Online Appendix

Optimal Taxation in the Life Cycle with Human Capital Investment†

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This Appendix offers proofs for the propositions and corollaries in the text.

A.1 Hamiltonian method of the simple two-period model and proofs of propositions in Section 2

A.1.1 *The Hamiltonian of the relaxed planning in Section 2*

Let λ_t be the shadow price of the resource constraint in period $t = 1, 2$ and $\mu(\theta)$ be the co-state variable associated with $\dot{W}(\theta)$. Moreover, we use (5c) to replace $c_2(\theta)$ by

$$u^{-1}\left\{\frac{1}{\beta}\left[W(\theta) + \phi\big(l_1(\theta)\big) - u\big(c_1(\theta)\big)\right] + \phi\big(l_2(\theta)\big)\right\}$$

Then, the Hamiltonian of the relaxed planning problem is given by

$$\begin{aligned}
\mathcal{H} &= \pi(\theta)W(\theta) \\
&+ \lambda_1[F(K_1, Z_1) + (1-\delta_k)K_1 - G_1 - \pi(\theta)[c_1(\theta) + y_1(\theta) - x_1(\theta)] - K_2\,] \\
&+ \lambda_2\left[F(K_2, Z_2) + (1-\delta_k)K_2 - G_2 - \pi(\theta)u^{-1}\left\{\tfrac{1}{\beta}\left[W(\theta) + \phi\big(l_1(\theta)\big) - u\big(c_1(\theta)\big)\right] + \phi\big(l_2(\theta)\big)\right\}\right] \\
&+ \mu(\theta)\left[-u'\big(c_1(\theta)\big)\frac{\partial y_1(\theta)}{\partial\theta} + \beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)^2}\frac{\partial h_2(\theta)}{\partial\theta}\right],
\end{aligned}$$

The first-order conditions with respect to $c_1(\theta), z_1(\theta), z_2(\theta), K_2$ and $x_1(\theta)$ are as follows:

$$\frac{\partial\mathcal{H}}{\partial c_1(\theta)} = -\lambda_1\pi(\theta) + \lambda_2\pi(\theta)\frac{u'(c_1(\theta))}{\beta u'(c_2(\theta))} - \mu(\theta)u''\big(c_1(\theta)\big)\frac{\partial y_1(\theta)}{\partial\theta} = 0, \tag{A1a}$$

$$\frac{\partial\mathcal{H}}{\partial z_1(\theta)} = \lambda_1 F_z(K_1, Z_1)\pi(\theta) - \lambda_2\pi(\theta)\frac{\phi'\left(\frac{z_1(\theta)}{h_1}\right)\frac{1}{h_1}}{\beta u'(c_2(\theta))} = 0, \tag{A1b}$$

$$\frac{\partial\mathcal{H}}{\partial z_2(\theta)} = \lambda_2\pi(\theta)\left[F_z(K_2, Z_2) - \frac{\phi'\left(\frac{z_2(\theta)}{h_2}\right)\frac{1}{h_2}}{u'\big(c_2(\theta)\big)}\right] + \frac{\mu(\theta)\beta\left[\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + \phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right]\frac{\partial h_2(\theta)}{\partial\theta}}{[h_2(\theta)]^2} = 0, \tag{A1c}$$

$$\frac{\partial\mathcal{H}}{\partial x_1(\theta)} = -\lambda_1\pi(\theta) + \frac{\lambda_2\pi(\theta)\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)H_x}{u'\big(c_2(\theta)\big)[h_2(\theta)]^2} - \frac{\beta\mu(\theta)z_2(\theta)H_x\left\{\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + 2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}}{[h_2(\theta)]^3}\frac{\partial h_2(\theta)}{\partial\theta} = 0, \tag{A1d}$$

$$\frac{\partial\mathcal{H}}{\partial K_2} = -\lambda_1 + \lambda_2[F_k(K_2, Z_2) + 1 - \delta_k] = 0. \tag{A1e}$$

Note that the choice of capital in the last condition is the same as the corresponding condition in the Ramsey model. Moreover, if the IC constraint is not binding and thus $\mu(\theta) = 0$, the first 4 conditions reduce to standard conditions for consumption, effective labor, and education investment in the Ramsey model, wherein the discounted marginal utility of consumption and effective labor for each type is equal to the marginal cost. However, if the IC constraint binds and thus $\mu(\theta) \neq 0$, these conditions differ from those in the Ramsey model.

Optimal conditions of constrained efficient allocation. We can rearrange above the first-order conditions to obtain the following four conditions, which characterize the constrained efficient allocation in the social planner's problem.

$$\frac{u'(c_1(\theta))}{\beta u'(c_2(\theta))} - \frac{\mu(\theta)}{\lambda_2}\frac{u''(c_1(\theta))\frac{\partial y_1(\theta)}{\partial\theta}}{\pi(\theta)} = F_k(K_2, Z_2) + 1 - \delta_k, \tag{A2a}$$

$$\frac{\phi'\left(\frac{z_1(\theta)}{h_1}\right)\frac{1}{h_1}}{u'(c_1(\theta))}\left(1 - \frac{\mu(\theta)\phi'\left(\frac{z_1(\theta)}{h_1}\right)u''(c_1(\theta))\frac{\partial y_1(\theta)}{\partial\theta}}{\lambda_1\pi(\theta)u'(c_1(\theta))F_z(K_1,Z_1)h_1}\right)^{-1} = F_z(K_1, Z_1), \tag{A2b}$$

$$\frac{\phi'\left(\frac{z_2(\theta)}{h_2}\right)\frac{1}{h_2}}{u'(c_2(\theta))} - \frac{\beta\mu(\theta)}{\lambda_2\pi(\theta)[h_2(\theta)]^2}\left[\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + \phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right]\frac{\partial h_2(\theta)}{\partial\theta} = F_z(K_2, Z_2). \tag{A2c}$$

$$u'(c_1(\theta)) = \frac{\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)u'(c_1(\theta))H_x}{[F_k(K_2,Z_2)+1-\delta_k]u'(c_2(\theta))[h_2(\theta)]^2} - \frac{\beta\mu(\theta)u'(c_1(\theta))z_2(\theta)H_x\left\{\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)}+2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}}{\lambda_1\pi(\theta)[h_2(\theta)]^3}\frac{\partial h_2(\theta)}{\partial\theta}. \tag{A2d}$$

Condition (A2a) connects the household's *marginal rate of substitution* (henceforth, MRS) between consumption today and tomorrow with the *marginal rate of transformation* (henceforth, MRT) between consumption and investment today. As investment today accumulates physical capital tomorrow, the MRT is the firm's *marginal product of capital* (henceforth, MPK) tomorrow. Moreover, (A2b) and (A2c) also link the household's MRS between leisure and consumption to the MRT between labor and consumption. This MRT is the firm's *marginal product of labor* (henceforth, MPL). Finally, (A2d) links the marginal cost with the marginal benefit of observable education expenditure in human capital investment. These conditions determine the optimal implicit marginal tax rates on capital income and labor income, as well as the optimal implicit subsidy rate on observable human capital investment. These conditions are also known as wedges, and are defined in the next subsection.

Computation of equations (A2a)-(A2d). First, (A2a) can be derived by using (A1a) and (A1e). Next, using (A1a) and (A1b), we obtain

$$\frac{\phi'\left(\frac{z_1(\theta)}{h_1}\right)\frac{1}{h_1}}{u'(c_1(\theta))}\left(1 - \beta u'(c_2(\theta))\frac{\mu(\theta)u''(c_1(\theta))\frac{\partial y_1(\theta)}{\partial\theta}}{\lambda_2\pi(\theta)u'(c_1(\theta))}\right)^{-1} = F_z(K_1, Z_1), \tag{A2e}$$

which, based on (A1b) and (A1e), gives

$$\beta[F_k(K_2, Z_2) + 1 - \delta_k]u'(c_2(\theta)) = \phi'\left(\frac{z_1(\theta)}{h_1}\right)\frac{1}{F_z(K_1,Z_1)h_1}. \tag{A2f}$$

Equation (A2b) is derived by using (A2e), (A2f) and (A1e). Moreover, equation (A2c) is derived by using (A1c). Finally, using (A1e), (A1d) becomes

$$1 = \frac{\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)H_x}{[F_k(K_2,Z_2)+1-\delta_k]u'(c_2(\theta))[h_2(\theta)]^2} - \frac{\beta\mu(\theta)z_2(\theta)H_x}{\lambda_1\pi(\theta)[h_2(\theta)]^3}\left\{\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + 2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}\frac{\partial h_2(\theta)}{\partial\theta}. \quad \text{(A2g).}$$

Then, (A2d) is derived by multiplying $u'(c_1(\theta))$ on the both sides of (A2g).

Co-state variable associated with the envelope condition. The boundary conditions of the Hamiltonian are

$$\mu(\underline{\theta}) = \lim_{\theta\to\underline{\theta}}\mu(\theta) = 0 \text{ and } \mu(\bar{\theta}) = \lim_{\theta\to\bar{\theta}}\mu(\theta) = 0. \tag{A3a}$$

In optimum, changes in the co-state $\mu(\theta)$ with respect to skill types are equal to the negative effect of the state variable $W(\theta)$ upon the Hamiltonian $\mathcal{H}$; that is,

$$\dot{\mu}(\theta) = -\frac{\partial \mathcal{H}}{\partial W(\theta)} = -\pi(\theta)\left[1 - \frac{\lambda_2}{\beta u'(c_2(\theta))}\right]. \tag{A3b}$$

Note that if the IC constraint is not binding and thus $\mu(\theta) = 0$, (A3b) does not apply and the first-order conditions reduce to standard conditions in the Ramsey model. However, if the IC constraint binds and thus $\mu(\theta) \neq 0$, (A3b) applies, then the first-order conditions differ from those in the Ramsey model.

Based on (A3a) and (A3b), we prove the following lemma. First, it is standard to obtain a positive multiplier λ_2 of the resource constraint. Moreover, according to (A3a), the co-state of the information frictions μ is zero at the top and the bottom of the type distribution, which is standard. However, outside the top and bottom types in the distribution, the co-state of the IC constraint is negative, $\mu(\theta) < 0$, which suggests a welfare cost due to information frictions.[1]

Lemma A1. *Suppose that $c_2(\theta)$ is monotone increasing in θ. Then, $\lambda_2 > 0$ and $\mu(\theta) < 0$ for $\theta \in (\underline{\theta}, \overline{\theta})$.*

Proof of Lemma A1. This proof is similar to that in Stantcheva (2017). With boundary conditions $\mu(\overline{\theta}) = \mu(\underline{\theta}) = 0$ in (A3a), according to Rolle's theorem, there exists $\theta' \in (\underline{\theta}, \overline{\theta})$ such that $\frac{d}{d\theta}\mu(\theta') = 0$. The law of motion (A3b) gives

$$\lambda_2 = \beta u'(c_2(\theta')) > 0.$$

Because $c_2(\theta)$ is monotone increasing in θ, the above equation implies that $\lambda_2 < \beta u'(c_2(\theta))$ for $\theta < \theta'$ and $\lambda_2 > \beta u'(c_2(\theta))$ for $\theta > \theta'$ According to the law of motion in (A3b), this implies that $\frac{d}{d\theta}\mu(\theta) < 0$ for $\theta < \theta'$ and $\frac{d}{d\theta}\mu(\theta) > 0$ for $\theta > \theta'$ That is, the derivative of $\mu(\theta)$ is negative for small values of θ and positive for large values of θ. With $\mu(\overline{\theta}) = \mu(\underline{\theta}) = 0$ this ensures that $\mu(\theta)$ is negative for $\theta \in (\underline{\theta}, \overline{\theta})$, as illustrated in Figure A1.

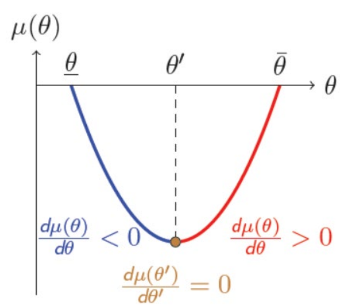

Figure A1. Non-positive co-state $\mu(\theta)$

Alternatively, if we integrate (A3b) and use the boundary condition, $\mu(\overline{\theta}) = 0$, we obtain

[1] A negative co-state $\mu(\theta)$ may be viewed as the marginal welfare loss for the social planner to choose allocations that are incentive-compatible. If the IC constraint is not binding, which arises when agents have no incentives to cheat, then $\mu(\theta) = 0$ and the social planner does not have to sacrifice the welfare.

$$\mu(\theta) = \int_{\theta}^{\bar{\theta}} \left(1 - \frac{\lambda_2}{\beta u'(c_2(\theta''))}\right) \pi(\theta'') d\theta''.$$

Note that, for $\theta \in [\theta', \bar{\theta})$, due to the fact that $\lambda_2 > \beta u'(c_2(\theta))$ for $\theta > \theta'$the above equation integrates over non-positive variables only. Thus, the above equation implies $\mu(\theta) < 0$ for $\theta \in [\theta', \bar{\theta})$.

Similarly, integrating (A3b) and using the boundary condition $\mu(\underline{\theta}) = 0$ yields

$$\mu(\theta) = \int_{\underline{\theta}}^{\theta} \left(-1 + \frac{\lambda_2}{\beta u'(c_2(\theta''))}\right) \pi(\theta'') d\theta''.$$

Since $\lambda_2 < \beta u'(c_2(\theta))$ for $\theta \in (\underline{\theta}, \theta']$, the above equation integrates over non-negative variables only when $\theta < \theta'$, which implies $\mu(\theta) < 0$ for any $\theta \in (\underline{\theta}, \theta']$, and hence, $\mu(\theta) < 0$ for any $\theta \in (\underline{\theta}, \overline{\theta})$. □

A.1.2 *Proof of Proposition 1*

Suppose that the constrained efficient allocation $\{c_1^\sigma(\theta), y_1^\sigma(\theta), h_2^\sigma(\theta)\}$ is the solution for the problem in (3). Then, by using the constraints in (3), we can replace $c_1^\sigma(\theta)$ by $c_1(\sigma) + y_1(\sigma) - y_1^\sigma(\theta)$ and $h_2^\sigma(\theta)$ by $H(x_1(\sigma), y_1^\sigma(\theta), \theta, h_1)$. Then, the problem in (3) becomes:

$$\max_{y_1^\sigma(\theta)} u\big(c_1(\sigma) + y_1(\sigma) - y_1^\sigma(\theta)\big) - \phi\left(\frac{z_1(\sigma)}{h_1}\right) + \beta\left[u\big(c_2(\sigma)\big) - \phi\left(\frac{z_2(\sigma)}{H\big(x_1(\sigma), y_1^\sigma(\theta), \theta, h_1\big)}\right)\right]. \tag{A4a}$$

We denote $\phi_h\left(\frac{z_t}{h_t}\right) \equiv -\phi'\left(\frac{z_t}{h_t}\right)\frac{z_t}{(h_t)^2} < 0$ and $\phi_{hh}\left(\frac{z_t}{h_t}\right) \equiv \phi''\left(\frac{z_t}{h_t}\right)\frac{(z_t)^2}{(h_t)^4} + 2\phi'\left(\frac{z_t}{h_t}\right)\frac{z_t}{(h_t)^3} > 0$. Then, the first-order condition of the problem (A4a) is

$$-u'\big(c_1(\sigma) + y_1(\sigma) - y_1^\sigma(\theta)\big) - \beta\phi_h\left(\frac{z_2(\sigma)}{H\big(x_1(\sigma), y_1^\sigma(\theta), \theta, h_1\big)}\right) H_y\big(x_1(\sigma), y_1^\sigma(\theta), \theta, h_1\big) = 0, \tag{A4b}$$

and the second order condition of the problem (A4a) is

$$u''\big(c_1^\sigma(\theta)\big) - \beta\phi_{hh}\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right) H_y\big(x_1(\sigma), y_1^\sigma(\theta), \theta, h_1\big) - \beta\phi_h\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right) H_{yy}\big(x_1(\sigma), y_1^\sigma(\theta), \theta, h_1\big) < 0$$

which is negative due to the facts that $u'' < 0$, $\phi_{hh} > 0, H_y > 0, \phi_h < 0$, and $H_{yy} < 0$. This completes the proof of Proposition 1. □

A.1.3 *Proof of Lemma 1*

(i) Differentiating (A4b) with respect to θ yields

$$u''\big(c_1^\sigma(\theta)\big)\frac{\partial y_1^\sigma(\theta)}{\partial\theta} = \beta\left\{\phi_{hh}\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\left[H_y\frac{\partial y_1^\sigma(\theta)}{\partial\theta} + H_\theta\right]H_y + \phi_h\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\left(H_{yy}\frac{\partial y_1^\sigma(\theta)}{\partial\theta} + H_{y\theta}\right)\right\},$$

which gives

$$\frac{\partial}{\partial\theta}y_1^\sigma(\theta) = \frac{\beta\left[\phi_{hh}\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)H_\theta H_y + \phi_h\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)H_{y\theta}\right]}{u''\big(c_1^\sigma(\theta)\big) - \beta\left[\phi_{hh}\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\big(H_y\big)^2 + \phi_h\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)H_{yy}\right]}$$

$$= \frac{\beta\frac{z_2(\sigma)}{\big(h_2^\sigma(\theta)\big)^3}\left[\phi''\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\frac{z_2(\sigma)}{h_2^\sigma(\theta)}H_\theta H_y + \phi'\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\big(2H_\theta H_y - H_{y\theta}H\big)\right]}{u''\big(c_1^\sigma(\theta)\big) - \beta\left[\phi_{hh}\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)\big(H_y\big)^2 + \phi_h\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)H_{yy}\right]}, \tag{A5a}$$

As $u'' < 0,\ \phi'',\phi',\ \phi_{hh} > 0,\ \phi_h < 0$ and, $H_y > 0$, $H_\theta > 0$, and $H_{yy} < 0$, (A5a) implies that $\frac{\partial y_1^\sigma(\theta)}{\partial\theta} < 0$ if $\rho_{y\theta} = \frac{H_{y\theta}H}{H_\theta H_y} \leq 2$.

(ii) When $\rho_{y\theta} \geq \frac{H_{yy}H}{(H_y)^2}$, definition $\rho_{y\theta} \equiv \frac{H_{y\theta}H}{H_yH_\theta}$ and $H_y > 0$ imply that $H_{y\theta}H_y \geq H_{yy}H_\theta$. Based on the fact that $\frac{\partial h_2^\sigma(\theta)}{\partial\theta} = H_y\frac{\partial y_1^\sigma(\theta)}{\partial\theta} + H_\theta$, we obtain

$$\frac{\partial h_2^\sigma(\theta)}{\partial\theta} = H_y\frac{\partial y_1^\sigma(\theta)}{\partial\theta} + H_\theta = \frac{u''(c_1^\sigma(\theta))H_\theta + \beta\phi_h\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)[H_{y\theta}H_y - H_{yy}H_\theta]}{u''\left(c_1^\sigma(\theta)\right) - \beta\left[\phi_{hh}\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)(H_y)^2 + \phi_h\left(\frac{z_2(\sigma)}{h_2^\sigma(\theta)}\right)H_{yy}\right]}. \qquad \text{(A5b)}$$

As $u'' < 0,\ \phi_{hh} > 0,\ \phi_h < 0$ and, $H_y > 0$, $H_\theta > 0$, and $H_{yy} < 0$, (A5b) implies that $\frac{\partial h_2^\sigma(\theta)}{\partial\theta} > 0$, if $H_{y\theta}H_y \geq H_{yy}H_\theta$.

(iii) If there are only verifiable education expenses, then $y_1^\sigma(\theta) = 0$ for all θ and σ, and thus,

$$\frac{\partial y_1^\sigma(\theta)}{\partial\theta} = 0 \text{ and } \frac{\partial h_2^\sigma(\theta)}{\partial\theta} = H_y\frac{\partial y_1^\sigma(\theta)}{\partial\theta} + H_\theta = H_\theta > 0. \ \square$$

A.1.4 *Proof of Proposition 2*

This proof is like the proof of Theorem 5 in Kapicka (2013). The envelope condition (5d) implies that

$$W(\theta) - W(\sigma) = \int_\sigma^\theta -u'\left(c_1(\varepsilon)\right)\frac{\partial y_1(\varepsilon)}{\partial\varepsilon} + \beta\phi'\left(\frac{z_2(\varepsilon)}{h_2(\varepsilon)}\right)\frac{z_2(\varepsilon)}{(h_2(\varepsilon))^2}\frac{\partial h_2(\varepsilon)}{\partial\varepsilon}d\varepsilon$$

$$\geq \int_\sigma^\theta -u'\left(c_1^\sigma(\varepsilon)\right)\frac{\partial y_1^\sigma(\varepsilon)}{\partial\varepsilon} + \beta\phi'\left(\frac{z_2(\sigma)}{h_2^\sigma(\varepsilon)}\right)\frac{z_2(\sigma)}{(h_2^\sigma(\varepsilon))^2}\frac{\partial h_2^\sigma(\varepsilon)}{\partial\varepsilon}d\varepsilon$$

$$= W^\sigma(\theta) - W(\sigma)$$

which implies that $W(\theta) \geq W^\sigma(\theta)$ and then the incentive compatible constraint (5b) holds. The first equality follows from the envelope condition (5d) and the fundamental theorem of calculus. The second inequality follows from the facts that the monotonicity condition (5e) is non-decreasing in σ and that any element ε in the range of integral $[\sigma,\theta]$ is not less that σ. Finally, the last equality follows from the fundamental theorem of calculus and the fact that $W^\varepsilon(\theta)$ is differentiable in θ. □

A.1.5 *Proof of Proposition 3*

Based on equation (A2a) and (A1e), we have

$$\frac{1}{u'(c_1(\theta))} = \frac{1}{\beta R_2 u'(c_2(\theta))} - \frac{\mu(\theta)u''(c_1(\theta))}{\lambda_1\pi(\theta)u'(c_1(\theta))}\frac{\partial y_1(\theta)}{\partial\theta}.\square$$

A.1.6 *Proof of Corollary 1*

(i) According to Lemma A1 and Lemma 1(i), we have (with $\sigma = \theta$), $\lambda_2 > 0$, $\mu(\theta) < 0$ for any $\theta \in (\underline{\theta},\overline{\theta})$ and $\frac{\partial y_1(\theta)}{\partial\theta} < 0$ if $\rho_{y\theta} \geq 2$. Along with the fact that $u'' < 0$ and (7a), the proof is complete.

(ii) When there are only verifiable education expenses, according to Lemma 1(iii), we have $\frac{\partial y_1(\theta)}{\partial\theta} = 0$. Then, equation (7b) implies that the standard (inverse) Euler equation holds. □

A.1.7 *Proof of Proposition 4*

The labor wedge in the first period can be derived by equation (A2b):

$$\tau_{z_1}(\theta) \equiv 1 - \frac{\phi'\left(\frac{z_1(\theta)}{h_1}\right)\frac{1}{w_1 h_1}}{u'(c_1(\theta))} = \frac{\mu(\theta)\phi'\left(\frac{z_1(\theta)}{h_1}\right)u''(c_1(\theta))}{\lambda_1\pi(\theta)u'(c_1(\theta))w_1 h_1}\frac{\partial y_1(\theta)}{\partial\theta} \ \ \forall\, \theta \in [\underline{\theta}, \overline{\theta}]. \tag{A6a}$$

In addition, the labor wedge in the second period can be derived by equation (A2c):

$$\tau_{z_2}(\theta) \equiv 1 - \frac{\phi'\left(\frac{z_2(\theta)}{h_2}\right)\frac{1}{w_2 h_2}}{u'(c_2(\theta))} = \frac{-\ \beta\mu(\theta)}{\lambda_2\pi(\theta)[h_2(\theta)]^2 w_2}\left[\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + \phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right]\frac{\partial h_2(\theta)}{\partial\theta} \ \forall\, \theta \in [\underline{\theta}, \overline{\theta}].\ \square \tag{A6b}$$

A.1.8 *Proof of Corollary 2*

(i) According to Lemma A1, we have $\lambda_2 > 0,\ \mu(\theta) < 0$ for any $\theta \in (\underline{\theta}, \overline{\theta})$. By Lemma 1(i), we have $\frac{\partial y_1(\theta)}{\partial\theta} < 0$ if $\rho_{y\theta} \le 2$. Along with the fact that $u'' < 0$ and $u' > 0$, (A6a) implies that the skill-fostering effect is negative for any $\theta \in (\underline{\theta}, \overline{\theta})$:

$$\frac{\mu(\theta)\phi'\left(\frac{z_1(\theta)}{h_1}\right)u''(c_1(\theta))}{\lambda_1\pi(\theta)u'(c_1(\theta))w_1 h_1}\frac{\partial y_1(\theta)}{\partial\theta} < 0$$

and thus, the labor wedge in the first period is negative $\tau_{z_1}(\theta) < 0$ for any $\theta \in (\underline{\theta}, \overline{\theta})$.

(ii) By Lemma 1(ii), we have $\frac{\partial h_2(\theta)}{\partial\theta} > 0,$ if $\rho_{y\theta} \ge \frac{H_{yy}H}{(H_y)^2}$. Along with the fact that $\phi' > 0$ and $\phi'' > 0$, (A6b) implies that the shirk-preventing effect is positive for any $\theta \in (\underline{\theta}, \overline{\theta})$:

$$\frac{-\ \beta\mu(\theta)}{\lambda_2\pi(\theta)[h_2(\theta)]^2 w_2}\left[\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + \phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right]\frac{\partial h_2(\theta)}{\partial\theta} > 0,$$

and thus, the labor wedge in the second period is positive $\tau_{z_2}(\theta) > 0$ for any $\theta \in (\underline{\theta}, \overline{\theta})$.

(iii) When there are only verifiable education expenses, by Lemma 1(iii), we have $\frac{\partial y_1(\theta)}{\partial\theta} = 0$ and $\frac{\partial h_2(\theta)}{\partial\theta} > 0$, which implies $\tau_{z_1}(\theta) = 0$ in equation (A6a) and $\tau_{z_2}(\theta) > 0$ in equation (A6b). □

A.1.9 *Proof of Proposition 5*

Adding $\frac{-\beta H_x z_2(\theta)}{[h_2(\theta)]^2}\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)$ on the both sides of (A2d), then equation (A2d) gives

$$u'(c_1(\theta)) - \frac{\beta H_x z_2(\theta)}{[h_2(\theta)]^2}\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)$$
$$= \frac{\beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)H_x}{[h_2(\theta)]^2}\left[\frac{u'(c_1(\theta))}{\beta R_2 u'(c_2(\theta))} - 1\right] - \frac{\beta\mu(\theta)u'(c_1(\theta))z_2(\theta)H_x\left\{\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + 2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}\frac{\partial h_2(\theta)}{\partial\theta}}{\lambda_1\pi(\theta)[h_2(\theta)]^3}. \tag{A7a}$$

Dividing $u'(c_1(\theta))$ on the both sides of (A7a) and using the definition in (6c), equation (A7a) becomes

$$\tau_{x_1}(\theta) = 1 - \frac{\beta H_x\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)}{u'(c_1(\theta))[h_2(\theta)]^2}$$
$$= \frac{\beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)H_x}{[h_2(\theta)]^2}\left[\frac{1}{\beta R_2 u'(c_2(\theta))} - \frac{1}{u'(c_1(\theta))}\right] - \frac{\beta\mu(\theta)z_2(\theta)H_x}{\lambda_1\pi(\theta)[h_2(\theta)]^3}\left\{\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)} + 2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}\frac{\partial h_2(\theta)}{\partial\theta}.$$

Using equation (7b) and replacing $\frac{1}{u'(c_1(\theta))} - \frac{1}{\beta R_2 u'(c_2(\theta))}$ by Ω_1, the above expression becomes

$$\tau_{x_1}(\theta)=\frac{-\beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)H_x}{[h_2(\theta)]^2}\Omega_1-\frac{\beta\mu(\theta)z_2(\theta)H_x}{\lambda_1\pi(\theta)[h_2(\theta)]^3}\left\{\phi''\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{z_2(\theta)}{h_2(\theta)}+2\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\right\}\frac{\partial h_2(\theta)}{\partial\theta}.$$

Using Proposition 4 and equation (A1e), the above equation can be rewritten as

$$\tau_{x_1}(\theta)=\frac{-\beta\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)z_2(\theta)H_x}{[h_2(\theta)]^2}\Omega_1+\frac{z_2(\theta)w_2H_x}{R_2h_2(\theta)}\tau_{z_2}(\theta)-\frac{\beta\mu(\theta)z_2(\theta)H_x}{\lambda_1\pi(\theta)[h_2(\theta)]^3}\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{\partial h_2(\theta)}{\partial\theta}. \tag{A7b}$$

Using the notations in Definition 1, the first term in the righthand side of (A7b) can be replaced by $-\mathcal{K}_2(\theta)$, and the second term of (A7b) can be replaced by $\mathcal{N}_2(\theta)$. Therefore, equation (A7b) can be rewritten as

$$\tau_{x_1}(\theta)=-\mathcal{K}_2(\theta)+\mathcal{N}_2(\theta)-\frac{\beta\mu(\theta)z_2(\theta)H_x}{\lambda_1\pi(\theta)[h_2(\theta)]^3}\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{\partial h_2(\theta)}{\partial\theta}.$$

Hence, according to Definition 1, the above equation implies that the net human capital wedge is as follows:

$$\tau_{x_1}^n(\theta)\equiv\tau_{x_1}(\theta)-\mathcal{N}_2(\theta)+\mathcal{K}_2(\theta)=\frac{-\beta\mu(\theta)z_2(\theta)H_x}{\lambda_1\pi(\theta)[h_2(\theta)]^3}\phi'\left(\frac{z_2(\theta)}{h_2(\theta)}\right)\frac{\partial h_2(\theta)}{\partial\theta}.\quad\square \tag{A7c}$$

A.1.10 *Proof of Corollary 3*

(i) By Lemma A1, $\mu(\theta)<0$ *for* $\theta\in(\underline{\theta},\overline{\theta})$. By Lemma 1 (ii), $\frac{\partial h_2(\theta)}{\partial\theta}>0$ if $\rho_{y\theta}\geq\frac{H_{yy}H}{(H_y)^2}$. Therefore, equation (A7c) implies that the net human capital wedge is positive for any $\theta\in(\underline{\theta},\overline{\theta})$.

(ii) If there are only verifiable education expenses, by Lemma 1(iii), $\frac{\partial h_2(\theta)}{\partial\theta}>0,$ and then equation (A7c) implies that the net human capital wedge is positive for any $\theta\in(\underline{\theta},\overline{\theta})$. □

A.2 Hamiltonian method of general T-period model and proofs of propositions in Section 3

A.2.1 *The derivation of the recursive relaxed social planning problem*

Given a targeted threshold of the expected continuation utility $\underline{v}$, the planner's problem is to minimize the expected discounted resource cost, subject to the expected social welfare above $\underline{v}$ and incentive compatibility. Formally, the social planner's problem, denoted by P^{IC}, is

$$\min\sum_{t=1}^{T}\frac{1}{R_2R_3\ldots R_T}\int[c(\theta^t)+x(\theta^t)+y(\theta^t)-w_tz(\theta^t)]\pi(\theta^t)d\theta^t, \tag{A8a}$$

$$\text{subject to}\quad \int W(\theta)\pi(\theta)d\theta\geq\underline{v},$$

$$\text{and}\quad W(\theta^t)\geq W^{\sigma}(\theta^t),\forall\sigma\in\mathcal{R}.\ \text{(IC)}.$$

Using the first-order approach, we replaces the IC constraint (A8a) by the envelope condition (9b) to form the relaxed planning problem, which is denoted by P^{FOA}. Through a suitable definition of state variables, we may rewrite the relaxed planning problem in the following recursive forms.

Building on the relaxed planning problem P^{FOA}, and Bellman equations (9b) and (9c), the problem in $t=1$ is reformulated as follows:

$$\mathcal{K}(\underline{v},h_1,1)=\min\int\left[c(\theta)+x(\theta)+y(\theta)-w_1z(\theta)+\frac{\mathcal{K}(v(\theta),\Delta(\theta),h(\theta),\theta,2)}{R_2}\right]\pi(\theta)d\theta, \tag{A8b}$$

$$\text{subject to}\quad W(\theta)=u\big(c(\theta)\big)-\phi\left(\frac{z(\theta)}{h_1}\right)+\beta v(\theta),$$

$$\dot{W}(\theta) = -u'\big(c(\theta)\big)\frac{\partial y(\theta)}{\partial \theta} + \beta\Delta(\theta),$$

$$\int W(\theta)\pi(\theta)d\theta \geq \underline{v},$$

where the minimization is taken over $c(\theta), x(\theta), z(\theta), W(\theta), v(\theta)$ and $\Delta(\theta)$, and $\mathcal{K}(v(\theta), \Delta(\theta), h(\theta), \theta, 2)$ is solved by the following problems recursively.

For periods $t = 2, 3, \ldots, T-1$, given past history θ^{t-1}, we take as given previous values for state variables $v(\theta^{t-1})$, $\Delta(\theta^{t-1})$, and $h(\theta^{t-1})$. For any period t, conditional on the history of shocks in one period earlier θ_{t-1}, the entire history of shocks θ^{t-2} is redundant. Thus, the recursive relaxed planning problem is:

$$\mathcal{K}(v, \Delta, h, \theta_{-}, t) = \min \int \left[c(\theta) + y(\theta) + x(\theta) - w_t z(\theta) + \frac{\mathcal{K}(v(\theta), \Delta(\theta), h(\theta), \theta, t+1)}{R_{t+1}}\right]\pi(\theta)d\theta, \quad \text{(A8c)}$$

subject to
$$W(\theta) = u\big(c(\theta)\big) - \phi\left(\frac{z(\theta)}{h(\theta_{-})}\right) + \beta v(\theta),$$

$$\dot{W}(\theta) = -u'\big(c(\theta)\big)\frac{\partial y(\theta)}{\partial \theta} + \beta\Delta(\theta),$$

where $v = \int W(\theta)\pi(\theta)\, d\theta$ and $\Delta = \int \phi'\left(\frac{z(\theta)}{h(\theta_{-})}\right)\frac{z(\theta)\frac{\partial h(\theta_{-})}{\partial \theta_{-}}}{[h(\theta_{-})]^2}\pi(\theta)d\theta$, with θ_{-} denoting past shocks and the minimization being taken over $c(\theta), x(\theta), z(\theta), W(\theta), v(\theta)$ and $\Delta(\theta)$. Denote by P^{*FOA} the set of the recursive planning problem in (A8b) and (A8c) and by X^{*FOA} the set of the resulting constrained efficient allocation.

We are ready to derive the relaxed social planning problem. Since the utility is given by $W(\theta) = u\big(c(\theta)\big) - \phi\left(\frac{z(\theta)}{h(\theta_{-})}\right) + \beta v(\theta)$, we can replace $c(\theta)$ by $u^{-1}\left[W(\theta) - \beta v(\theta) + \phi\left(\frac{z(\theta)}{h(\theta_{-})}\right)\right]$. Then, for periods $t = 2, 3, \ldots, T$, the Hamiltonian is as follows:

$$\begin{aligned}
&\mathcal{K}(v(\theta_{-}), \Delta(\theta_{-}), h(\theta_{-}), \theta_{-}, t) \\
&\quad = \left\{u^{-1}\left[W(\theta) - \beta v(\theta) + \phi\left(\frac{z(\theta)}{h(\theta_{-})}\right)\right] + x(\theta) + y(\theta) - w_t z(\theta) + \frac{\mathcal{K}(v(\theta), \Delta(\theta), h(\theta), \theta, t+1)}{R_{t+1}}\right\}\pi(\theta) \\
&\quad + \lambda(\theta_{-})[v(\theta_{-}) - W(\theta)\pi(\theta)] \\
&\quad + \gamma(\theta_{-})\left[\Delta(\theta_{-}) - \frac{z(\theta)\frac{\partial h(\theta_{-})}{\partial \theta_{-}}}{[h(\theta_{-})]^2}\phi'\left(\frac{z(\theta)}{h(\theta_{-})}\right)\pi(\theta)\right] \\
&\quad + \hat{\mu}(\theta)\left[-u'\left(u^{-1}\left[W(\theta) - \beta v(\theta) + \phi\left(\frac{z(\theta)}{h(\theta_{-})}\right)\right]\right)\frac{\partial y(\theta)}{\partial \theta} + \beta\Delta(\theta)\right].
\end{aligned}$$

where $\lambda(\theta_{-})$ and $\gamma(\theta_{-})$ are the shadow price associated with $v(\theta_{-})$ and $\Delta(\theta_{-})$ respectively, and $\hat{\mu}(\theta)$ is the co-state variable associated with $\dot{W}(\theta)$.

The first-order conditions are

$$\frac{\partial \mathcal{K}}{\partial z(\theta)} = \pi(\theta)\left[\frac{\phi'\left(\frac{z(\theta)}{h(\theta_{-})}\right)\frac{1}{h(\theta_{-})}\left[1 - \frac{\hat{\mu}(\theta)}{\pi(\theta)}u''\big(c(\theta)\big)\frac{\partial y(\theta)}{\partial \theta}\right]}{u'\big(c(\theta)\big)} - w_t - \frac{\gamma(\theta_{-})\frac{\partial h(\theta_{-})}{\partial \theta_{-}}\left[\phi'\left(\frac{z(\theta)}{h(\theta_{-})}\right) + \frac{z(\theta)}{h(\theta_{-})}\phi''\left(\frac{z(\theta)}{h(\theta_{-})}\right)\right]}{[h(\theta_{-})]^2}\right] = 0, \quad \text{(A9a)}$$

$$\frac{\partial \mathcal{K}}{\partial x(\theta)} = \pi(\theta)\left[1 + \frac{\mathcal{K}_h(v(\theta), \Delta(\theta), h(\theta), \theta, t+1)\mathbb{H}_{x,t}^t}{R_{t+1}}\right] = 0, \quad \text{(A9b)}$$

$$\frac{\partial \mathcal{K}}{\partial \Delta(\theta)} = \frac{\pi(\theta)}{R_{t+1}}K_\Delta(v(\theta), \Delta(\theta), h(\theta), \theta_t, t+1) + \beta\hat{\mu}(\theta) = 0, \quad \text{(A9c)}$$

$$\frac{\partial \mathcal{K}}{\partial v(\theta)} = \frac{-\beta\pi(\theta)}{u'\big(c(\theta)\big)} + \frac{\pi(\theta)}{R_{t+1}}K_v(v(\theta), \Delta(\theta), h(\theta), \theta_t, t+1) + \frac{\hat{\mu}(\theta)\beta u''\big(c(\theta)\big)}{u'\big(c(\theta)\big)}\frac{\partial y(\theta)}{\partial \theta} = 0. \quad \text{(A9d)}$$

Moreover, the law of motion for the co-state $\hat{\mu}(\theta)$ is:

$$\frac{\partial \mathcal{K}}{\partial W(\theta)} = -\dot{\hat{\mu}}(\theta) \equiv -\frac{\partial \hat{\mu}(\theta)}{\partial \theta} = \left[\frac{1}{u'(c(\theta))} - \lambda(\theta_-)\right]\pi(\theta) - \frac{\hat{\mu}(\theta)u''(c(\theta))}{u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta}, \tag{A9e}$$

with boundary conditions $\hat{\mu}(\bar{\theta}) = \hat{\mu}(\underline{\theta}) = 0$.

Envelope conditions are as follows:

$$\mathcal{K}_\Delta(v(\theta_-), \Delta(\theta_-), h(\theta_-), \theta_-, t) = \gamma(\theta_-), \tag{A10a}$$

$$\mathcal{K}_v(v(\theta_-), \Delta(\theta_-), h(\theta_-), \theta_-, t) = \lambda(\theta_-), \tag{A10b}$$

$$\mathcal{K}_h(v(\theta_-), \Delta(\theta_-), h(\theta_-), \theta_-, t) = \frac{z(\theta)\pi(\theta)}{[h(\theta_-)]^2}\left[\frac{-\phi'\left(\frac{z(\theta)}{h(\theta_-)}\right)\left[1-\frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)}\frac{\partial y(\theta)}{\partial \theta}\right]}{u'(c(\theta))} + \frac{\gamma(\theta_-)\frac{\partial h(\theta_-)}{\partial \theta_-}\left[2\phi'\left(\frac{z(\theta)}{h(\theta_-)}\right)+\phi''\left(\frac{z(\theta)}{h(\theta_-)}\right)\frac{z(\theta)}{h(\theta_-)}\right]}{h(\theta_-)}\right]. \tag{A10c}$$

Lagging conditions (A10a), (A10b) and (A10c) by one period, we use (A9b), (A9c) and (A9d) to obtain

$$\gamma(\theta) = \frac{-R_{t+1}\beta\hat{\mu}(\theta)}{\pi(\theta)}, \tag{A10d}$$

$$\lambda(\theta) = \beta R_{t+1}\left[\frac{1}{u'(c(\theta))} - \frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta}\right], \tag{A10e}$$

$$\frac{R_{t+1}}{\mathbb{H}^t_{x,t}} = \frac{z(\theta')\pi(\theta')}{[h(\theta)]^2}\left\{\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\left[\frac{1}{u'(c(\theta'))} - \frac{\hat{\mu}(\theta')u''(c(\theta'))}{\pi(\theta')u'(c(\theta'))}\frac{\partial y(\theta')}{\partial \theta'}\right] - \frac{\gamma(\theta)\frac{\partial h(\theta)}{\partial \theta}\left[2\phi'\left(\frac{z(\theta')}{h(\theta)}\right)+\phi''\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')}{h(\theta)}\right]}{h(\theta)}\right\}. \tag{A10f}$$

In order to determine the sign of the optimal wedge, following Stantcheva (2017) we make the following assumption.[2]

Assumption A1. $v(\theta)$ is increasing in θ and $\mathcal{K}(v, \Delta, h, \theta, t)$ is increasing and convex in v. That is, for all θ, $(\partial/\partial v)\mathcal{K} \geq 0$ and $(\partial^2/\partial v^2)\mathcal{K} \geq 0$.

According to Assumption A1, agents with a higher shock θ today tend to get the allocation yielding a higher expected future utility $v(\theta)$. Agents with a higher expected future utility v will have a higher expected discounted cost of providing an allocation $\mathcal{K}$, and moreover, the cost is increasing in v.

For any $\theta_0 \in [\underline{\theta}, \bar{\theta}]$, let the corresponding expected future utility be denoted by $v(\theta_0) \equiv v_0$. Then, the inverse function of v implies that $\theta_0 = v^{-1}(v_0)$. Based on the envelope condition (A10b), we find that

$$\frac{\partial \mathcal{K}}{\partial v_0} = \lambda(\theta_0) = \lambda\left(v^{-1}(v_0)\right). \tag{A11a}$$

Then, taking the derivative with respect to v_0 on both sides of the equation (A11a) gives

$$\frac{\partial^2 \mathcal{K}}{\partial v_0^2} = \lambda'\left(v^{-1}(v_0)\right) \cdot \frac{d}{dv_0}v^{-1}(v_0) = \lambda'(\theta_o)\frac{1}{v'(\theta_0)}. \tag{A11b}$$

By Assumption A1, (A11b) implies that $\lambda'(\theta_0) > 0$ for any $\theta_0 \in [\underline{\theta}, \bar{\theta}]$.

Building on Assumption A1, we easily establish Lemma A2 in the following, which proves that the co-state $\hat{\mu}(\theta)$ is positive, indicating a marginal cost caused by information frictions.[3]

[2] See Assumption 3 in Stantcheva (2017, p. 1978).

[3] Information frictions reduce the social welfare or raises the social cost. That is why the co-states are negative in the utility maximization problem (c.f. Lemma A1) and are positive in the cost minimization problem (c.f. Lemma A2.)

Lemma A2. *Under Assumption A1*, $\gamma(\theta) < 0 < \hat{\mu}(\theta)$ *for any* $\theta \in (\underline{\theta}, \overline{\theta})$

Proof of Lemma A2. From (A10e) and the envelope condition (A10b), we obtain

$$\frac{\partial \mathcal{K}}{\partial v} = \lambda(\theta) = \beta R_{t+1}\left[\frac{1}{u'\big(c(\theta)\big)} - \frac{\hat{\mu}(\theta)u''\big(c(\theta)\big)}{\pi(\theta)u'\big(c(\theta)\big)}\frac{\partial y(\theta)}{\partial \theta}\right].$$

By Assumption A1, we know that $\lambda(\theta)$ is increasing in θ, so the term $\frac{1}{u'(c(\theta))} - \frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta}$ increases in θ as well.

With boundary conditions $\hat{\mu}(\overline{\theta}) = \hat{\mu}(\underline{\theta}) = 0$, according to Rolle's theorem, there exists $\theta^* \in (\underline{\theta}, \overline{\theta})$ such that $\dot{\hat{\mu}}(\theta^*) = 0$. Then, the law of motion (A9e) gives

$$\lambda(\theta_-) = \frac{1}{u'(c(\theta^*))} - \frac{\hat{\mu}(\theta^*)u''(c(\theta^*))}{\pi(\theta^*)u'(c(\theta^*))}\frac{\partial y(\theta^*)}{\partial \theta^*} \tag{A12a}$$

Because $\frac{1}{u'(c(\theta))} - \frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta}$ is monotone increasing in θ, (A12a) implies that

$$\frac{1}{u'(c(\theta))} - \frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta} < \lambda(\theta_-) \qquad \forall \theta < \theta^*,$$

$$\frac{1}{u'(c(\theta))} - \frac{\hat{\mu}(\theta)u''(c(\theta))}{\pi(\theta)u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta} > \lambda(\theta_-) \qquad \forall \theta > \theta^*.$$

Integrating (A9e) and using the boundary condition $\hat{\mu}(\overline{\theta}) = 0$ we obtain

$$\hat{\mu}(\theta) = \int_{\theta}^{\overline{\theta}} \left[\frac{1}{u'(c(\theta''))} - \lambda(\theta_-)\right]\pi(\theta'') - \frac{\hat{\mu}(\theta'')u''\big(c(\theta'')\big)}{u'(c(\theta''))}\frac{\partial y(\theta'')}{\partial \theta''}d\theta''.$$

Note that this equation integrates over non-negative variables only for $\theta \in [\theta', \overline{\theta})$. Thus, the above equation implies $\hat{\mu}(\theta) > 0$ for $\theta \in [\theta', \overline{\theta})$.

Similarly, using the boundary condition $\hat{\mu}(\underline{\theta}) = 0$, we get

$$\hat{\mu}(\theta) = \int_{\underline{\theta}}^{\theta} \left[\frac{-1}{u'(c(\theta''))} + \lambda(\theta_-)\right]\pi(\theta'') + \frac{\hat{\mu}(\theta'')u''\big(c(\theta'')\big)}{u'(c(\theta''))}\frac{\partial y(\theta'')}{\partial \theta''}d\theta'',$$

which also implies $\hat{\mu}(\theta) > 0$ for $\theta \in (\underline{\theta}, \theta']$. From (A10d), $\hat{\mu}(\theta) > 0$ implies $\gamma(\theta) < 0$ for $\theta \in (\underline{\theta}, \theta']$. □

A.2.2 *Proof of Proposition 6*

Now, we prove Proposition 6 in Subsection 3.1.

(i) **Capital wedge.** First, integrating (A9e) and using the boundary condition $\hat{\mu}(\bar{\theta}) = 0$ yields:

$$\hat{\mu}(\theta) = \int_{\theta}^{\overline{\theta}} \left[\frac{1}{u'(c(\theta''))} - \lambda(\theta_-)\right]\pi(\theta'') - \frac{\hat{\mu}(\theta'')u''\big(c(\theta'')\big)}{u'(c(\theta''))}\frac{\partial y(\theta'')}{\partial \theta''}d\theta''. \tag{A13a}$$

Next, using (A10e) to replace $\lambda(\theta_-)$ in (A13a), we get

$$\hat{\mu}(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi(\theta'')}{u'(c(\theta''))} - \frac{\hat{\mu}(\theta'')u''\big(c(\theta'')\big)}{u'(c(\theta''))}\frac{\partial y(\theta'')}{\partial \theta''}d\theta'' - \beta R_t\left[\frac{1}{u'(c(\theta_-))} - \frac{\hat{\mu}(\theta_-)u''(c(\theta_-))}{\pi(\theta_-)u'(c(\theta_-))}\frac{\partial y(\theta_-)}{\partial \theta_-}\right]. \tag{A13b}$$

Moreover, using boundary condition $\hat{\mu}\big(\underline{\theta}\big) = 0$, (A13b) leads to

$$\frac{1}{u'(c(\theta_-))} - \frac{1}{\beta R_t}\int_{\underline{\theta}}^{\overline{\theta}} \frac{\pi(\theta'')}{u'(c(\theta''))} d\theta'' = \underbrace{\frac{\hat{\mu}(\theta_-)u''(c(\theta_-))}{\pi(\theta_-)u'(c(\theta_-))}\frac{\partial y(\theta_-)}{\partial \theta_-}}_{\text{Current period's HCI effect}} - \frac{1}{\beta R_t}\int_{\underline{\theta}}^{\overline{\theta}} \underbrace{\frac{\hat{\mu}(\theta'')u''\big(c(\theta'')\big)}{u'(c(\theta''))}\frac{\partial y(\theta'')}{\partial \theta''}}_{\text{Next period's HCI effect}} d\theta''. \tag{A13c}$$

With some manipulation, the above equation gives the inverse Euler equation as follows:

$$\frac{1}{u'(c(\theta_-))} - \frac{1}{\beta R_t}E\left[\frac{1}{u'(c(\theta))}\right] = \underbrace{\frac{\hat{\mu}(\theta_-)u''(c(\theta_-))}{\pi(\theta_-)u'(c(\theta_-))}\frac{\partial y(\theta_-)}{\partial \theta_-}}_{\text{Current period's HCI effect}} - \frac{1}{\beta R_t}E\left[\underbrace{\frac{\hat{\mu}(\theta)u''\big(c(\theta)\big)}{\pi(\theta)u'\big(c(\theta)\big)}\frac{\partial y(\theta)}{\partial \theta}}_{\text{Next period's HCI effect}}\right]. \tag{A13d}$$

Replacing the notation θ_- and θ by θ^{t-1} and θ^t, respectively, the above equation completes the proof of Proposition 6(i).

(ii) **Labor wedge.** For the first period $t = 1$, the Hamiltonian of the social planning problem is:

$$\mathcal{K}(\underline{v}, h_1, 1) = \left\{u^{-1}\left[W(\theta) - \beta v(\theta) + \phi\left(\frac{z(\theta)}{h_1}\right)\right] + x(\theta) + y(\theta) - w_1 z(\theta) + \frac{\mathcal{K}(v(\theta),\Delta(\theta),h(\theta),\theta,2)}{R_2}\right\}\pi(\theta)$$
$$+\lambda[\underline{v} - W(\theta)\pi(\theta)] + \hat{\mu}(\theta)\left[-u'\left(u^{-1}\left[W(\theta) - \beta v(\theta) + \phi\left(\frac{z(\theta)}{h_1}\right)\right]\right)\frac{\partial y(\theta)}{\partial \theta} + \beta \Delta(\theta)\right].$$

The first-order conditions are

$$\frac{\partial \mathcal{K}}{\partial z(\theta)} = \left[\frac{\phi'\left(\frac{z(\theta)}{h_1}\right)\frac{1}{h_1}}{u'(c(\theta))} - w_1\right]\pi(\theta) - \hat{\mu}(\theta)\frac{u''(c(\theta))\phi'\left(\frac{z(\theta)}{h_1}\right)\frac{1}{h_1}}{u'(c(\theta))}\frac{\partial y(\theta)}{\partial \theta} = 0.$$

The above condition gives the following labor wedge in the first period.

$$\tau_z(\theta) = 1 - \frac{\phi'\left(\frac{z(\theta)}{h_1}\right)\frac{1}{w_1 h_1}}{u'(c(\theta))} = \underbrace{\frac{-\hat{\mu}(\theta)u''(c(\theta))\phi'\left(\frac{z(\theta)}{h_1}\right)\frac{\partial y(\theta)}{\partial \theta}}{\pi(\theta)u'(c(\theta))w_1 h_1}}_{\text{skill-fostering effect}},$$

which shows that only the skill-fostering effect is present in the first period. For periods $t = 2, 3, \dots, T$, from (A9a), it is easy to obtain the labor wedge as follows:

$$\tau_{z_t}(\theta) = \underbrace{\frac{-\gamma(\theta_-)\frac{\partial h(\theta_-)}{\partial \theta_-}\left[\phi'\left(\frac{z(\theta)}{h(\theta_-)}\right) + \frac{z(\theta)}{h(\theta_-)}\phi''\left(\frac{z(\theta)}{h(\theta_-)}\right)\right]}{w_t[h(\theta_-)]^2}}_{\text{shirk-preventing effect}} + \underbrace{\frac{-\hat{\mu}(\theta)u''(c(\theta))\phi'\left(\frac{z(\theta)}{h(\theta_-)}\right)\frac{\partial y(\theta)}{\partial \theta}}{\pi(\theta)u'(c(\theta))w_t h(\theta_-)}}_{\text{skill-fostering effect}}.$$

Because there is no need to invest in HCI in the terminal period, we have $\frac{\partial y(\theta^T)}{\partial \theta_T} = 0$, and thus the skill-fostering effect vanishes in the terminal period. This completes the proof of Proposition 6(ii).

(iii) **Net human capital wedge.** Divided $\frac{R_{t+1}}{\mathbb{H}_x^t}$ on the both sides of (A10f), we obtain the following equation

$$1 = \frac{z(\theta')\pi(\theta')\mathbb{H}_{x,t}^t}{[h(\theta)]^2 R_{t+1}}\left\{\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\left[\frac{1}{u'(c(\theta'))} - \frac{\hat{\mu}(\theta')u''\big(c(\theta')\big)}{\pi(\theta')u'(c(\theta'))}\frac{\partial y(\theta')}{\partial \theta'}\right] - \frac{\gamma(\theta)\frac{\partial h(\theta)}{\partial \theta}\left[2\phi'\left(\frac{z(\theta')}{h(\theta)}\right) + \phi''\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')}{h(\theta)}\right]}{h(\theta)}\right\}. \tag{A14a}$$

Adding $\frac{-\beta \mathbb{H}_{x,t}^t}{u'(c(\theta))}\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')\pi(\theta')}{[h(\theta)]^2}$ on the both sides of (A14a), then we have

$$1 - \frac{\beta \mathbb{H}_{x,t}^t}{u'(c(\theta))}\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')\pi(\theta')}{[h(\theta)]^2}$$
$$= \frac{z(\theta')\pi(\theta')\mathbb{H}_{x,t}^t\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\left[\frac{1}{u'\big(c(\theta')\big)} - \frac{\beta R_{t+1}}{u'(c(\theta))} - \frac{\hat{\mu}(\theta')u''\big(c(\theta')\big)}{\pi(\theta')u'\big(c(\theta')\big)}\frac{\partial y(\theta')}{\partial \theta'}\right]}{[h(\theta)]^2 R_{t+1}} - \frac{\gamma(\theta)z(\theta')\pi(\theta')\mathbb{H}_{x,t}^t\frac{\partial h(\theta)}{\partial \theta}\left[2\phi'\left(\frac{z(\theta')}{h(\theta)}\right) + \phi''\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')}{h(\theta)}\right]}{[h(\theta)]^3 R_{t+1}}. \tag{A14b}$$

Integrating (A14b) with respect to θ' and according to the definition (10c), we get

$$\tau_x(\theta) \equiv 1 - \frac{\beta \mathbb{H}_{x,t}^{t}}{u'(c(\theta))} \int \phi'\left(\frac{z(\theta')}{h(\theta)}\right) \frac{z(\theta')}{[h(\theta)]^2} \pi(\theta') d\theta'$$

$$= \int \frac{z(\theta')\mathbb{H}_{x,t}^{t}\phi'\left(\frac{z(\theta')}{h(\theta)}\right)\left[\frac{1}{u'(c(\theta'))} - \frac{\beta R_{t+1}}{u'(c(\theta))} - \frac{\hat{\mu}(\theta')u''(c(\theta'))}{\pi(\theta')u'(c(\theta'))}\frac{\partial y(\theta')}{\partial \theta'}\right] - z(\theta')\mathbb{H}_{x,t}^{t}\frac{\gamma(\theta)}{h(\theta)}\left[2\phi'\left(\frac{z(\theta')}{h(\theta)}\right) + \phi''\left(\frac{z(\theta')}{h(\theta)}\right)\frac{z(\theta')}{h(\theta)}\right]\frac{\partial h(\theta)}{\partial \theta}}{[h(\theta)]^2 R_{t+1}} \pi(\theta') d\theta'. \quad \text{(A14c)}$$

Replacing θ' and θ by θ^{t+1} and θ^{t}, respectively, and using notation $\Omega_t = \Omega_t(c(\theta^{t-1}), c(\theta^{t})$ defined in (7a), equation (A14c) becomes

$$\tau_x(\theta^t) = E_t\left[\frac{-\mathbb{H}_{x,t}^{t} z(\theta^{t+1})\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\left[\beta R_{t+1}\Omega_t + \frac{\hat{\mu}(\theta^{t+1})u''(c(\theta^{t+1}))}{\pi(\theta_{t+1})u'(c(\theta^{t+1}))}\frac{\partial y(\theta^{t+1})}{\partial \theta_{t+1}}\right]}{[h(\theta^t)]^2 R_{t+1}} - \frac{\mathbb{H}_{x,t}^{t} z(\theta^{t+1})\gamma(\theta^t)\frac{\partial h(\theta^t)}{\partial \theta_t}\left[2\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right) + \phi''\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\frac{z(\theta^{t+1})}{h(\theta^t)}\right]}{[h(\theta^t)]^3 R_{t+1}}\right].$$

Using equation (11b) in Proposition 6(ii), the above equation can be rewritten as follows:

$$\tau_x(\theta^t) = E_t\left[\frac{-\mathbb{H}_{x,t}^{t} z(\theta^{t+1})}{[h(\theta^t)]^2}\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\beta\Omega_t + \frac{\mathbb{H}_{x,t}^{t} z(\theta^{t+1})}{h(\theta^t)R_{t+1}} w_t \tau_z(\theta^{t+1}) - \frac{\mathbb{H}_{x,t}^{t} z(\theta^{t+1})\gamma(\theta^t)}{[h(\theta^t)]^3 R_{t+1}}\frac{\partial h(\theta^t)}{\partial \theta_t}\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\right]. \quad \text{(A14d)}$$

Using notations $\mathcal{N}_{t+1} = \frac{w_{t+1}z_{t+1}}{R_{t+1}h_{t+1}}\mathbb{H}_{x,t}^{t}\tau_{z_{t+1}}$ and $\mathcal{K}_{t+1} = \frac{\beta z_{t+1}}{(h_{t+1})^2}\phi'\left(\frac{z_{t+1}}{h_{t+1}}\right)\mathbb{H}_{x,t}^{t}\Omega_t$ defined in Definition 1, then (A14d) becomes

$$\tau_x(\theta^t) = E_t\left[-\mathcal{K}_{t+1}(\theta^{t+1}) + \mathcal{N}_{t+1}(\theta^{t+1}) - \frac{\mathbb{H}_{x,t}^{t} z(\theta^{t+1})\gamma(\theta^t)}{[h(\theta^t)]^3 R_{t+1}}\frac{\partial h(\theta^t)}{\partial \theta_t}\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\right]. \quad \text{(A14e)}$$

Based on the Definition 1 and equation (A14e), the net human capital wedge is derived as follows:

$$\tau_x^n(\theta^t) \equiv E_t[\tau_x(\theta^t) - \mathcal{N}_{t+1}(\theta^{t+1}) + \mathcal{K}_{t+1}(\theta^{t+1})] = E_t\left[\frac{-\mathbb{H}_{x,t}^{t} z(\theta^{t+1})\gamma(\theta^t)}{[h(\theta^t)]^3 R_{t+1}}\frac{\partial h(\theta^t)}{\partial \theta_t}\phi'\left(\frac{z(\theta^{t+1})}{h(\theta^t)}\right)\right],$$

which completes the proof of Proposition 6(iii). □

A.2.3 *Proof of Corollary 4*

(i) When there are non-verifiable education expenses, in the terminal period, there is only the current HCI effect, so (A13d) becomes

$$\frac{1}{u'(c(\theta_-))} - \frac{1}{\beta R_T} E\left[\frac{1}{u'(c(\theta))}\right] = \frac{\hat{\mu}(\theta_-)u''(c(\theta_-))}{\pi(\theta_-)u'(c(\theta_-))}\frac{\partial y(\theta_-)}{\partial \theta_-}. \quad \text{(A15a)}$$

By Lemma A2 and Lemma 2(i), we have $\hat{\mu}(\theta) > 0$ and $\frac{\partial y(\theta)}{\partial \theta} < 0$ if $\rho_{y\theta,T-1}^{T-1} \leq 2$,. Then, (A15a) gives

$$\frac{1}{u'(c(\theta_-))} > \frac{1}{\beta R_T} E\left[\frac{1}{u'(c(\theta))}\right],$$

which implies a larger capital wedge than the case with no non-verifiable education expenses.

(ii) If there are only verifiable education expenses, $\frac{\partial y(\theta_-)}{\partial \theta_-} = \frac{\partial y(\theta)}{\partial \theta} = 0$ and the right-hand side of (A13d) is zero. Then, equation (A13d) reduces to the inverse Euler equation and the capital wedge is positive that involves the effect of insurance purposes. This completes the proof of Corollary 4. □

A.2.4 *Proof of Corollary 5*

(i) When there are non-verifiable education expenses, based on Lemma 2(i), we obtain $\frac{\partial y(\theta^t)}{\partial \theta_t} < 0$ if $\rho_{y\theta,t}^{t} \leq 2$. Similar to the proof of Lemma A2, it is easy to show that $\hat{\mu}(\theta) > 0, \forall \theta \in (\underline{\theta}, \overline{\theta})$. Thus, equation (11b) in

Proposition 8 implies that in the first period, the labor wedge is negative, i.e., $\tau_z(\theta) < 0,$ *for* $\theta \in (\underline{\theta}, \overline{\theta})$.

(ii) When there are non-verifiable education expenses, based on Lemma 2 (ii), we obtain $\frac{\partial h(\theta^t)}{\partial \theta_t} > 0$ if $\rho^t_{y\theta,t} \geq \frac{\mathbb{H}^t_{yy}\mathbb{H}^t}{(\mathbb{H}^t_y)^2}$. Besides, based on Lemma A2, we have $\gamma(\theta^{T-1}) < 0$, and thus, equation (11b) in Proposition 8 implies that in the terminal period, the labor wedge is positive, i.e., $\tau_z(\theta^T) > 0,$ *for* $\theta \in (\underline{\theta}, \overline{\theta})$.

(iii) When there are only verifiable education expenses, by Lemma 2 (iii), we obtain $\frac{\partial y(\theta^t)}{\partial \theta_t} = 0$ *and* $\frac{\partial h(\theta^t)}{\partial \theta_t} > 0$, which implies that the last term of (11b) is zero, and the labor wedge is zero in the first period. Besides, because of $\frac{\partial h(\theta^t)}{\partial \theta_t} > 0$ and $\gamma(\theta^{t-1}) < 0$ in Lemma A2, the first term of (11b) is positive, and thus the labor wedge is positive for all except for the first period. That is, $\tau_z(\theta^t) > 0$ *for* $\forall \theta_t \in (\underline{\theta}, \overline{\theta})$ *and* $t \geq 2$ □

A.2.5 *Proof of Corollary 6*

(i) When there are non-verifiable education expenses, based on Lemma 2(ii), we obtain $\frac{\partial h(\theta^t)}{\partial \theta_t} > 0$ if $\rho^t_{y\theta,t} \geq \frac{\mathbb{H}^t_{yy,t}\mathbb{H}^t}{(\mathbb{H}^t_{y,t})^2}$. In addition, based on Lemma A2, we have $\gamma(\theta^t) < 0$, and thus, Proposition 9 implies that the net human capital wedge is positive. That is, $\tau^n_x(\theta^t) > 0, \forall \theta_t \in (\underline{\theta}, \overline{\theta})$.

(ii) When there are only verifiable education expenses, by Lemma 2(iii), we obtain $\frac{\partial h(\theta^t)}{\partial \theta_t} > 0$, and thus, Proposition 9 implies that the net human capital wedge is positive. That is, $\tau^n_x(\theta^t) > 0, \forall \theta_t \in (\underline{\theta}, \overline{\theta})$. □

A.2.6 *Proof of Proposition 7*

Suppose that an agent with the type history $\theta^t = (\theta_1, \theta_2, \ldots, \theta_t)$ reports type $\sigma = (\sigma_1, \sigma_2, .., \sigma_t)$. The agent chooses non-verifiable education expenses $y^\sigma(\theta^t)$ to maximize the following problem:

$$W(\theta^t) = \max_{y^\sigma(\theta^t)} u\big(c(\sigma^t) + y(\sigma^t) - y^\sigma(\theta^t)\big) - \phi\left(\frac{z(\sigma^t)}{h^\sigma(\theta^{t-1})}\right)$$
$$+ \textstyle\sum_{s=t+1}^T \beta^{s-t} \int \left[u\big(c(\theta^s)\big) - \phi\left(\frac{z(\theta^s)}{h^\sigma(\theta^{s-1})}\right)\right] \pi(\theta^s_{t+1}) d\theta^s_{t+1}$$

The first-order condition with respect to $y^\sigma(\theta^t)$ is

$$u'\big(c(\sigma^t) + y(\sigma^t) - y^\sigma(\theta^t)\big) = -\textstyle\sum_{s=t+1}^T \beta^{s-t} \int \left[\phi_h\left(\frac{z(\sigma^s)}{h^\sigma(\theta^{s-1})}\right) \mathbb{H}^{s-1}_{y,t}\right] \pi(\theta^s_{t+1}) d\theta^s_{t+1}. \qquad \text{(A16a)}$$

Using the fact that $c^\sigma(\theta^t) = c(\sigma^t) + y(\sigma^t) - y^\sigma(\theta^t)$, $\mathbb{H}^{s-1}_{y,t} \equiv \frac{\partial \mathbb{H}^{s-1}(\mathbf{X}(\sigma^{s-1}), \mathbf{Y}^\sigma(\theta^{s-1}), \theta^{s-1})}{\partial y^\sigma(\theta^t)}$, and $\phi_h\left(\frac{z(\sigma^s)}{h^\sigma(\theta^{s-1})}\right) = -\phi'\left(\frac{z(\sigma^s)}{h^\sigma(\theta^{s-1})}\right)\frac{z(\sigma^s)}{h^\sigma(\theta^{s-1})^2}$, the proof is complete. □

A.2.7 *Proof of Lemma 3*

(i) First, Proposition 7 indicates that the optimal choice of the agent with reporting strategy σ^t satisfies the optimal condition (A16a). Differentiating equation (A16a) with respect to θ_t yields $u''\big(c(\sigma^t) + y(\sigma^t) - y^\sigma(\theta^t)\big)\frac{\partial y^\sigma(\theta^t)}{\partial \theta_t} = \sum_{s=t+1}^T \beta^{s-t} \int \left[\phi_{hh}\left(\frac{z(\sigma^s)}{h^\sigma(\theta^{s-1})}\right) \mathbb{H}^{s-1}_{y,t}\left[\mathbb{H}^{s-1}_{\theta,t} + \mathbb{H}^{s-1}_{y,t}\frac{\partial y^\sigma(\theta^t)}{\partial \theta_t}\right]\right.$

$$+ \phi_h\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\left[\mathbb{H}_{y\theta,t}^{s-1} + \mathbb{H}_{yy,t}^{s-1}\frac{\partial y^\sigma(\theta^t)}{\partial\theta_t}\right]\Bigg]\pi(\theta_{t+1}^s)d\theta_{t+1}^s, \tag{A16b}$$

where $h^\sigma(\theta^{s-1}) = \mathbb{H}^{s-1}(\mathbf{X}(\sigma^{s-1}), \mathbf{Y}^\sigma(\theta^{s-1}), \theta^{s-1})$, $\mathbb{H}_{y,t}^{s-1}, \mathbb{H}_{\theta,t}^{s-1} > 0$ and $\mathbb{H}_{yy,t}^{s-1} < 0$.

Manipulation of this above condition (A16b) gives

$$\frac{\partial y^\sigma(\theta^t)}{\partial\theta_t} = \frac{\sum_{s=t+1}^T \beta^{s-t}\int\left[\phi_{hh}\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\mathbb{H}_{y,t}^{s-1}\mathbb{H}_{\theta,t}^{s-1} + \phi_h\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\mathbb{H}_{y\theta,t}^{s-1}\right]\pi(\theta_{t+1}^s)d\theta_{t+1}^s}{u''(c^\sigma(\theta^t)) - \sum_{s=t+1}^T \beta^{s-t}\int\left[\phi_{hh}\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\left(\mathbb{H}_{y,t}^{s-1}\right)^2 + \phi_h\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\mathbb{H}_{yy,t}^{s-1}\right]\pi(\theta_{t+1}^s)d\theta_{t+1}^s}, \tag{A16c}$$

where $\pi(\theta_{t+1}^s) \equiv \pi(\theta_s)..\pi(\theta_{t+1})$ and $d\theta_{t+1}^s \equiv d\theta_s \dots d\theta_{t+1}$

Using the fact that $\phi_{hh}\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right) = \phi''\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\frac{[z(\sigma^S)]^2}{[h^\sigma(\theta^{s-1})]^4} + 2\phi'\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\frac{z(\sigma^S)}{[h^\sigma(\theta^{s-1})]^3} > 0$, and $\phi_h\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right) = -\phi'\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\frac{z(\sigma^S)}{[h^\sigma(\theta^{s-1})]^2} < 0$, equation (A16c) can be written as follows:

$$\frac{\partial y^\sigma(\theta^t)}{\partial\theta_t} = \frac{\sum_{s=t+1}^T \beta^{s-t}\int\frac{z(\sigma^S)}{[h^\sigma(\theta^{s-1})]^3}\left[\phi''\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\mathbb{H}_{y,t}^{s-1}\mathbb{H}_{\theta,t}^{s-1} + \phi'\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\left[2\mathbb{H}_{y,t}^{s-1}\mathbb{H}_{\theta,t}^{s-1} - \mathbb{H}^{s-1}\mathbb{H}_{y\theta,t}^{s-1}\right]\right]\pi(\theta_{t+1}^s)d\theta_{t+1}^s}{u''(c^\sigma(\theta^t)) - \sum_{s=t+1}^T \beta^{s-t}\int\left[\phi_{hh}\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\left(\mathbb{H}_{y,t}^{s-1}\right)^2 + \phi_h\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\mathbb{H}_{yy,t}^{s-1}\right]\pi(\theta_{t+1}^s)d\theta_{t+1}^s}. \tag{A16d}$$

Since $h^\sigma(\theta^{s-1}) = \mathbb{H}^{s-1}(\mathbf{X}(\sigma^{s-1}), \mathbf{Y}^\sigma(\theta^{s-1}), \theta^{s-1})$ and $\mathbb{H}_{yy,t}^{s-1} < 0 < \mathbb{H}_{y,t}^{s-1}$, along with the fact that $\phi_h < 0$ and $\phi_{hh}, \phi'', \phi' > 0$, the above equation (A16d) implies that $\frac{\partial y^\sigma(\theta^t)}{\partial\theta_t} < 0$, if $2\mathbb{H}_{y,t}^s\mathbb{H}_{\theta,t}^s \geq \mathbb{H}^s\mathbb{H}_{y\theta,t}^s$ for all $s \geq t$. That is $\rho_{y\theta,t}^s \leq 2$ for all $s \geq t$.

(ii) Moreover, using (A16d) and the fact that $h^\sigma(\theta^t) = \mathbb{H}^t(\mathbf{X}(\sigma^t), \mathbf{Y}^\sigma(\theta^t), \theta^t)$, we obtain

$$\frac{dh^\sigma(\theta^t)}{d\theta_t} = \mathbb{H}_{y,t}^t\frac{\partial y^\sigma(\theta^t)}{\partial\theta_t} + \mathbb{H}_{\theta,t}^t$$

$$= \frac{\mathbb{H}_{\theta,t}^t u''(c^\sigma(\theta^t)) + \sum_{s=t+1}^T \beta^{s-t}\int\left[\phi_{hh}\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\left(\mathbb{H}_{y,t}^{s-1}\right)^2\mathbb{H}_{y,t}^t\left[\frac{\mathbb{H}_{\theta,t}^{s-1}}{\mathbb{H}_{y,t}^{s-1}} - \frac{\mathbb{H}_{\theta,t}^t}{\mathbb{H}_{y,t}^t}\right] + \phi_h\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\mathbb{H}_{yy,t}^{s-1}\mathbb{H}_{y,t}^t\left[\frac{\mathbb{H}_{y\theta,t}^{s-1}}{\mathbb{H}_{yy,t}^{s-1}} - \frac{\mathbb{H}_{\theta,t}^t}{\mathbb{H}_{y,t}^t}\right]\right]\pi(\theta_{t+1}^s)d\theta_{t+1}^s}{u''(c^\sigma(\theta^t)) - \sum_{s=t+1}^T \beta^{s-t}\int\left[\phi_{hh}\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\left(\mathbb{H}_{y,t}^{s-1}\right)^2 + \phi_h\left(\frac{z(\sigma^S)}{h^\sigma(\theta^{s-1})}\right)\mathbb{H}_{yy,t}^{s-1}\right]\pi(\theta_{t+1}^s)d\theta_{t+1}^s}. \tag{A16e}$$

When human capital is not fully depreciated, conditions $\frac{\mathbb{H}_{\theta,t}^t}{\mathbb{H}_{y,t}^t} \geq \frac{\mathbb{H}_{\theta,t}^S}{\mathbb{H}_{y,t}^S}$ and $\rho_{y\theta,t}^s \geq \frac{\mathbb{H}_{yy,t}^S\mathbb{H}^S}{\left(\mathbb{H}_{y,t}^S\right)^2}$ imply that

$$\frac{\mathbb{H}_{\theta,t}^t}{\mathbb{H}_{y,t}^t} \geq \frac{\mathbb{H}_{\theta,t}^S}{\mathbb{H}_{y,t}^S} \geq \frac{\mathbb{H}_{y\theta,t}^S}{\mathbb{H}_{yy,t}^S} \text{ for any } s \geq t.$$

Based on equation (A16e), the above condition implies that $\frac{dh^\sigma(\theta^t)}{d\theta_t} > 0$.

(iii) If there are only verifiable education expenses, then $y^\sigma(\theta^t) = 0$ for all $\sigma(\theta^t)$, and hence

$$\frac{\partial y^\sigma(\theta^t)}{\partial\theta_t} = 0 \text{ and } \frac{dh^\sigma(\theta^t)}{d\theta_t} = \mathbb{H}_{y,t}^t\frac{\partial y^\sigma(\theta^t)}{\partial\theta_t} + \mathbb{H}_{\theta,t}^t = \mathbb{H}_{\theta,t}^t > 0. \ \square$$

A.3 Extension of a one-deviation strategy to multi-deviation strategies

The Appendix shows that, under proper conditions, the incentive compatibility of the one-deviation strategy can be extended to a multi-deviation strategy. The goal is to show that if a one-deviation strategy cannot generate a higher lifetime utility than a truth-telling strategy, then neither can a multi-deviation strategy. In other words, given the true type θ^T, suppose that the best strategy among all one-deviation strategies $\hat{\sigma} = (\theta_1, \dots, \hat{\theta}_t, \dots, \theta_T)$ is to tell the truth; namely,

$$W(\theta^t) = \max_{\hat{\theta}} W^{\hat{\sigma}}(\theta^t).$$

Then, telling the truth is also the best strategy among all multi-deviation strategies.

Without loss of generality, consider a strategy σ that reports the type truthfully in all periods except in $t = t_1, .., t_m$. To be more specific, denote the reported type under strategy σ as σ^T, which follows the following rule.

$$\sigma^T = \{(\sigma_1, \dots, \sigma_T) | \sigma_s = \theta_s,\ \forall s \notin \{t_1, \dots, t_m\}\}.$$

We will show that, even allowing for the possibility of deviating in the future periods, the agent will choose to tell the truth; that is,

$$W(\theta^{t_1}) = \max_{\sigma_{t_1}, \dots, \sigma_{t_m}} W^{\sigma}(\theta^{t_1}).$$

Before we proceed to the proof, we establish a result in the following Lemma.

Lemma A3. *With human capital technology (8b),* $y^{\sigma}(\theta^t) = y^{\sigma}(\tilde{\theta}^{t-1}, \theta_t)$ *and* $c^{\sigma}(\theta^t) = c^{\sigma}(\tilde{\theta}^{t-1}, \theta_t)$ *for any* $\tilde{\theta}^{t-1} \in \Theta^{t-1}$ *and any given strategy* σ.

Proof of Lemma A3: According to the optimal condition (8c), the optimal condition to determine the value of $y = y^{\sigma}(\theta^t)$ is

$$u'(c(\sigma^t) + y(\sigma^t) - y) = -\beta \int \phi_h \left(\frac{z(\sigma^{t+1})}{\mathbb{H}^t(x(\sigma^t), y, \theta_t)} \right) \frac{\partial}{\partial y} \mathbb{H}^t(x(\sigma^t), y, \theta_t) \pi(\theta_{t+1}) d\theta_{t+1}.$$

As can be seen, this condition does not depend on the past history of true types θ^{t-1}, and hence the optimal condition for $y = y^{\sigma}(\tilde{\theta}^{t-1}, \theta_t)$ is the same as the above equation. Therefore, $y^{\sigma}(\theta^t) = y^{\sigma}(\tilde{\theta}^{t-1}, \theta_t)$. Based on the fact that $c^{\sigma}(\theta^t) = c(\sigma^t) + y(\sigma^t) - y^{\sigma}(\theta^t)$, the property $y^{\sigma}(\theta^t) = y^{\sigma}(\tilde{\theta}^{t-1}, \theta_t)$ implies $c^{\sigma}(\theta^t) = c^{\sigma}(\tilde{\theta}^{t-1}, \theta_t)$. □

Assumption A2. Suppose that one-deviation strategies satisfy the following conditions.

(i) Among all one-deviation strategies $\hat{\sigma} = (\theta_1, \dots, \hat{\theta}_t, \dots, \theta_T)$, telling the truth is the best reporting strategy:

$$W(\theta^t) \geq W^{\hat{\sigma}}(\theta^t).$$

(ii) Given any current human capital level h_t, the lifetime utility $\widetilde{W}^{\hat{\sigma}}(\theta^t; h_t)$ is defined as

$$\widetilde{W}^{\hat{\sigma}}(\theta^t; h_t) = u\left(c^{\hat{\sigma}}(\theta^t)\right) - \phi\left(\frac{z(\theta^{t-1}, \hat{\theta}_t)}{h_t}\right) + \beta \int W^{\hat{\sigma}}(\theta^{t+1}) \pi(\theta_{t+1}) d\theta_{t+1}. \tag{A17a}$$

Condition (i) still applies to $\widetilde{W}^{\hat{\sigma}}(\theta^t; h_t)$; that is, $\widetilde{W}(\theta^t; h_t) \geq \widetilde{W}^{\hat{\sigma}}(\theta^t; h_t)$.

The incentive compatibility condition in the text assures that Condition (i) is met. Moreover, Condition (ii) assumes that, just like a predetermined human capital in the first period, when the human capital level is given in period t, then a one-deviation strategy cannot generate a higher utility than a truth-telling strategy.

We are ready to prove that if a one-deviation strategy cannot lead to a higher utility than a truth-telling strategy, then neither can a multi-deviation strategy.

Proposition A1. *With human capital technology (8b) and Assumption A2, for any multi-deviation strategy* σ*, which reports the type truthfully in all periods except* $t = t_1, .., t_m$*, where* $t_m > \cdots > t_1$*, the best reported type in the first* *possible-deviation period* $t = t_1$ *is to tell the truth. That is,* $\sigma^*_{t_1} = \theta_{t_1}$ *and thus,* $W(\theta^{t_1}) = \max_{\sigma_{t_1},\ldots,\sigma_{t_m}} W^\sigma(\theta^{t_1})$.

Proof of Proposition A1: We start from the last possible deviation period t_m.

First, we show that the best strategy in period t_m is to report the true type; that is, $\sigma^*_{t_m} = \theta_{t_m}$. Given a human capital level h_{t_m}, the lifetime utility of an agent with type θ^{t_m} in period t_m is

$$\begin{aligned}\widetilde{W}^\sigma(\theta^{t_m}; h_{t_m}) &= u\big(c^\sigma(\theta^{t_m})\big) - \phi\left(\frac{z(\sigma^{t_m})}{h_{t_m}}\right) + \beta \int W^\sigma(\theta^{t_m+1})\pi(\theta_{t_m+1})d\theta_{t_m+1} \\ &= u\big(c^\sigma(\theta^{t_m})\big) - \phi\left(\frac{z(\sigma^{t_m})}{h_{t_m}}\right) \\ &\quad + \beta^{s-t_m}\int \sum_{s=t_m+1}^{T}\left[u\big(c^\sigma(\theta^s)\big) - \phi\left(\frac{z(\sigma^s)}{\mathbb{H}^{s-1}(x(\sigma^{s-1}), y^\sigma(\theta^{s-1}), \theta_{s-1})}\right)\right]\pi(\theta^s_{t_m+1})d\theta^s_{t_{m+1}}.\end{aligned}$$

The lifetime utility of another agent with type $(\sigma^{t_m-1}, \theta_{t_m})$ in period t_m is

$$\begin{aligned}&\widetilde{W}^\sigma\big((\sigma^{t_m-1}, \theta_{t_m}); h_{t_m}\big) = u\big(c^\sigma(\sigma^{t_m-1}, \theta_{t_m})\big) - \phi\left(\frac{z(\sigma^{t_m})}{h_{t_m}}\right) + \beta\int W^\sigma(\sigma^{t_m-1}, \theta_{t_m}, \theta_{t_m+1})\pi(\theta_{t_m+1})d\theta_{t_m+1} \\ &= u\big(c^\sigma(\sigma^{t_m-1}, \theta_{t_m})\big) - \phi\left(\frac{z(\sigma^{t_m})}{h_{t_m}}\right) \\ &\quad + \beta^{s-t_m}\int \sum_{s=t_m+1}^{T}\left[u\big(c^\sigma(\sigma^{t_m-1}, \theta_{t_m}, \ldots \theta_s)\big) - \phi\left(\frac{z(\sigma^s)}{\mathbb{H}^{s-1}\big(x(\sigma^{s-1}), y^\sigma(\sigma^{t_m-1}, \theta_{t_m}, \ldots \theta_{s-1})\big) + \theta_{s-1}}\right)\right]\pi(\theta^s_{t_m+1})d\theta^s_{t_{m+1}}.\end{aligned}$$

By Lemma A3, $y^\sigma(\theta^s) = y^\sigma(\sigma^{s-1}, \theta_s)$ and $c^\sigma(\theta^s) = c^\sigma(\sigma^{s-1}, \theta_s)$. Then, from the above equations, one can easily show that, given the same reporting strategy σ in which t_m is the last possible deviation period, the agent with type θ^{t_m} and the other agent with type $(\sigma^{t_m-1}, \theta_{t_m})$ have the same the lifetime utility; that is,

$$\widetilde{W}^\sigma(\theta^{t_m}; h_{t_m}) = \widetilde{W}^\sigma\big((\sigma^{t_m-1}, \theta_{t_m}); h_{t_m}\big).$$

Note that, for the agent with type $(\sigma^{t_m-1}, \theta_{t_m})$, his or her reporting strategy σ can be seen as a one-deviation strategy, which reports the type truthfully for all periods except t_m. Then, Condition (ii) in Assumption A2 applies; that is,

$$\theta_{t_m} \in \operatorname*{argmax}_{\sigma_{t_m}} \widetilde{W}^\sigma\left((\sigma^{t_m-1}, \theta_{t_m}); h_{t_m}\right).$$

Since $\widetilde{W}^\sigma(\theta^{t_m}; h_{t_m}) = \widetilde{W}^\sigma\big((\sigma^{t_m-1}, \theta_{t_m}); h_{t_m}\big)$, we also obtain

$$\theta_{t_m} \in \operatorname*{argmax}_{\sigma_{t_m}} \widetilde{W}^\sigma(\theta^{t_m}; h_{t_m}). \tag{A17b}$$

This completes the proof that, in the last possible deviation period t_m, the best reporting strategy is to tell the truth. Thus, no matter whether an agent reports his types truthfully or not in periods before t_m, the agent obtains the same utility if the type is reported truthfully in period t_m. Since the condition (ii) in Assumption A2 assumes that the (temporary) incentive compatibility holds for any given level of human capital, the optimal condition in (A17b) is also true for any given level of human capital h_{t_m} in period t_m. That is, $\widetilde{W}^\sigma(\theta^{t_m}; h_{t_m}) = \widetilde{W}^\sigma\big((\sigma^{t_m-1}, \theta_{t_m}); h_{t_m}\big)$. Then, it suffices to show that in period t_m, the best reporting strategy is to tell the truth. That is, when $h_{t_m} = h(\theta^{t_m-1})$, based on the definition in (A17a) $\widetilde{W}^\sigma(\theta^{t_m}; h_{t_m}) = W^{\hat{\sigma}}(\theta^{t_m})$ and (A17b) implies that

$$\theta_{t_m} \in \underset{\sigma_{t_m}}{\operatorname{argmax}} \; W^{\hat{\sigma}}(\theta^{t_m}).$$

Even if deviating in t_m may generate a higher utility from the perspectives in earlier periods, it is not a time consistent strategy, because it cannot be carried out in period t_m.

The intuition behind this result is that, when the human capital technology is fully depreciated after one period, if deviating in period t_m, the effects of the past reported types $\theta^{t_{m-1}}$ are cancelled out in period t_m due to the full depreciation. In this circumstance, the past reported types do not affect the future human capital level, the future consumption and the future non-verifiable human capital investment in period t_m. In other words, these private decisions are independent of the past history of reported types $\theta^{t_{m-1}}$. Therefore, misreporting in the past and truth-telling in the past do not affect private decisions in period t_m. Hence, in the last possible deviation period t_m, the muti-deviation policy is equivalent to the one-deviation policy. If t_m is the final period, the situation is the same as that in any other period, except for no investment in human capital in the terminal period.

Next, following the same method, we can easily prove that, for all periods before t_m except t_1 (i.e., $t = t_{m-1}, \ldots, t_2$), the best reporting strategy is to tell the truth.

Finally, given that the agent tells the truth in all future possible deviation periods $t = t_{m-1}, \ldots, t_2$, then in period t_1, the reporting strategy σ is a one-deviation strategy and thus, using Condition (i) in Assumption A2, the best strategy is to tell the truth. This completes the proof. □

A.4 Tax implementation in an Equilibrium

While it is tempting to interpret the capital and labor wedges defined in (10a) and (10b) as capital and labor taxes, because there is a *double deviation* problem,[4] the relationship between wedges and taxes is not straightforward.[5] In this appendix, we build a tax system to implement the associated constrained efficient allocation in a decentralized economy. Such tax implementations usually are not unique. In general, there are different tax systems that can implement constrained efficient allocations as an equilibrium in a decentralized economy. See, e.g., Albanesi and Sleet (2006), Kocherlakota (2005) and Grochulski and Piskorski (2010). In our implementation, we show that the capital wedge and labor wedge are not just implicit marginal tax rates, but they are also explicit marginal tax rates in our tax system.

Golosov et al. (2006) pointed out that the simplest method of implementation is to assign arbitrarily high punishments if an agent's observable allocation in any period is different from the constrained efficient

[4] Intuitively, each wedge controls only one aspect of a worker's behavior (labor in a period, or savings) taking all other choices fixed *at the optimal level*. For example, assuming that an agent supplies the socially optimal amount of labor, a capital tax defined by a capital wedge would ensure that the agent also makes a socially optimal amount of savings. However, agents choose labor and savings jointly; if an agent considers changing her labor, then, in general, she also considers changing her savings. Because the wedges are not constant values, joint changes in savings and labor may change the wedges to other values, which possibly gives allocations that are better than socially optimal allocations. Thus, there are *double deviations*. Kocherlakota (2005) and Albanesi and Sleet (2006) showed that such double deviations would give an agent a higher utility than the utility from the socially optimal allocations, and therefore the optimal tax system must be enriched with additional elements in order to implement the optimal allocations.

[5] Moreover, the wedges may also differ from taxes because of the general-equilibrium effects, but these effects are shut down in the model.

allocation. Yet, this way severely limits an agent's choices and may be unrealistic. To relax the limitation and create a direct connection between wedges and taxes, we provide a simple tax system with linear income taxes to implement the constrained efficient allocation in a market economy. Different from Grochulski and Piskorski (2010), deferred capital taxes are not necessary for our linear capital income taxes. We show that these optimal linear capital and labor income tax rates are exactly the same as the optimal wedges established by the social planner.

A class of the tax system. Our tax system is described as follows. The tax system $\{\mathcal{T}_t\}$ includes linear labor and capital income tax rates ($\hat{\tau}_z$ and $\hat{\tau}_k$) and lump-sum taxes (Γ). Note that the government observes verifiable education expenses, effective labor and capital ($\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1}$), but cannot observe non-verifiable human capital investment. These linear labor and capital income tax rates thus depend on agents' history of verifiable education expenses, with the tax rates being $\hat{\tau}_z(\tilde{x}^t)$ and $\hat{\tau}_k(\tilde{x}^t)$ which are different for different agents. Moreover, the lump-sum tax depends on these three observable allocations,[6] since the lump-sum tax $\Gamma(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1})$ is different for different agents.

Therefore, for period t, the tax revenue of an agent with observable allocation $(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1})$ in our tax system is

$$\mathcal{T}_t \equiv \Gamma\left(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1}\right) + \hat{\tau}_z(\tilde{x}^t) w_t \tilde{z}_t + \hat{\tau}_k(\tilde{x}^{t-1}) R_t \tilde{k}_t. \tag{A18a}$$

Income condition and reduced forms of taxes. Since a part of HCIs is verifiable, our tax system punishes agents whose verifiable HCIs deviate from the constrained efficient allocation. Let the set of recommended verifiable education expenses be $X^t \equiv \{x^t : \exists \hat{\theta}^t \in \Theta^t \; s.t. \; x^t = x^t(\hat{\theta}^t)\}$, and agents are required to choose one of the allocations in X^t; if otherwise, they will face severe punishment. As for capital and effective labor, agents are not obligated to choose constrained efficient allocations, but are expected to make sure agents' after-tax income consistent with the constrained efficient allocation. Thus, we impose the following "income" condition:

$$S\left(x^t(\hat{\theta}^t), \tilde{z}_t, \tilde{k}_{t+1}\right) \equiv \left(1 - \hat{\tau}_z\left(x^t(\hat{\theta}^t)\right)\right) w_t\left(z(\hat{\theta}^t) - \tilde{z}_t\right) - \left(k(\hat{\theta}^t) - \tilde{k}_{t+1}\right) = 0, \text{ for all } \hat{\theta}^t \in \Theta^t. \tag{A18b}$$

To avoid punishment, agents have to choose the observable allocation $\left(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1}\right)$ such that $\tilde{x}^t = x^t(\hat{\theta}^t)$ and the income condition $S\left(x^t(\hat{\theta}^t), \tilde{z}_t, \tilde{k}_{t+1}\right) = 0$ is met for some $\hat{\theta}^t \in \Theta^t$; if otherwise, they face a severe punishment $\Gamma\left(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1}\right) = \infty$. Given such conditions, our tax system specifies the following reduced-form taxes for linear labor and capital income tax rates and lump-sum taxes.[7]

$$\begin{cases} \hat{\tau}_z\left(x^t(\hat{\theta}^t)\right) = \tilde{\tau}_z\left(\hat{\theta}^t\right) \\ \hat{\tau}_k\left(x^t(\hat{\theta}^t)\right) = \tilde{\tau}_k\left(\hat{\theta}^t\right) \\ \Gamma\left(x^t(\hat{\theta}^t), \tilde{z}_t, \tilde{k}_{t+1}\right) = \tilde{\Gamma}\left(\hat{\theta}^t\right) \qquad if \quad S\left(x^t(\hat{\theta}^t), \tilde{z}_t, \tilde{k}_{t+1}\right) = 0 \end{cases}$$

[6] These three observable allocations $(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1})$ in the lump-sum tax Γ are used to check whether the income condition (A18b), to be specified below, is satisfied or not. If the income condition is not met, then agents would be severely punished through the lump-sum tax. Once the income condition is met, then the lump-sum tax Γ is not affected by the value of these three observable allocations $\left(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1}\right)$.

[7] Although Γ depends on the current labor z, it is a lump-sum tax, because once the income condition is satisfied, Γ depends on only the reported type.

Implementation with the tax system. Given physical and human capital $(\tilde{k}_t, \tilde{h}_t)$ accumulated from previous periods, under our tax system $\{\mathcal{T}_t\}$, the problem of type θ_t agent in the decentralized economy in period t is:

$$\tilde{U}^t(\tilde{k}_t, \tilde{h}_t, \theta_t) = \max u(\tilde{c}_t) - \phi\left(\frac{\tilde{z}_t}{\tilde{h}_t}\right) + \beta E_t[\tilde{U}^{t+1}(\tilde{k}_{t+1}, \tilde{h}_{t+1}, \theta_{t+1})], \tag{A18c}$$

$$\text{subject to} \quad \tilde{c}_t + \tilde{x}_t + \tilde{y}_t + \tilde{k}_{t+1} \leq w_t \tilde{z}_t + R_t \tilde{k}_t - \mathcal{T}_t,$$
$$\tilde{h}_{t+1} = \mathbb{H}^t(\tilde{x}_t, \tilde{y}_t, \theta_t),$$

where $E_t[\tilde{U}^{t+1}(\tilde{k}_{t+1}, \tilde{h}_{t+1}, \theta_{t+1})] = \int \tilde{U}^{t+1}(\tilde{k}_{t+1}, \tilde{h}_{t+1}, \theta_{t+1})\pi(\theta_{t+1})d\theta_{t+1}$, and the maximization is taken over $\{\tilde{c}_t, \tilde{z}_t, \tilde{x}_t, \tilde{y}_t, \tilde{h}_{t+1}, \tilde{k}_{t+1}\}$. Note that the tax policies (A18a) are substituted into the constraints above, and the income condition (A18b) is used in solving the problem.

In the following proposition, we prove that with the income condition (A18b), the wedges (τ_z, τ_k) as defined in the planning problem in subsection 3.1 can implement the constrained efficient allocation in our tax system.

Proposition A2. *Under the tax system $\{\mathcal{T}_t\}$, the constrained efficient allocations can be implemented in a decentralized economy, and the corresponding two linear capital and labor income tax rates are consistent with the wedges. That is, $\tilde{\tau}_k(\theta^t) = \tau_k(\theta^t)$ and $\tilde{\tau}_z(\theta^t) = \tau_z(\theta^t)$, $t = 1, 2, \dots, T$.*

Proof. See Appendix A.4.1.

The necessity of the income condition. The intuition concerning the role of the income condition (A18b) that helps our tax system implement the constrained efficient allocation in our tax system is as follows. Without the income condition (A18b) in our tax system in a decentralized economy, the optimal allocation must satisfy the following Euler equation.

$$u'(\tilde{c}_t) = \beta R_{t+1}\left(1 - \tilde{\tau}_k(\hat{\theta}^t)\right) E[u'(\tilde{c}_{t+1})] \text{ for some } \hat{\theta}^t \in \Theta^t. \tag{A19a}$$

In order to implement the constrained efficient allocation in this tax system, we must find an appropriate linear capital tax $\tilde{\tau}_k(\hat{\theta}^t)$ such that, for any reporting strategy $\hat{\sigma}(\theta^t) = \hat{\theta}^t$ in the planning problem, the allocations $\{c^{\hat{\sigma}}(\theta^t), c^{\hat{\sigma}}(\theta^{t+1})\}$ satisfy equation (A19a).[8] However, this is impossible, because the government cannot tell mis-reporting from truth-telling agents. For the same reported type $\hat{\theta}^t$, mis-reporting and truth-telling agents must have the same linear capital tax rate $\tilde{\tau}_k(\hat{\theta}^t)$. Suppose that the human capital is fully depreciated and the mis-reporting agent only misreports in period t. Under such a deviating strategy $\hat{\sigma} \in \hat{R}_t(\theta^T)$, this mis-reporting agent does not misreport in period $t+1$, and, based on the optimal condition in (8c), both mis-reporting and truth-telling agents have the same consumption in period $t+1$; i.e. $c^{\hat{\sigma}}(\theta^{t+1}) = c(\theta^{t+1})$. This indicates that the discounted, post-tax marginal utility of consumption in the right-hand side of Euler equation (A19a) is the

[8] Grochulski and Piskorski (2010) pointed out that if (A19a) does not hold for some reporting strategies, then this reporting strategy is not individually optimal, and thus it cannot be an equilibrium strategy, and the constrained efficient allocation is not implemented.

same for both mis-reporting and truth-telling agents. However, by deviating in period t, the mis-reporting agent can consume more than a truth-telling agent, so the marginal utility of consumption in period t in the left-hand side of (A19a) for the mis-reporting agent is different from that of the truth-telling agent. As such, there is no capital tax rate $\tilde{\tau}_k(\hat{\theta}^t)$ that satisfies Euler equation (A19a) for both mis-reporting and truth-telling agents.

To resolve the problem, we need the income condition (A18b) in our tax system. By denoting $\eta^{\hat{\sigma}}(\theta^t)$ as the multiplier of the income condition (A18b) in the agents' problem, we revise the Euler equation (A19a) as follows:

$$u'(\tilde{c}_t) = \beta R_{t+1}\big(1 - \tilde{\tau}_k(\hat{\theta}^t)\big)E_t[u'(\tilde{c}_{t+1})] - \eta^{\hat{\sigma}}(\theta^t). \tag{A19b}$$

Agents of different types have different shadow prices $\eta^{\hat{\sigma}}(\theta^t)$. Moreover, as the linear labor income tax rate $\tilde{\tau}_z(\hat{\theta}^t)$ directly affects the income condition (A18b), the shadow price of the income condition $\eta^{\hat{\sigma}}(\theta^t)$ changes with different values of $\tilde{\tau}_z(\hat{\theta}^t)$. Hence, this income condition makes room for the constrained efficient allocation to satisfy (A19b) for both mis-reporting and truth-telling agents. In other words, different multipliers of the income conditions make it possible to find linear income tax rates $(\tilde{\tau}_k(\hat{\theta}^t), \tilde{\tau}_z(\hat{\theta}^t))$ so constrained efficient allocations resulting from reporting strategies specified by shirking and truth-telling agents satisfy (A19b).

Our implementation of the constrained efficient allocations in terms of linear capital and labor income taxes is different from the implementation in terms of a linear capital income tax in Grochulski and Piskorski (2010). To avoid an agent from deviating from labor and saving jointly, these two authors used a non-linear labor income tax to restrict the agent's labor to the constrained efficient level. Under the restriction, however, their linear capital income tax-adjusted Euler equations associated with truth-telling and mis-reporting strategies can be consistent with each other only if there exists a deferred capital tax.

By contrast, we do not restrict agents' labor choices. Instead, we only impose an income condition $S(\tilde{x}^t, \tilde{z}_t, \tilde{k}_{t+1}) = 0$ to restrict the post-tax total income to be consistent with the constrained efficient level. Our tax system has two merits. First, it allows agents to choose their labor and savings jointly without the concern of a double deviation problem, provided that agents' post-tax income is consistent with the constrained efficient level. Second, a deferred capital tax is not necessary, because our linear labor income tax rate can function like the deferred capital tax in Grochulski and Piskorski (2010). Our linear labor income taxes play a role that makes our linear capital income tax-adjusted Euler equations associated with truth-telling and shirking strategies consistent with each other.

A.4.1 *Proof of Proposition A2*

In the problem (A18c), the agent of type θ^t is to maximize the following life-time utility

$$\tilde{U}^t(\tilde{k}_t, \tilde{h}_t, \theta_t) = \max u(\tilde{c}_t) - \phi\left(\frac{\tilde{z}_t}{\tilde{h}_t}\right) + \beta E_t\big[\tilde{U}^{t+1}(\tilde{k}_{t+1}, \tilde{h}_{t+1}, \theta_{t+1})\big]$$

$$\text{subject to} \qquad \tilde{c}_t + \tilde{x}_t + \tilde{y}_t + \tilde{k}_{t+1} \leq w_t\tilde{z}_t + R_t\tilde{k}_t - \mathcal{T}_t,$$

$$\tilde{h}_{t+1} = \mathbb{H}^t(\tilde{x}_t, \tilde{y}_t, \theta_t),$$

where $E_t\big[\tilde{U}^{t+1}(\tilde{k}_{t+1}, \tilde{h}_{t+1}, \theta_{t+1})\big] = \int \tilde{U}^{t+1}(\tilde{k}_{t+1}, \tilde{h}_{t+1}, \theta_{t+1})\pi(\theta_{t+1})d\theta_{t+1}$, and the maximization is taken over

$\{c_t, z_t, x_t, y_t, h_{t+1}, k_{t+1}\}$.

If we restrict $x_t = x(\hat{\theta}^t)$ for some reporting strategy $\hat{\sigma} = (\hat{\theta}_1, \ldots, \hat{\theta}_t) \in \hat{R}_t(\theta^t)$, then given the previous choice of capital and human capital $(\tilde{k}(\hat{\theta}^{t-1}), \tilde{h}(\hat{\theta}^{t-1}))$, the agent of type $\theta^t \in \Theta^t$ chooses allocations $\{\tilde{c}, \tilde{z}, \tilde{y}, \tilde{h}, \tilde{k}\}$ to maximize the following life-time utility

$$\widetilde{U}^t(\tilde{k}(\hat{\theta}^{t-1}), \tilde{h}(\hat{\theta}^{t-1}), \theta_t) = \max u(\tilde{c}) - \phi\left(\frac{\tilde{z}}{\tilde{h}(\hat{\theta}^{t-1})}\right) + \beta \int \widetilde{U}^{t+1}(\tilde{k}, \tilde{h}, \theta_{t+1}) \pi(\theta_{t+1}) d\theta_{t+1},$$

subject to $$(1 - \tilde{\tau}_z(\hat{\theta}^t)) w_t \tilde{z} + R_t(1 - \tilde{\tau}_k(\hat{\theta}^{t-1})) \tilde{k}(\hat{\theta}^{t-1}) - \tilde{c} - x(\hat{\theta}^t) - \tilde{y} - \tilde{k} - \Gamma(\hat{\theta}^t) \geq 0, \quad \text{(A20a)}$$

$$\tilde{h} = \mathbb{H}^t(x(\hat{\theta}^t), \tilde{y}_t, \theta_t), \quad \text{(A20b)}$$

$$S(x(\hat{\theta}^t), \tilde{z}, \tilde{k}) = (1 - \tilde{\tau}_z(\hat{\theta}^t)) w_t(\tilde{z} - z(\hat{\theta}^t)) - (\tilde{k} - k(\hat{\theta}^t)) = 0, \quad \text{(A20c)}$$

where the maximization is taken over $\{\tilde{c}, \tilde{z}, \tilde{k}, \tilde{y}, \tilde{h}\}$, and the lump-sum tax $\Gamma(\hat{\theta}^t)$ is as follows:

$$\Gamma(\hat{\theta}^t) = (1 - \tau_z(\hat{\theta}^t)) w_t z(\hat{\theta}^t) + R_t(1 - \tau_k(\hat{\theta}^{t-1})) k(\hat{\theta}^{t-1}) - c(\hat{\theta}^t) - x(\hat{\theta}^t) - y(\hat{\theta}^t) - k(\hat{\theta}^t). \quad \text{(A20d)}$$

Let the multipliers respect to the constraints (A20a)-(A20c) be $\lambda^{\hat{\sigma}}(\theta^t)$, $\mu^{\hat{\sigma}}(\theta^t)$ and $\eta^{\hat{\sigma}}(\theta^t)$. The Hamiltonian of the above problem is

$$\begin{aligned}
\mathcal{H} &= \max U^t(\tilde{k}(\hat{\theta}^{t-1}), \tilde{h}(\hat{\theta}^{t-1}), \theta_t) \\
&= \max u(\tilde{c}) - \phi\left(\tfrac{\tilde{z}}{\tilde{h}(\hat{\theta}^{t-1})}\right) + \beta \int U^{t+1}(\tilde{k}, \tilde{h}, \theta_{t+1}) \pi(\theta_{t+1}) d\theta_{t+1} \\
&+ \lambda^{\hat{\sigma}}(\theta^t)\left[(1 - \tilde{\tau}_z(\hat{\theta}^t)) w_t \tilde{z} + R_t(1 - \tilde{\tau}_k(\hat{\theta}^{t-1})) \tilde{k}(\hat{\theta}^t) - \tilde{c} - \tilde{x}(\hat{\theta}^t) - \tilde{y} - \tilde{k} - \Gamma_t(\hat{\theta}^t)\right] \\
&+ \mu^{\hat{\sigma}}(\theta^t)\left[\mathbb{H}^t(x(\hat{\theta}^t), \tilde{y}_t, \theta_t) - \tilde{h}\right] \\
&+ \eta^{\hat{\sigma}}(\theta^t)\left[\left(1 - \tau_z(\hat{\theta}^t)\right) w_t \left(\tilde{z} - z(\hat{\theta}^t)\right) - \left(\tilde{k} - k(\hat{\theta}^t)\right)\right].
\end{aligned}$$

The first-order conditions with respect to $\{\tilde{c}, \tilde{z}, \tilde{k}, \tilde{y}, \tilde{h}\}$ are

$$\frac{\partial \mathcal{H}}{\partial \tilde{c}} = u'(\tilde{c}) - \lambda^{\hat{\sigma}}(\theta^t) = 0,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{z}} = -\phi'\left(\tfrac{\tilde{z}}{\tilde{h}(\hat{\theta}^{t-1})}\right) \frac{1}{\tilde{h}(\hat{\theta}^{t-1})} + w_t\left(1 - \tau_{z_1}(\hat{\theta}^t)\right)\left(\lambda^{\hat{\sigma}}(\theta^t) + \eta^{\hat{\sigma}}(\theta^t)\right) = 0,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{k}} = \beta \int U_k^{t+1}(\tilde{k}, \tilde{h}, \theta_{t+1}) \pi(\theta_{t+1}) d\theta_{t+1} - \left(\lambda^{\hat{\sigma}}(\theta^t) + \eta^{\hat{\sigma}}(\theta^t)\right) = 0,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{y}} = -\lambda^{\hat{\sigma}}(\theta^t) + \mu^{\hat{\sigma}}(\theta^t) \mathbb{H}_y^t(x(\hat{\theta}^t), \tilde{y}_t, \theta_t) = 0,$$

$$\frac{\partial \mathcal{H}}{\partial \tilde{h}} = \beta \int U_h^{t+1}(\tilde{k}, \tilde{h}, \theta_{t+1}) \pi(\theta_{t+1}) d\theta_{t+1} - \mu^{\hat{\sigma}}(\theta^t) = 0.$$

Moreover, the envelope conditions are

$$U_k^t(\tilde{k}(\hat{\theta}^{t-1}), \tilde{h}(\hat{\theta}^{t-1}), \theta_t) = \lambda^{\hat{\sigma}}(\theta^t) R_t\left(1 - \tilde{\tau}_k(\hat{\theta}^{t-1})\right),$$

$$U_h^t(\tilde{k}(\hat{\theta}^{t-1}), \tilde{h}(\hat{\theta}^{t-1}), \theta_t) = \phi'\left(\tfrac{\tilde{z}}{\tilde{h}(\hat{\theta}^{t-1})}\right) \frac{\tilde{z}}{[\tilde{h}(\hat{\theta}^{t-1})]^2}.$$

Shifting backward by one period, these envelope conditions are

$$U_k^{t+1}(\tilde{k}, \tilde{h}, \theta_{t+1}) = \lambda^{\hat{\sigma}}(\theta^{t+1}) R_{t+1}(1 - \tilde{\tau}_k(\hat{\theta}^t)),$$

$$U_h^{t+1}(\tilde{k}, \tilde{h}, \theta_{t+1}) = \phi'\left(\tfrac{\tilde{z}(\theta_{t+1})}{\tilde{h}}\right) \frac{\tilde{z}(\theta_{t+1})}{[\tilde{h}]^2}.$$

Besides, (A19b) is easily derived by the envelope and the first-order conditions with respect to $\tilde{c}$ and $\tilde{k}$.

Also, because the preference is concave and the constraint set is a convex set, the first-order conditions are both necessary and sufficient for the maximum, and thus there is a unique solution $\{\tilde{c}, \tilde{z}, \tilde{y}, \tilde{h}, \tilde{k}\}$ to this problem. In combination with the constraints that hold with an equality, these first-order conditions and envelope conditions above give the following conditions for any $s \geq t$.

$$\frac{\phi'\left(\frac{\tilde{z}}{\tilde{h}(\hat{\theta}^{s-1})}\right)\frac{1}{\tilde{h}(\hat{\theta}^{s-1})}}{w_s\left(1-\tilde{\tau}_z(\hat{\theta}^s)\right)} = \beta R_{s+1}\left(1-\tilde{\tau}_k\left(\hat{\theta}^s\right)\right)\int u'\left(\tilde{c}(\theta_{s+1})\right)\pi(\theta_{s+1})d\theta_{s+1}, \tag{A21a}$$

$$u(\tilde{c}) = \beta \mathbb{H}_y^s\left(x\left(\hat{\theta}^s\right), \tilde{y}, \theta_s\right)\int \phi'\left(\frac{\tilde{z}(\theta_{s+1})}{\tilde{h}}\right)\frac{\tilde{z}(\theta_{s+1})}{[\tilde{h}]^2}\pi(\theta_{s+1})d\theta_{s+1}, \tag{A21b}$$

$$\tilde{h} = \mathbb{H}^s\left(x\left(\hat{\theta}^s\right), \tilde{y}, \theta_s\right) \tag{A21c}$$

$$\left(1-\tilde{\tau}_z(\hat{\theta}^s)\right)w\tilde{z} + R\left(1-\tilde{\tau}_k(\hat{\theta}^{s-1})\right)\tilde{k}(\hat{\theta}^{s-1}) - \tilde{c} - x\left(\hat{\theta}^s\right) - \tilde{y} - \tilde{k} - \Gamma(\hat{\theta}^s) = 0, \tag{A21d}$$

$$\left(1-\tilde{\tau}_z(\hat{\theta}^s)\right)w\left(\tilde{z} - z(\hat{\theta}^s)\right) - \left(\tilde{k} - k(\hat{\theta}^s)\right) = 0. \tag{A21e}$$

Suppose that the linear tax rates $\left\{\tilde{\tau}_k\left(\hat{\theta}^s\right), \tilde{\tau}_z\left(\hat{\theta}^s\right)\right\}$ are set as the wedges defined in (10a)-(10b). Next, we show that, for the agent whose reporting strategy is $\hat{\sigma} \in \hat{R}_t(\theta^T)$, if the resulting constrained efficient allocation $\{c^{\hat{\sigma}}(\theta^s), y^{\hat{\sigma}}(\theta^s), k(\hat{\theta}^s), z(\hat{\theta}^s), h^{\hat{\sigma}}(\theta^s)\}_{s\geq t}$ satisfies all (A21a)-(A21e), then the allocation must be the unique solution of the above decentralized problem.

Because the reporting strategy $\hat{\sigma} \in \hat{R}_t(\theta^T)$ only possibly deviates at period t, Corollary 6 suggests that $y^{\hat{\sigma}}(\theta^s) = y(\theta^s)$ and $c^{\hat{\sigma}}(\theta^s) = c(\theta^s)$ for any $s > t$. From the definitions in (10a)-(10b), we can easily verify that the tuple $\left\{z\left(\hat{\theta}^s\right), c^{\hat{\sigma}}(\theta^{s+1})\right\}$ satisfies (A21a). Also, based on the optimal condition (8c), the tuple $\{c^{\hat{\sigma}}(\theta^s)$, $y^{\hat{\sigma}}(\theta^s)$, $h^{\hat{\sigma}}(\theta^s)$, $z\left(\hat{\theta}^{s+1}\right)\}$ satisfies (A21b). Besides, based on human capital technology (8b), we know that the tuple $\{y^{\hat{\sigma}}(\theta^s), h^{\hat{\sigma}}(\theta^s)\}$ satisfies (A21c), and based on the definition of $\Gamma(\hat{\theta}^s)$ in (A20d) and the fact that $c^{\hat{\sigma}}(\theta^s) + y^{\hat{\sigma}}(\theta^s) = c(\hat{\theta}^s) + y(\hat{\theta}^s)$, the tuple $\{c^{\hat{\sigma}}(\theta^s), y^{\hat{\sigma}}(\theta^s), k(\hat{\theta}^s), z(\hat{\theta}^s)\}$ satisfies (A21d). Finally, it is obvious that the tuple $\{z(\theta^s), k(\theta^s)\}$ satisfies (A21e). Therefore, for agents whose reporting strategy is $\hat{\sigma}$, the constrained efficient allocation indeed satisfies all (A21a)-(A21e), which implies that conditions (A21a)-(A21e) yield the allocations that are the same as the constrained efficient allocation $\{c^{\hat{\sigma}}(\theta^s)$, $y^{\hat{\sigma}}(\theta^s)$, $k(\hat{\theta}^s)$, $z(\hat{\theta}^s)$, $h^{\hat{\sigma}}(\theta^s)\}$.

Finally, the IC constraint (9a) implies that when type θ_t agents choose a truth-telling strategy, i.e., $\hat{\theta}_t = \theta_t$, the constrained efficient allocation $\{c^{\hat{\sigma}}(\theta^s)$, $y^{\hat{\sigma}}(\theta^s)$, $k(\hat{\theta}^s)$, $z(\hat{\theta}^s)$, $h^{\hat{\sigma}}(\theta^s)\}$ gives the highest lifetime utility. Thus, the best strategy is to report the type truthfully, and the constrained efficient allocation $\{c^{\hat{\sigma}}(\theta^s), y^{\hat{\sigma}}(\theta^s), k(\hat{\theta}^s), z(\hat{\theta}^s), h^{\hat{\sigma}}(\theta^s)\}$ indeed solves the optimization problem for type θ_t agents under the tax system. Then, our tax system implements the constrained efficient allocation in an equilibrium. □