b}(\theta_{T^*}))^{-1}\Gamma_a \\ &= DT_a(\bar{a},\bar{b})\bar{a}_{t-1} + (I - B_a\bar{b}(\theta_{T^*}))^{-1}\Gamma_a\end{aligned}$$

As shown above, the eigenvalues of $DT_a(\bar{a},\bar{b})$ are strictly inside the unit circle and $\Gamma_a = 0_{2\times 1}$ for all $j > 1$. Therefore $\bar{a}_j \to 0_{2\times 1}$ as $j \to \infty$. This proves that $\Phi_j \to \bar{\Phi}(\theta_{T^a})$ as $j \to \infty$ given $\Phi_0 = \Phi_0(\theta_{T^*})$. The forward guidance puzzle is absent.

Case. Now suppose $\phi = \phi^* > 1$ and $\delta = \delta^* < 1$ for $t \geq T^*$. Then the unique dynamically stable MSV solution for $t \geq T^*$ is given by[3]

$$y_t = \begin{pmatrix} 0 & 0 \\ 0 & \delta^* \end{pmatrix} y_{t-1} = \tilde{b}(\theta_{T^*})y_{t-1}$$

Thus the unique stable terminal solution is given by $\Phi_0(\theta_{T^*}) = (0_{2\times 1}, \tilde{b}(\theta_{T^*}))$. We now obtain the FGA solution. We first obtain the solution for $\bar{b}_j$. Starting from $j = 1$:

$$\bar{b}_1 = (I - B_a\tilde{b}(\theta_{T^*}))^{-1}A_a = \begin{pmatrix} 0 & 0 \\ 0 & \delta_a \end{pmatrix} = \tilde{b}(\theta_{T^*})$$

Iterating forward, we see that $\bar{b}_j = \tilde{b}(\theta_{T^*})$ for $j \geq 0$. This shows that Φ_j does not converge to $\bar{\Phi}(\theta_{T^a})$; condition (3) of Proposition 2 fails. Moreover, one can easily verify that $B_a\bar{b}_j = 0_{2\times 2}$ for $j \geq 0$ and therefore $\bar{a}_j$ evolves according to

$$\bar{a}_j = B_a\bar{a}_{j-1} + \Gamma_a = B_a^{j-1}\Gamma_a = \begin{pmatrix} -\frac{\bar{i}}{(\phi^a)^j} \\ \frac{\bar{i}(1-\beta\phi^a)}{(\phi^a)^j} \end{pmatrix}$$

where the second and third equality follow from the fact that $\Gamma_a = 0$ for $j > 1$ and $a_0 = 0_{2\times 1}$. Notice that $\pi_{T^a} = -\frac{\bar{i}}{(\phi^a)^{\Delta_p}} \to +/-\infty$ as $\Delta_p = T^a - T^* \to \infty$. We obtain the same solution as in the section 2 example under RE and the puzzle emerges.[4]

[3] The θ_{T^*} regime admits two MSV solutions. The second is given by $\pi_t = \frac{\phi^*-\delta^*}{\phi^*-\beta^{-1}}b_{t-1}$, and $b_t = \phi^* b_{t-1}$. Because $\phi^* > 1$ the second MSV solution is not a dynamically stable equilibrium.

[4] Note that here we assume the shock occurs at $t = T^* - 1$ whereas the shock is assumed to occur at T^* in

As emphasized in the text, $\delta > 1$ is a consequence of fiscal irresponsibility. If $\delta > 1$ in all periods, then debt-stabilizing inflation is necessary for a stable equilibrium to exist. This debt-stabilizing role for inflation rules out unbounded unanticipated changes in inflation, e.g. following a forward guidance announcement, and this solves the puzzle (as found by Cochrane (2017)). However, this is not the case if the fiscal authority is expected to stabilize the debt stock in the future ($\delta < 1$ for $t \geq T^*$). Agents understand that debt stability comes from fiscal revenues in the long run ($t \geq T^*$) and this permits a unique equilibrium in which inflation plays no systematic debt-stabilizing role in any period. It is the standard "Ricardian" solution considered in our section 2 example and by most other papers in the literature. Fiscal sustainability considerations do not pin down inflation in this equilibrium, and this permits unbounded changes in inflation at the time of announcement. The puzzle emerges.

We can also directly appeal to the mathematical structure of the system to characterize why the puzzle arises when $\delta < 1$ for $t \geq T^*$. The model admits a "block-recursive" structure; the equation for inflation does not depend directly on lagged, current or expected debt, and hence equilibrium inflation will not depend on debt in the T^a regime unless it also depends on debt in the T^* regime. Cho (2021) describes how the block-recursive structure of the same model *without* an anticipated structural change makes it difficult to solve the model forward.[5] Specifically, in the case $\delta > 1$ and $\phi < 1$, one cannot obtain the unique dynamically stable MSV solution by solving the model forward (i.e. by iterating on the T-map; see Cho and Moreno (2011) and Cho (2016)) unless the solution recursion is modified to ensure that expected inflation depends on debt in the recursion. In contrast, our model is a piecewise linear model involving an anticipated structural change, and importantly, we obtain a unique rational expectations equilibrium in both cases described above. The block-recursive structure is an important feature of our model that identifies the assumptions needed in the announcement and terminal regimes to ensure that the puzzle is absent in the resulting unique equilibrium.

the section 2 example. The results are not sensitive to the exact timing of the shock.

[5] Cho (2021)'s insights apply to both the model with fixed parameters, δ and ϕ, and to models that feature Markov-switching in those parameters (e.g. see section 6 of this paper).

2 Simple sunspot example

Consider the following bivariate system of expectational difference equations

$$y_t = \frac{1}{\phi_y}\mathbb{E}_t y_{t+1} + \frac{1}{\phi_x}\mathbb{E}_t x_{t+1} + \epsilon_t \tag{1}$$

$$x_t = \frac{1}{\phi_x}\mathbb{E}_t x_{t+1}, \tag{2}$$

where $\phi_y \neq \phi_x$. To capture both determinate and indeterminate FGA solutions, we study the BN augmented system. Specifically, and without loss of generality, we assume that expectations of x_t may be driven by a sunspot process v_t. In order to account for this possibility, we append

$$s_t = \frac{1}{\alpha}s_{t-1} - v_t + x_t - \mathbb{E}_{t-1}x_t, \tag{3}$$

to the system of equations. By choosing the value of α and ϕ_x or ϕ_y appropriately, we can move between the determinate and indeterminate solutions of the model.

The system is written compactly as

$$\tilde{A}_0 z_t = \tilde{A}_1 z_{t-1} + \tilde{B}_0 \mathbb{E}_t z_{t+1} + \tilde{D}_0 \varepsilon_t,$$

where $z_t = (y_t, x_t, s_t, \mathbb{E}_t x_{t+1})'$, $\varepsilon_t = (\epsilon_t, v_t)'$,

$$\tilde{A}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B}_0 = \begin{pmatrix} \phi_y^{-1} & \phi_x^{-1} & 0 & 0 \\ 0 & \phi_x^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{D}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

RE solutions to the above take the form of

$$z_t = a + b z_{t-1} + c\varepsilon_t$$

and must satisfy the following the conditions: $a = (I - \tilde{B}b)^{-1}\tilde{B}a$, $b = (I - \tilde{B}b)^{-1}\tilde{A}$, and $c = (I - \tilde{B}b)^{-1}\tilde{D}$, where $\tilde{A} = \tilde{A}_0^{-1}\tilde{A}_1$, $\tilde{B} = \tilde{A}_0^{-1}\tilde{B}_0$, and $\tilde{D} = \tilde{A}_0^{-1}\tilde{D}_0$. The quadratic in b has closed form solutions, which define the RE solutions:

$$\bar{b}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \bar{b}^2 = \begin{pmatrix} 0 & 0 & -\frac{\phi_y}{\alpha(\phi_y - \phi_x)} & \frac{\phi_y}{\phi_y - \phi_x} \\ 0 & 0 & -\frac{1}{\alpha} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\phi_x}{\alpha} & \phi_x \end{pmatrix}$$

The first solution, $\bar{b}^1$, is the unique RE solution when the model is determinate, which implies $y_t = \epsilon_t$ and $x_t = 0$. The s_t variable is an exogenous process that does not affect y_t or x_t in this case. The second solution, $\bar{b}^2$, is an indeterminate solution that permits coordination on the sunspot v_t. The solution in this case is

$$y_t = \frac{\phi_y}{\phi_y - \phi_x} x_t + \epsilon_t \tag{4}$$

$$x_t = \phi_x x_{t-1} + v_t. \tag{5}$$

2.1 Example of IE-stability analysis

To analyse the Forward Guidance properties of the model, we first calculate the IE-stability conditions for each equilibria using

$$DT_b(\bar{b}) = \left[(I - \tilde{B}\bar{b})^{-1}\tilde{A}\right]' \otimes \left[(I - \tilde{B}\bar{b})^{-1}\tilde{B}\right]$$

and

$$DT_a(\bar{a}, \bar{b}) = (I - \tilde{B}\bar{b})^{-1}\tilde{B}.$$

The eigenvalues for the two solutions are

$$Eig(DT_b(\bar{b}^1)) = \left\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{\alpha\phi_x},\frac{1}{\alpha\phi_y}\right\}, \; Eig(DT_a(\bar{a}^1,\bar{b}^1)) = \left\{0,0,\frac{1}{\phi_x},\frac{1}{\phi_y}\right\}$$

$$Eig(DT_b(\bar{b}^2)) = \left\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,\alpha\phi_x,\frac{\phi_x}{\phi_y}\right\}, \; Eig(DT_a(\bar{a}^2,\bar{b}^2)) = \left\{0,0,\alpha,\frac{1}{\phi_y}\right\}.$$

When $\alpha > 1$, the determinate θ_{T^a} solution, $\bar{b}^1$, is selected when using standard solution techniques. IE-stability requires that $|\phi_x| > 1$ and $|\phi_y| > 1$, which coincide exactly with the conditions for determinacy.

When $\alpha < 1$ and $|\phi_x| < 1$, standard solution techniques select the sunspot solution. The IE-Stability conditions in this case reduces to $|\phi_y| > 1$. When this solution is selected it is possible that the forward guidance puzzle is not present. The indeterminacy implied by ϕ_x no longer matters for the IE-stability condition.

This illustrates the confusion discussed in the introduction, which ascribes indeterminacy to the forward guidance puzzle. The determinacy or indeterminacy of the θ_{T^a} regime just happen to coincide with IE-stability conditions in the first case. But in the second case, IE-stability and determinacy do not imply the same conditions.

To see that the IE-stability condition is the relevant condition for diagnosing a forward guidance puzzle, it is useful to work through an example. Consider an FGA where a temporary change to x_t is announced at date $t = T^a$ but which occurs at time $t = T^*$.[6] To induce indeterminacy in the θ_{T^*} regime, we assume that $|\phi_x| < 1$ such that

$$x_t = \phi_x x_{t-1} + v_t$$

for $t > T^*$. The sunspot solution for $t > T^*$ follows equations (4) and (5). The structural equations at the $t = T^*$ are

$$\begin{aligned} y_t &= \frac{1}{\phi_y}\mathbb{E}_t y_{t+1} + \frac{1}{\phi_x}\mathbb{E}_t x_{t+1} + \epsilon_t \\ x_t &= \gamma + \frac{1}{\phi_x}\mathbb{E}_t x_{t+1}, \end{aligned}$$

where γ is the anticipated change. The backward recursion starts by constructing the expectations that prevail in time period $t = T^*$ given the above equations and the sunspot terminal solution:

$$\mathbb{E}_t x_{T^*+1} = \phi_x(x_{T^*} - \gamma) \text{ and } \mathbb{E}_t y_{T^*+1} = \frac{\phi_y}{\phi_y - \phi_x}\phi_x(x_{T^*} - \gamma).$$

[6] Whether the change occurs in y_t or x_t does not affect the conclusion of this analysis.

Then, substituting these beliefs into the structural equations, we recover

$$
\begin{aligned}
\mathbb{E}_{T^a} y_{T^*} &= \frac{1}{\phi_y}\left(\frac{\phi_y}{\phi_y-\phi_x}\phi_x(x_{T^*}-\gamma)\right)+\frac{1}{\phi_x}\left(\phi_x(x_{T^*}-\gamma)\right)\\
\mathbb{E}_{T^a} x_{T^*} &= \gamma+\frac{1}{\phi_x}\phi_x(x_{T^*}-\gamma)=\phi_x x_{T^*-1},
\end{aligned}
$$

where the last equality is obtained because we assume that agents coordinate on a sunspot. Finally, simplifying and working our way back through time we can recover the following solution path

$$
\begin{aligned}
\mathbb{E}_{T^a} y_{T^*-1} &= \frac{1}{\phi_y}\left(\frac{\phi_y}{\phi_y-\phi_x}\right)(-\gamma)+\frac{\phi_y}{\phi_y-\phi_x}x_{T*-1}\\
\mathbb{E}_{T^a} x_{T^*-1} &= \phi_x x_{T^*-2}\\
&\dots\\
y_{T^a} &= \left(\frac{1}{\phi_y}\right)^{T^*-T^a}\left(\frac{\phi_y}{\phi_y-\phi_x}\right)(-\gamma)+\frac{\phi_y}{\phi_y-\phi_x}x_{Ta}\\
x_{T^a} &= \phi_x x_{T^a-1}+\epsilon_{T^a},
\end{aligned}
$$

where the last period is the contemporaneous effect of the announcement. The impact of the FGA is given by

$$
\left(\frac{1}{\phi_y}\right)^{\Delta_p}\left(\frac{\phi_y}{\phi_y-\phi_x}\right)\gamma.
$$

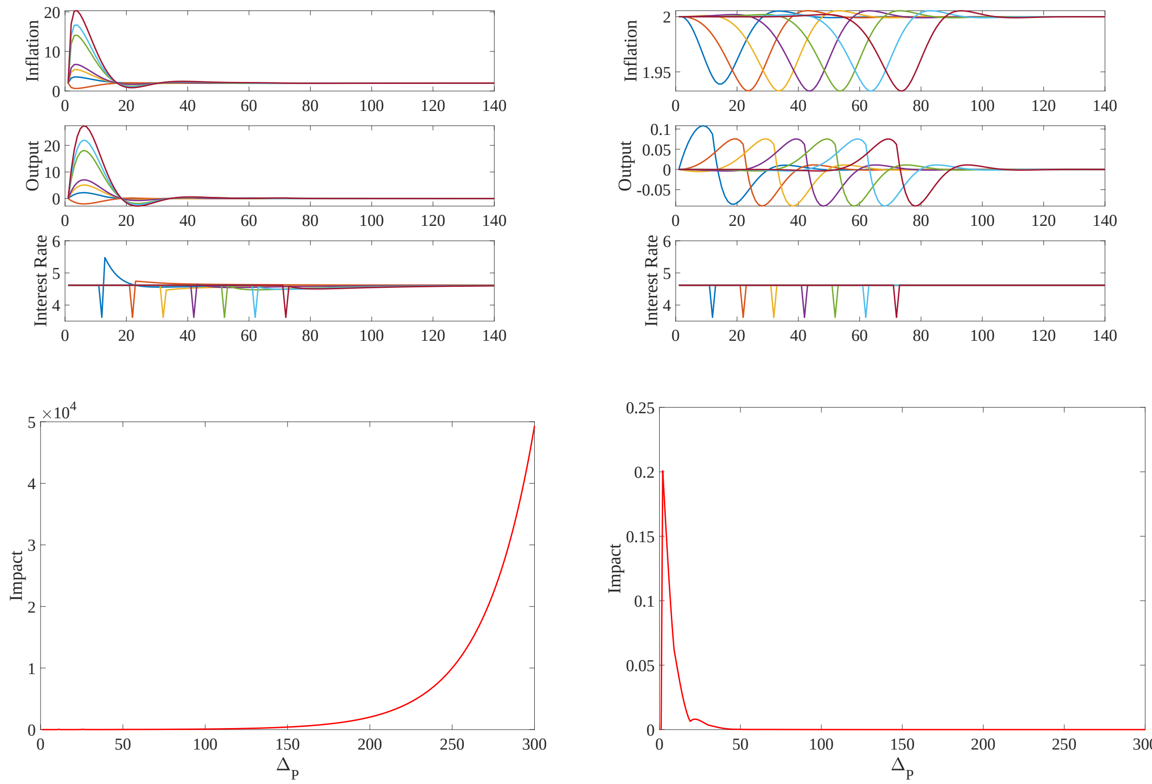
The impact is bounded as $\Delta_p = T^* - T^a \to \infty$ when $|\phi_y| > 1$. Exactly as predicted by IE-stability. The indeterminacy caused by ϕ_x no longer plays a role. The reason that this occurs is because we assume that the agents coordinate on the sunspot in the terminal regime. The agents' beliefs about the evolution of x_t from that point on are pinned down by the sunspot process. They are "predetermined" by last period's realizations in $t = T^a$. They no longer respond at all on announcement of the policy. From the standpoint of calculating the expectations of y_t, the evolution of x_t is exogenously determined.

3 Medium Scale model robustness

Figure 1 shows the simulated response of inflation, output, and the interest rate to anticipated 100-basis point decreases in the interest rate at different horizons under an interest rate peg in the θ_{T^a} regime combined with either a determinate monetary policy announced in the θ_{T^*} regime or a continuation of the interest rate peg in a sunspot equilibrium. The value of the reaction coefficient for inflation in the determinate case is set at the posterior mean reported by Smets and Wouters (2007) ($\phi_\pi^* = 2.04$), while the other parameters of the policy rule are set to zero. For the sunspot case, all parameters of the policy rule are set to zero. The remaining parameters of the model are set to the mean posterior values reported in Smets and Wouters (2007).

Figure 1: Forward Guidance Announcements in the Smets and Wouter's model

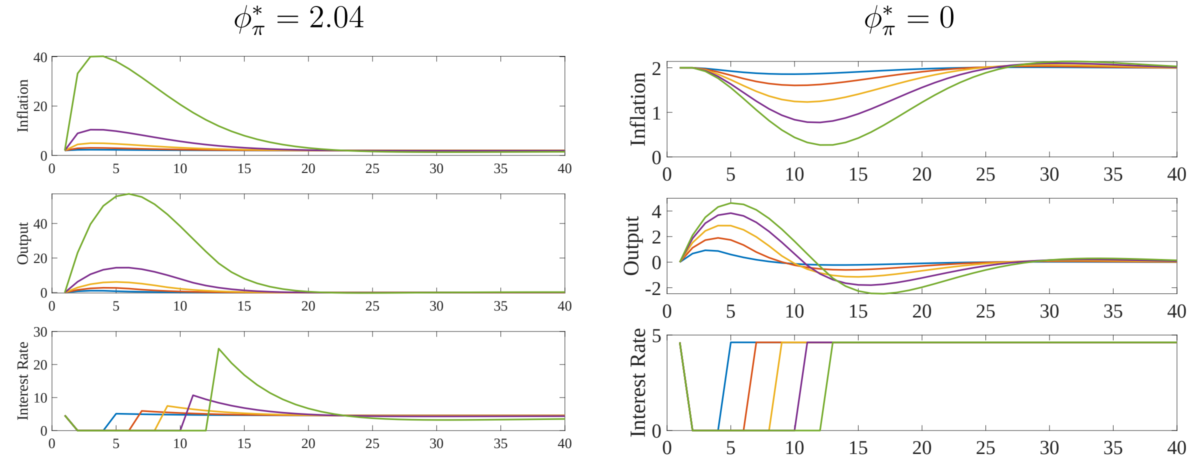
$\phi_\pi^* = 2.04$ $\qquad\qquad$ $\phi_\pi^* = 0$



Notes: The top figures shows the paths of output, inflation, and interest rates for anticipated 100-basis point monetary policy shocks with $\Delta_p = 10, 20, \ldots, 70$. The bottom figures show the impact using the L_∞ norm for $\Delta_p = 1, \ldots, 300$.

The figure shows that the assumption of a return to a determinate policy regime leads to the forward guidance puzzle. It also generates a reversal puzzle in this case. For the case where the anticipated interest rate change is in 20 quarters, inflation falls at the announcement rather than rises. This additional puzzle was also pointed out by Carlstrom, Fuerst and Paustian (2015). It occurs when the unstable eigenvalues are complex. The assumption of an indefinite interest rate peg in the sunspot equilibrium does not lead to the forward guidance puzzle.

Figure 2: Forward Guidance Announcements at the ZLB in the Smets and Wouter's model



Notes: The figure shows the paths of output, inflation, and interest rates for anticipated bind of the zero lower bound for $\Delta_p = 4$, 6, 8, 10, and 12 quarters.

Figure 2 shows the same experiment but for a case of an actual announced zero lower bound regime. Here the interest rate is dropped from a positive steady state to zero and it is announced that it will remain at zero for 4, 6, 8, 10, or 12 quarters. The drop in the interest rate in this case is 400+ basis points, which causes a much larger output and inflation response in the determinate terminal regime case. This illustrates the type of stimulus that would be provided by making these announcements at the zero lower bound. Promises to keep interest rates at zero beyond the duration of a shock that caused the zero lower bound to bind are in effect promises of future 400+ basis point monetary policy interventions in the absence of further shocks in this model. A forward guidance announcement of this kind is highly stimulatory in the determinate case. In the indeterminate case, policy is an interest rate peg in both the zero lower bound regime and afterwards. The central bank moves the interest rate peg to zero and announces a promised duration of zero interest rates. It then moves the interest rate back to steady state and pegs the rate there. In this case, the policy increases output but results in a mild deflation. There is no forward guidance puzzle.

4 Forward Guidance in Markov-switching models

Consider the model (33)-(34) described in the main text. As in Section 3, McClung (2021) obtains the RE solution to a FGA by using the method of undetermined coefficients and backward induction (i.e. the approach is a backward application of techniques developed in Cho, 2016). This appendix provides additional details about the solution approach used for the

Markov-switching model analysis.

Starting from the θ_{T^*} regime, a (MSV) RE solution for $t = T^*$ takes the form of

$$y_t = a(\xi_t) + b(\xi_t)y_{t-1} + c(\xi_t)\omega_t \tag{6}$$

This implies that the expectation of y_{t+1} in time T^* is given by

$$\begin{aligned} \mathbb{E}_t y_{t+1} &= \mathbb{E}_t a(\xi_{t+1}) + \mathbb{E}_t b(\xi_{t+1}) y_t + \mathbb{E}_t c(\xi_{t+1})\rho(\theta_{T^*}, \xi_{t+1})\omega_t & (7) \\ &= \sum_{j=1}^{S} p_{\xi_t j} \left(a(j) + b(j)y_t + c(j)\rho(\theta_{T^*}, j)\omega_t\right) & (8) \end{aligned}$$

Define $b = (b(1), \ldots, b(S))$, $\Xi_*(\xi_t, b) = (I - B_*(\xi_t)\mathbb{E}_t b(\xi_{t+1}))$, and $B_*(\xi_t) = B(\theta_{T^*}, \xi_t)$, $B_a(\xi_t) = B(\theta_{T^a}, \xi_t)$, etc. Substituting equation (7) into equation (33) and rearranging yields the following equivalences

$$\begin{aligned} a(\xi_t) &= \Xi_*(\xi_t, b)^{-1} \left(\Gamma_*(\xi_t) + B_*(\xi_t)\mathbb{E}_t a(\xi_{t+1})\right) & (9) \\ b(\xi_t) &= \Xi_*(\xi_t, b)^{-1} A_*(\xi_t) & (10) \\ c(\xi_t) &= \Xi_*(\xi_t, b)^{-1} \left(B_*(\xi_t)\mathbb{E}_t c(\xi_{t+1})\rho_*(\xi_{t+1}) + D_*(\xi_t)\right) & (11) \end{aligned}$$

where the MSV RE solution is given by $\bar{a}(\theta_{T^*}, \xi_t)$, $\bar{b}(\theta_{T^*}, \xi_t)$, and $\bar{c}(\theta_{T^*}, \xi_t)$ satisfying equations (9), (10), and (11) for $\xi_t = 1, \ldots, S$. This completes the derivation of (35).

In period $t = T^* - 1$, the MSV solution again takes the same form as equation (6). Expectations of y_{t+1} at time $t = T^* - 1$, however, are no longer unknown. They are given by

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t \bar{a}(\theta_{T^*}, \xi_{t+1}) + \mathbb{E}_t \bar{b}(\theta_{T^*}, \xi_{t+1}) y_t + \mathbb{E}_t \bar{c}(\theta_{T^*}, \xi_{t+1})\rho_*(\xi_{t+1})\omega_t$$

Continuing to work backwards in time, the RE Markov-switching solution for the FGA can be written recursively as

$$\begin{aligned} \bar{a}_j(\xi_t) &= \Xi_a(\xi_t, \bar{b}_{j-1})^{-1} \left(\Gamma_a(\xi_t) + B_a(\xi_t)\mathbb{E}_t \bar{a}_{j-1}(\xi_{t+1})\right) & (12) \\ \bar{b}_j(\xi_t) &= \Xi_a(\xi_t, \bar{b}_{j-1})^{-1} A_a(\xi_t) & (13) \\ \bar{c}_j(\xi_t) &= \Xi_a(\xi_t, \bar{b}_{j-1})^{-1} \left(B_a(\xi_t)\mathbb{E}_t \bar{c}_{j-1}(\xi_{t+1})\rho_{j-1}(\xi_{t+1}) + D_a(\xi_t)\right) & (14) \end{aligned}$$

which yields the coefficients in equation (36) of section 6. The T-map difference equations are therefore (12), (13), and (14), or equivalently, (37), (38), and (39).