# Misallocation and Intersectoral Linkages
## Online Appendix

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## Contents

# A Complement to the theory section

We provide 9 additional theoretical results. We show that we can assume, without loss of generality, that there are no direct distortions on the consumption allocation. We also present the solution of the model with Cobb-Douglas production functions and the expression of the Domar weight in the absence of distortions. In addition, we show that uniform labor distortions have no effect on aggregate TFP, we present the counterpart of Section 3.3's results for labor distortions, we provide the expression of TFP when distortions are random, and we study the interaction of distortions. The proofs of the additional propositions are all in Appendix B. Here, we also use the examples of Section 3.1.3 to illustrate Proposition 4, we explain how our production specification relates to the specification proposed by Klump et al. (2012), and we show the relation between the importance of a sector as an input supplier and the sector's uptreamness and centrality. Finally, we discuss the relation between our derivations and the recent contribution by Baqaee and Farhi (2020).

## A.1 Allocation with consumption frictions

If a planner maximizes output (and welfare) subject to the production functions (1)-(3) and resource constraints (4) - (5), the optimal allocation will satisfy:

$$\frac{dY}{dC_i} = \lambda_i$$

$$\lambda_i \frac{dQ_i}{dX_{ij}} = \lambda_j$$

$$\lambda_i \frac{dQ_i}{dL_i} = \eta.$$

This implies that conditions (6)- (8) can be considered as deviations from the frictionless allocation. There may be distortions that do not occur during the production process but change the consumption mix; for example consumption taxes or subsidies. Therefore, we may want to modify (6) as follows:

$$\frac{dY}{dC_i} = \lambda_i(1 + \tau_{Ci}),$$

with $\tau_{Ci}$ being the consumption distortion.

The following proposition implies that we can assume $\tau_{Ci} = 0$ without loss of generality. Moreover, any allocation that satisfies the physical feasibility constraints can be rationalized by a set of distortions in our framework.

**Proposition A.1** *Suppose that an allocation* $\{C_i, Q_i, X_{ij}, L_i\}$ *satisfies* (7), (8) *and:*

$$\frac{dY}{dC_i} = \lambda_i(1 + \tau_{Ci})$$

*for some vectors* $(\lambda, \tau_X, \tau_L, \tau_C)$. *Then there exist vectors* $(\lambda', \tau'_X, \tau'_L)$ *such that the allocation satisfies*

(6)- (8). *Suppose that the allocation* $\{C_i, Q_i, X_{ij}, L_i\}$ *satisfies* (1)-(3) *and* (4)-(5). *Then there exist* $\{\lambda_i, \tau_{Xij}, \tau_{Li}\}$ *such that the allocation satisfies* (6)-(8). *Moreover,* $\{\lambda_i, \tau_{Xij}, \tau_{Li}\}$ *is unique up to an affine transformation of* $\{\tau_{Li}\}$.

## A.2 Specification of the production functions

We show here how to derive the production function (1) using the normalization procedure proposed by Klump et al. (2012). We normalize the production function with the equilibrium allocation without distortions, which we denote by bars. Also, we choose units in such a way that the price of all goods is equal to one. This choice of units implies $\bar{X}_i = \sum_j \bar{X}_{ij}$.
Define $v_{ij} = \bar{X}_{ij}/\sum_j \bar{X}_{ij} = \bar{X}_{ij}/\bar{X}_i$. Then, from the Klump et al. (2012) normalization we obtain:

$$X_i = \bar{X}_i \left( \sum_j v_{ij} \left( \frac{X_{ij}}{\bar{X}_{ij}} \right)^{\frac{\varepsilon_\rho - 1}{\varepsilon_\rho}} \right)^{\frac{\varepsilon_\rho}{\varepsilon_\rho - 1}} = \bar{X}_i \left( \sum_j v_{ij} \left( \frac{X_{ij}}{v_{ij}\bar{X}_i} \right)^{\frac{\varepsilon_\rho - 1}{\varepsilon_\rho}} \right)^{\frac{\varepsilon_\rho}{\varepsilon_\rho - 1}} = \left( \sum_j v_{ij}^{\frac{1}{\varepsilon_\rho}} X_{ij}^{\frac{\varepsilon_\rho - 1}{\varepsilon_\rho}} \right)^{\frac{\varepsilon_\rho}{\varepsilon_\rho - 1}}.$$

Next, we turn to the outer loop. The fact that all the commodity prices are one and there are no markups implies: $\bar{Q}_i = \bar{w}\bar{L}_i + \bar{X}_i$. Define $\alpha_i = X_i/Q_i$; it follows that $\bar{L}_i = (1-\alpha_i)\bar{Q}_i/\bar{w}$. Applying again the Klump et al. (2012) normalization, we obtain:

$$Q_i = \bar{Q}_i \left( (1-\alpha_i) \left( \frac{L_i}{\bar{L}_i} \right)^{\frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} + \alpha_i \left( \frac{X_i}{\bar{X}_i} \right)^{\frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} \right)^{\frac{\varepsilon_\sigma}{\varepsilon_\sigma - 1}}.$$

We can express $L_i/\bar{L}_i = \bar{w}L_i/[(1-\alpha_i)\bar{Q}_i]$ and $X_i/\bar{X}_i = X_i/[\alpha_i\bar{Q}_i]$. Then plugging in the expression above:

$$Q_i = \left[ (1-\alpha_i)^{\frac{1}{\varepsilon_\sigma}} (\bar{w}L_i)^{\frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} + \alpha_i^{\frac{1}{\varepsilon_\sigma}} X_i^{\frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} \right]^{\frac{\varepsilon_\sigma}{\varepsilon_\sigma - 1}}.$$

Note that the choice of units implicitly normalizes $A_i = 1$ for all $i$. Also since $\bar{w}$ is the real wage in the undistorted economy it is also equal to the GDP per unit of labor and hence to the labor-augmenting productivity term. In the data $L$ can be measured in employment, hours or payroll. In the last case the wage is simply 1. Thus this particular normalization implicitly chooses units in such a way that all the $B$ terms are identical, as proposition 2 allows us to do.

## A.3 Solution with Cobb-Douglas production functions

**Proposition A.2** *Suppose that* $\varepsilon_\sigma = \varepsilon_\rho = \varepsilon_\chi = 1$. *Then* $\lambda_i/\eta$ *is given by*

$$\begin{aligned} \log(\lambda/\eta) &= -(I - \tilde{\alpha}V)^{-1}((I-\tilde{\alpha})\log B + \log A) + \\ &\quad (I - \tilde{\alpha}V)^{-1}[(I-\tilde{\alpha})\Delta^{Lp} + \tilde{\alpha}(V \circ \Delta^{Xp})]\mathbf{1}, \end{aligned} \tag{A.1}$$

*and output by*

$$\log Y = \log \bar{L} + \beta'(I - \tilde{\alpha}V)^{-1}((I - \tilde{\alpha}) \log B + \log A) - \beta'(I - \tilde{\alpha}V)^{-1}[(I - \tilde{\alpha})\Delta^{Lp} + \tilde{\alpha}(V \circ \Delta^{Xp})]\mathbf{1}$$
$$- \log(\beta'(I - \tilde{\alpha}(V \circ \Delta^{Xq}))^{-1}\Delta^{Lq}(\mathbf{1} - \alpha)),$$

*where* $\Delta^{Xp}_{ij} = \log(1 + \tau_{Xij})$, $\Delta^{Lp}_{ii} = \log(1 + \tau_{Li})$; $\Delta^{Lp}_{ij} = 0, i \neq j$ *and* $\tilde{\alpha}$ *is a square matrix with* $\tilde{\alpha}_{ii} = \alpha_i$ *and* $\tilde{\alpha}_{ij} = 0$ *if* $i \neq j$ *and* $\Delta^{Xq}_{ij} = (1 + \tau_{Xij})^{-1}$; $\Delta^{Lq}_{ii} = (1 + \tau_{Li})^{-1}, \Delta^{Lq}_{ij} = 0, i \neq j$ *are matrices of distortions, and* $\circ$ *denotes the Hadamard (entrywise) product.*

## A.4 Domar weights in the frictionless economy

**Proposition A.3** *Consider an economy with* $A_i = B_i = 1, \forall i$ *and no distortions. Then* $Y = L, \lambda_i = 1, \eta = 1, L_i/Q_i = 1 - \alpha_i, X_{ij}/Q_i = \lambda_j X_{ij}/(\lambda_i Q_i) = \alpha_i v_{ij}$ *for all* $i$ *and* $j$*, and* $s_i = Q_i/Y = \lambda_i Q_i/Y$*.*

## A.5 Random distortions

If the number of sectors is sufficiently large and the distortions are independent of the production function and network parameters, and independent between each other, we can characterize the TFP loss in terms of the mean and the variance of the distortions.

**Proposition A.4** *Consider an economy with all-input distortions where* $TFP$ *is given by the approximation of Proposition 6. Suppose also that* $\tau_i$ *is identically and independently distributed across sectors and independent from any other model parameters, with mean* $\bar{\tau}$ *and cross-sectional variance* $\text{var}(\tau)$*. Then, aggregate TFP is approximatively equal to*

$$\log TFP \approx -\varepsilon_\chi \Gamma_\chi(\alpha, \beta, V)\bar{\tau}^2 - \varepsilon_\chi \Xi \chi(\alpha, \beta, V) \,\text{var}(\tau) - \varepsilon_\sigma \Gamma_\sigma(\alpha, \beta, V)\bar{\tau}^2 - \varepsilon_\sigma \Xi_\sigma(\alpha, \beta, V) \,\text{var}(\tau), \quad \text{(A.2)}$$

*where* $\Gamma_\chi(\alpha, \beta, V), \Gamma_\psi(\alpha, \beta, V), \Xi_\chi(\alpha, \beta, V), \Xi_\sigma(\alpha, \beta, V) \geq 0$*. If* $\alpha_i > 0$ *for some* $i$*, then* $\Gamma_\sigma > 0, \Xi_\sigma > 0$*.*

## A.6 Distortions on multiple sectors

How is the impact of a sectoral distortion modified when the distortion in another sector increases? Without sectoral linkages, a higher distortion in sector $j$ always reduces the effect of a distortion in sector $i$ because the two prices become more similar. However, with sectoral linkages, the two distortions may reinforce each other or offset each other, as we explain below. The analysis highlights the connection between the sectors as well as how similarly the two sectors' products are used by other sectors.

**Result A.1** *Consider an economy with moderate all-input distortions where* $\log TFP$ *is given by the approximation of Proposition 6. The TFP impact of distortions in two sector* $i$ *and* $j$*, is given by*

$$\frac{d^2 \log TFP}{d\tau_i d\tau_j} \approx -\varepsilon_\chi(-\beta_i\beta_j + \beta_i\left(x_{ij} - \gamma_j\right) + \beta_j\left(x_{ji} - \gamma_i\right) + \mathrm{cov}_\beta(x_{.i}, x_{.j})) \\ - \varepsilon_\sigma\left(-\gamma_i\gamma_j + \gamma_i x_{ij} + \gamma_j x_{ji} - \mathrm{cov}_\beta(x_{.i}, x_{.j})\right).$$

The parallels and differences between the consumption (first line) and production (second line) misallocation terms shed light on the roles of the elasticity of substitution in consumption ($\varepsilon_\chi$) and production ($\varepsilon_\sigma$) for TFP loss.

The term $-\beta_i\beta_j + \beta_i\left(x_{ij} - \gamma_j\right) + \beta_j\left(x_{ji} - \gamma_i\right)$ captures the interaction effect of the two distortions on the variance of final prices. The first part $-\beta_i\beta_j$ reduces the TFP loss: a higher distortion on sector $j$ reduces the impact of sector $i$'s distortion because it makes the prices of sectors $i$ and $j$ more similar. The remaining part $\beta_i\left(x_{ij} - \gamma_j\right) + \beta_j(x_{ji} - \gamma_i)$ emerges from input-output linkages: when the two sectors are heavy users of each other's goods for intermediate input, markups are charged on top of markups, increasing the variance of prices and hence the TFP loss. In the second line, the production inefficiency term, $-\gamma_i\gamma_j + \gamma_i x_{ij} + \gamma_j x_{ji}$ is the analogue of the term in the first line and corresponds to the effect of the interaction on the variance of input prices. Note that both in the first line and in the second line, this effect increases the TFP loss.

Finally, $\mathrm{cov}_\beta(x_{.i}, x_{.j})$ enters the two lines with opposite signs. The covariance is large when sectors $i$ and $j$ supply inputs to the same sectors. A higher covariance reduces input misallocation (since input prices become more similar), but it increases the variance of final consumption prices (since the input prices rise disproportionately for some sectors). For this effect, consumption and production elasticities work in opposite directions.

## A.7 Propositions and results for the labor distortions case

This section presents the result on the irrelevance of uniform labor distortions, the approximation of TFP when labor distortions are random, as well as the analogues of the results of Section 3.3, A.5, and A.6 in the labor distortions case.

### A.7.1 Irrelevance of uniform labor distortions

**Proposition A.5** *Consider two distorted economies where* $\log TFP$ *is given by Proposition 1. The two economies, denoted 1 and 2, are identical except for* $\tau_{2Li} = (1+z)(1+\tau_{1Li}) - 1$*,* $z > -1$*. Then* $\eta_2 = \eta_1/(1+z)$ *and all the other equilibrium variables are the same in the two economies.*

### A.7.2 Random labor distortions

**Proposition A.6** *Consider an economy with labor distortions and no intermediate-input distortions where $TFP$ is given by the approximation of Proposition 5. Suppose also that $\tau_{Li}$ is identically and independently distributed across sectors and independent from any other model parameters, with mean $\bar{\tau}_L$ and cross-sectional variance $\operatorname{var}(\tau_L)$. Then, aggregate TFP is approximatively equal to*

$$\log TFP \approx -\frac{1}{2}\varepsilon_\chi \psi_\chi(\alpha,\beta,V)\operatorname{var}(\tau_L) - \frac{1}{2}\varepsilon_\sigma \psi_\sigma(\alpha,\beta,V)\operatorname{var}(\tau_L), \tag{A.3}$$

*where $\psi_\chi(\alpha,\beta,V) > 0, \psi_\sigma(\alpha,\beta,V) \geq 0$. If $\alpha_i v_{ij} > 0$ for some $i \neq j$, then $\psi_\sigma(\alpha,\beta,V) > 0$.*

### A.7.3 Dampening conditions with labor distortions

The following result presents the conditions under which sectoral linkages dampen the effects of labor distortions. A key condition is that the input elasticity is lower than the consumption elasticity.

**Result A.2** *Consider an economy with labor distortions and no intermediate-input distortions where $TFP$ is given by the approximation of Proposition 5 and the TFP in an otherwise identical economy without sectoral linkages, $TFP|_{\alpha=0}$, is given by equation 15. If inputs are less substitutable than consumption goods ($\varepsilon_\sigma < \varepsilon_\chi$) and $(1-\alpha_i)s_i = \beta_i$, then $\log TFP \geq \log TFP|_{\alpha=0}$.*

To see where the result comes from, let us rewrite equation (13) as follows

$$\log TFP \approx -\frac{1}{2}\varepsilon_\chi \operatorname{var}_{s_j(1-\alpha_j)}(\tau_{Lj}) - \frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi)\sum_i \beta_i \operatorname{var}_{\omega_{ij}(1-\alpha_j)}(\tau_{Lj}). \tag{A.4}$$

Comparing this equation with the no-linkages economy (equation (15)), we see that linkages modify the weight of each sector. When the input and consumption elasticities are equal, $\log TFP \approx -\frac{1}{2}\varepsilon_\chi \operatorname{var}_{s_j(1-\alpha_j)}(\tau_{Lj})$ and the weight of the sector is its (undistorted) value added share ($s_j(1-\alpha_j)$) whereas the weight is the final consumption share ($\beta_i$) in the no-linkages economy. When $s_j(1-\alpha_j) = \beta_j$ and the two elasticities are equal, aggregate TFP is therefore identical to that of the economy without sectoral linkages. The irrelevance of sectoral linkages is the result of two counteracting effects. On the one hand, sectoral linkages generate an additional channel through which distortions reduce aggregate productivity as sectoral production can now be inefficient. On the other hand, the use of intermediate inputs reduces the effect of labor distortions since labor then represents a smaller share of the firms' inputs. When the input and consumption elasticities are equal, the two offsetting effects cancel out exactly. When inputs are less substitutable than consumption goods ($\varepsilon_\sigma < \varepsilon_\chi$), the second effect dominates, and sectoral linkages dampen the effects of labor distortions.

The case $s_i(1-\alpha_i) = \beta_i$ is particularly interesting because it corresponds to how models that do not explicitly account for intermediate inputs are generally calibrated (the sectors' weights are calibrated to their value added shares). Our finding implies that adjusting the calibration is not sufficient to take care of the sectoral-linkages effect. With $\varepsilon_\sigma < \varepsilon_\chi$, models that do not explicitly account for intermediate inputs overestimate the TFP loss from *labor* distortions.

### A.7.4 The TFP impact of a labor distortion on one sector

Next, we study what type of linkages leads to a larger TFP impact of a distortion on the labor decision of only one sector.

**Result A.3** *Consider an economy with labor distortions and no intermediate-input distortions where $TFP$ is given by the approximation of Proposition 5. The TFP impact of a labor distortion on sector $i$ only is given by*

$$\frac{d\log TFP}{d\tau_{Li}} \approx -\left[\varepsilon_{\chi}(1-\alpha_i)^2(s_{ii}-s_i^2)+\varepsilon_{\sigma}[(1-\alpha_i)s_i-(1-\alpha_i)^2 s_{ii}]\right]\tau_{Li}.$$

### A.7.5 Labor distortions on multiple sectors

We now consider how the TFP impact of a sectoral labor distortions is modified when the labor distortion increases in another sector.

**Result A.4** *Consider an economy with labor distortions and no intermediate-input distortions where $TFP$ is given by the approximation of Proposition 5. The TFP impact of distortions in two sector $i$ and $j$, is given by*

$$\frac{d^2\log TFP}{d\tau_{Li}d\tau_{Lj}} \approx -\left[\varepsilon_{\chi}\left(s_{ij}-s_i s_j\right)-\varepsilon_{\sigma}s_{ij}\right](1-\alpha_i)(1-\alpha_j).$$

## A.8 Examples and interpretation of the non-equal elasticities solution

In this section, we interpret the special case of equal elasticity presented in Proposition 4. Next, we elaborate on the example economies of Section 3.1.3.

### A.8.1 Interpretation of Proposition 4

We focus on the case where markups are the source of distortions. In this case $\eta$ can be interpreted as the real wage, $\lambda_i/\eta$ as the price of a good $i$ in terms of labor and $\lambda_i$ as the price of good $i$ relative to the final consumption good.

Let us denote the aggregate profit margin by $z$. Then, we have the following identity $\eta L + zY = Y$, so $Y/L = \eta\frac{1}{1-z}$ or

$$\log(Y/L) = \log\eta - \log(1-z). \tag{A.5}$$

Equivalently, if we want to express this relation using the aggregate markup $\tau^{aggregate}$, we can write $Y/L = \eta(1+\tau^{aggregate})$, where we used $z = \tau^{aggregate}/(1+\tau^{aggregate})$.

We can rewrite equation (11) as

$$
\begin{aligned}
\log TFP &= \frac{1}{\varepsilon_\chi - 1}\log\left(\sum_{i=1}^{n}\beta_i \hat{\lambda}_i^{\frac{\varepsilon_\chi-1}{\varepsilon_\sigma-1}}\right) \\
&- \left[\log(\hat{c}'[I-\tilde{\alpha}\hat{A}(\Delta^{Xq}\circ V)]^{-1}\Delta^{Lq}\hat{A}\hat{B}(\mathbf{1}-\alpha)) - \log\left(\sum_{i=1}^{n}\beta_i \hat{\lambda}_i^{\frac{\varepsilon_\chi-1}{\varepsilon_\sigma-1}}\right)\right] \\
&= \log\eta - \log(\tilde{c}'\Lambda[I-\tilde{\alpha}\hat{A}(\Delta^{Xq}\circ V)]^{-1}\Delta^{Lq}\hat{A}\hat{B}(\mathbf{1}-\alpha)), \qquad \text{(A.6)}
\end{aligned}
$$

where $\tilde{c}_i = \beta_i\lambda_i^{-\varepsilon_\chi}\eta = \beta_i(\lambda_i/\eta)^{-\varepsilon_\chi}\eta^{1-\varepsilon_\chi}$ is the expenditure on good $i$ out of labor income and $\Lambda$ is a square matrix with $\Lambda_{ii} = (\lambda_i/\eta)^{\varepsilon_\sigma} = \hat{\lambda}_i^{\frac{\varepsilon_\sigma}{1-\varepsilon_\sigma}}$. From equation (A.5), it follows that the second logarithm is the spending on labor generated by spending $\eta$ on the final good, or in other words $1-z$.

### A.8.2 Aggregate markup and profit margin

In the case without sectoral linkages,

$$\text{Profits} = \sum_i \tau_i w L_i.$$

And we also have,

$$\text{Profits} = \tau^{aggregate} wL,$$

so

$$\tau^{aggregate} = \frac{\sum_i \tau_i w L_i}{wL} = \sum_i \tau_i \frac{L_i}{L}.$$

Thus, the weights are simply the share of primary inputs used in each sector (since that is what is marked up).

In the general case, profits in sector $i$ are

$$\text{Profits}_i = \tau_i[wL_i + \sum_j \lambda_j X_{ij}] = \frac{\tau_i}{1+\tau_i}\lambda_i Q_i.$$

Then, the profit share is:

$$z = \sum_i \frac{\text{Profits}_i}{Y} = \sum_i \underbrace{\frac{\tau_i}{1+\tau_i}}_{\text{Profit margin } i} \underbrace{\frac{\lambda_i Q_i}{Y}}_{\text{Domar share}}.$$

### A.8.3 Horizontal economy

Here we continue our discussion on the horizontal economy, introduced in section 3.1.3.1. Every sector uses only one input (labor), so the distortion $\tau_i$ is equivalent to a sectoral markup.

### Nonmonotonicity of TFP loss with respect to the elasticity of substitution $\varepsilon_\chi$

For simplicity, we consider a horizontal economy with only two sectors.

First, consider the TFP of this economy with some fixed allocation of resources and varying elasticity of substitution. Define

$$m = \frac{C_2/\beta_2}{C_1/\beta_1}.$$

Efficiency requires $m = 1$. Without loss of generality, assume $m < 1$. Then TFP is simply:

$$TFP^{fixed}(m, \varepsilon_\chi) \equiv \frac{Y}{C_1 + C_2} = \frac{\left(\beta_1 \times 1^{\frac{\varepsilon_\chi - 1}{\varepsilon_\chi}} + \beta_2 m^{\frac{\varepsilon_\chi - 1}{\varepsilon_\chi}}\right)^{\frac{\varepsilon_\chi}{\varepsilon_\chi - 1}}}{\beta_1 + \beta_2 m}.$$

It is elementary to show that $TFP^{fixed}(m, \varepsilon_\chi)$ is increasing in $m$ on $(0, 1)$.

What about the effect of $\varepsilon_\chi$? The denominator is a generalized mean with exponent $\zeta = \frac{\varepsilon_\chi - 1}{\varepsilon_\chi}$ and it increases with $\zeta$ and therefore with $\varepsilon_\chi$ too. This is intuitive: as the elasticity of substitution increases the effect of *fixed* misallocation is reduced.

However, the size of the misallocation also depends on the elasticity of substitution. For some distortion $\tau_2 > \tau_1$, we have that $m(\varepsilon) = \left(\frac{1+\tau_2}{1+\tau_1}\right)^{-\varepsilon_\chi}$. So, we have:

$$TFP(\varepsilon_\chi) = TFP^{fixed}(m(\varepsilon_\chi), \varepsilon_\chi)$$

and we have

$$TFP(0) = TFP^{fixed}(0, 0) = 1.$$

Increasing $\varepsilon_\chi$ from zero allows misallocation to happen and hence lowers TFP. Result 1 goes further than that and shows that no matter how big $\varepsilon_\chi$ is, if $\tau$ is not too big, increasing $\varepsilon_\chi$ will lower TFP.

This simple environment gives some intuition for why we can obtain the nonmonotonicity of TFP with respect to the elasticity of substitution. For $\varepsilon_\chi > 1$,

$$TFP(\varepsilon_\chi) = TFP^{fixed}(m(\varepsilon_\chi), \varepsilon_\chi) > TFP^{fixed}(0, \varepsilon_\chi) = \beta_1^{\frac{1}{\varepsilon_\chi - 1}}$$

But the last expression tends to 1 as $\varepsilon_\chi$ tends to infinity, hence TFP eventually starts to increase for big enough $\varepsilon_\chi$.

### Specializing Proposition 4

Assuming $A = B = \mathbf{1}$, it is immediate that $\hat{\lambda}_i = (1+\tau_i)^{1-\varepsilon_\sigma}$ and $\lambda_i/\eta = 1 + \tau_i$.

Then $\log \eta = \frac{1}{\varepsilon_\chi - 1} \log\left(\sum_i \beta_i (\lambda_i/\eta)^{1-\varepsilon_\chi}\right) = \frac{1}{\varepsilon_\chi - 1} \log\left(\sum_i \beta_i (1+\tau_i)^{1-\varepsilon_\chi}\right)$, which is the first term.

Since $\alpha = \mathbf{0}$, we get

$$\tilde{c}'\Lambda[I - \tilde{\alpha}\hat{A}(\Delta^{Xq} \circ V)]^{-1}\Delta^{Lq}\hat{A}\hat{B}(\mathbf{1} - \alpha) = \tilde{c}'\Lambda\Delta^{Lq}\mathbf{1} = \sum_i \beta_i (1+\tau_i)^{-\varepsilon_\chi}\eta^{1-\varepsilon_\chi}.$$

Then,

$$\log TFP = \frac{\varepsilon_\chi}{\varepsilon_\chi - 1} \log\left(\sum_i \beta_i (1+\tau_i)^{1-\varepsilon_\chi}\right) - \log\left(\sum_i \beta_i (1+\tau_i)^{-\varepsilon_\chi}\right).$$

### Real wage and aggregate profit share

Even though the expression for the exact TFP is simple in this case, it is useful to see how the economic intuition of section A.8.1 is confirmed in this case and how TFP separates into the real wage component and the aggregate profit share.

From 1 unit of expenditure on the final good, we obtain $\beta_i \lambda_i^{-\varepsilon_\chi}$ spending on good $i$ and hence total profit in real terms is:

$$\begin{aligned} z &= \sum_i \beta_i \lambda_i^{-\varepsilon_\chi} \tau_i \eta = \sum_i \beta_i (\lambda_i/\eta)^{-\varepsilon_\chi} \tau_i \eta^{1-\varepsilon_\chi} = \sum_i \beta_i (1+\tau_i)^{-\varepsilon_\chi} \tau_i \eta^{1-\varepsilon_\chi}, \\ &= \sum_i \beta_i (1+\tau_i)^{1-\varepsilon_\chi} \eta^{1-\varepsilon_\chi} - \sum_i \beta_i (1+\tau_i)^{-\varepsilon_\chi} \eta^{1-\varepsilon_\chi} = 1 - \sum_i \beta_i (1+\tau_i)^{-\varepsilon_\chi} \eta^{1-\varepsilon_\chi}, \end{aligned}$$

so plugging into $\log TFP = \log \eta - \log(1-z)$, we obtain the same result.

### Second-order approximation of the Horizontal Economy

The second-order approximation of TFP is simply:

$$\log TFP \approx -\frac{1}{2} \varepsilon_\chi \operatorname{var}_\beta(\tau_i). \tag{A.7}$$

### A.8.4 One-sector economy

### Specializing Proposition 4

Again, let us assume $A = B = 1$ and the distortion is a markup. In this case, equation (12) immediately implies

$$\frac{\lambda}{\eta} = \left[1 - \alpha(1+\tau)^{1-\varepsilon_\sigma}\right]^{-\frac{1}{1-\varepsilon_\sigma}} (1-\alpha)^{\frac{1}{1-\varepsilon_\sigma}} (1+\tau).$$

Since by definition, $\lambda = 1$, $\eta$ is the reciprocal of the above.

Since $\tilde{c} = \beta \lambda^{-1} \eta = \eta$, the proposition implies that

$$\begin{aligned} \log TFP &= -\log\left((\lambda/\eta)^{\varepsilon_\sigma} (1 - \alpha(1+\tau)^{-\varepsilon_\sigma})^{-1} (1-\alpha)(1+\tau)^{-\varepsilon_\sigma}\right) \\ &= \frac{\varepsilon_\sigma}{1-\varepsilon_\sigma} \log[1 - \alpha(1+\tau)^{1-\varepsilon_\sigma}] + \frac{1}{1-\varepsilon_\sigma} \log(1-\alpha) + \log(1 - \alpha(1+\tau)^{-\varepsilon_\sigma}). \end{aligned}$$

The expression is easier to interpret if we use the formulation presented above:

$$\log TFP = \log \eta - \log(1-z) = \log \eta + \log(1 + \tau^{aggregate}).$$

Applying (A.6),

$$z = \frac{\tau}{1+\tau} \frac{1}{1-\alpha(1+\tau)^{-\varepsilon_\sigma}} = \frac{\tau}{1+\tau} \frac{Q}{Y}.$$

#### Second-order approximation of the one-sector economy

Applying Proposition 6, we obtain:

$$\log TFP \approx -\frac{1}{2}\varepsilon_\sigma \frac{\alpha\tau^2}{(1-\alpha)^2}. \tag{A.8}$$

### A.8.5 Star supplier economy

#### Second-order Approximation

Applying Proposition 6, we obtain:

$$\log TFP \approx -\frac{1}{2}\varepsilon_\chi \left( \sum_i \beta_i \left[\tau_i + \frac{\alpha_i}{1-\alpha_1}\tau_1\right]^2 - \left( \sum_i \beta_i \left[\tau_i + \frac{\alpha_i}{1-\alpha_1}\tau_1\right]\right)^2 \right)$$
$$- \frac{1}{2}\varepsilon_\sigma \frac{\tau_1^2}{(1-\alpha_1)^2} \sum_i \beta_i [\alpha_i + \alpha_i\alpha_1 - \alpha_i^2]. \tag{A.9}$$

The first term is still a variance of final prices, where $\tau_i + \frac{\alpha_i}{1-\alpha_1}\tau_1$ is the approximation to the cumulative markup. The second term is the productive inefficiency. Interesting to note: if $\alpha_i = 0$, then the expression multiplying $\beta_i$ is 0 since there is nothing to distort—just one way of doing production; on the other hand even if $\alpha_i = 1$, this expression will still be positive if $\alpha_1 > 0$. The distortion in sectors supplying inputs is inherited in the user sectors.

## A.9 The relation between the importance of a sector as an input supplier, its upstreamness and its centrality

### A.9.1 Relation to upstreamness

Are sectors that are important input suppliers necessarily more upstream? There are two reasons that a sector's relative $\gamma_i$ may be different from its relative upstreamness. First, a sector may have a large $\gamma_i$ simply by being large, even if the share of its output used for intermediate goods is not extraordinarily large. For example, in the U.S., the sector Primary Metals has a lower $\gamma_i$ than Utilities despite the fact that Primary Metals products are almost exclusively used as intermediate inputs.
Second, $\gamma_i$ measures how much of the sector's output is used as intermediate good, but not if the users of sector i are themselves upstream.
To address this question empirically, we generate an upstreamness score using the measure proposed by Antràs et al. (2012). First, for each country in our sample we calculate the correlation between $\gamma_i$ and the upstream score. The correlation varies between 0.19 to 0.65.

Second, we normalize $\gamma_i$ by the sector size $s_i$ by considering the variable $\gamma_i/s_i$. This is the share of output that is used for intermediate usage (at the undistorted equilibrium). Here the correlation is much higher, from a minimum of 0.93 to a maximum of 0.99. Thus, even though $\gamma_i/s_i$ can diverge from upstreamness in theory, quantitatively the divergence is negligible.

### A.9.2 Relation to centrality

The concept of Bonacich centrality (Bonacich, 1987) emerges very often in the economic analysis of networks (e.g. Acemoglu et al. (2012)). Roughly speaking, the concept of centrality in networks corresponds to the importance of a node due to its position in the network and its connections to other nodes.

In Bonacich (1987), for a given adjacency matrix $R$ and some scalars $\alpha_B, \beta_B$, the vector of centrality measures is given recursively as

$$c(\alpha_B, \beta_B) = \alpha_B R\mathbf{1} + \beta_B R c(\alpha_B, \beta_B),$$

that is the centrality of a node depends on the number of nodes it is connected to and the sum of the centrality of all the nodes it is connected to. This gives:

$$c(\alpha_B, \beta_B) = \alpha_B (I - \beta_B R)^{-1} R\mathbf{1}.$$

We can see that in our model, the vector $\gamma$ (which captures the importance of sectors as input suppliers) plays this role. Is is given by:

$$\gamma = (I - (\tilde{\alpha}V)')^{-1}(\tilde{\alpha}V)'\beta.$$

So the vector $\gamma$ is similar to the expression above with $\alpha_B = \beta_B = 1$ and the matrix $\tilde{\alpha}V$ instead of $R$. In network analysis $R_{ij}$ denotes the presence (or intensity) of a link from $i$ to $j$. In our model $\alpha_i V_{ij}$ is the cost share of input $j$ in sector $i$ sales, so it represents the strength of the link from $j$ to $i$. Hence the correct notion of adjacency matrix is $(\tilde{\alpha}V)'$. Finally in the Bonacich centrality measure, the paths are weighted the same regardless of the node they terminate in; in our model the paths are weighted by the importance of the terminating node for final consumption.

## A.10 Relation to Baqaee and Farhi (2020)

In a paper developed contemporaneously, Baqaee and Farhi (2020) (BF) also provide a second-order approximation of the TFP loss from distortions in a setting similar to ours. The main objectives of the two papers are however distinct. BF focus on deriving a nonparametric decomposition of aggregate productivity growth: most of their paper is devoted to deriving and explaining the decomposition and the TFP loss is only briefly discussed (on p. 140-142). In contrast, the focus of our paper is to provide an in-depth analysis of how input substitutability shapes the role of sectoral linkages. We provide analyses on the nonmonotonic effect of the input elasticity of substitution, the dampening effect of sectoral linkages, the relevant linkages governing the TFP impact of a sectoral distortion, and on the distinction between labor and all-input distortions. None of these analysis are provided by BF. Moreover, whereas BF study only the second-order approximation, we derive insights also from the exact solution of the model and use the solution to discuss the conditions under which the approximation is valid.

While our focus is distinct from BF, the approximation we rely on is similar to that of BF. In what follows, we explain that our approximation corresponds to a special case of BF's formula, when their approximation is taken at the undistorted allocation. We then explain how the importance of a sector ($s_i$) relates to the cost-based Domar weight highlighted in BF's analysis.

### Comparison with BF's approximation formula

To better see the correspondence with our formula, we rewrite below BF's formula (equation (18) in their paper) in our setting (using our own notation):

$$\log TFP \approx -\frac{1}{2}\sum_{j=0}^{n}\varepsilon_j \frac{p_j Q_j}{Y}\,\mathrm{var}_{\Lambda^{(j)}}\left(\sum_k \hat{\Omega}_{(k)}\Delta\log(1+\tau_k)\right), \tag{A.10}$$

with $1+\tau_k = \exp(\Delta\log(1+\tau_k))$, and where $\varepsilon_j$ is the elasticity of substitution between the CES aggregation $j$ (in BF's formulation each nested CES function is treated as a separate producer), $X^{(j)}$ indicates the j'th row of matrix X and $X_{(j)}$ indicates the j'th column, and $\mathrm{var}_x$ is the variance weighted by vector $x$, and $\hat{\Omega} \equiv (I-\Lambda)^{-1}$, where $\Lambda$ is $n+1 \times n+1$ matrix given by

$$\Lambda_{ij} \equiv \frac{p_j X_{ij}}{p_i Q_i},\ \text{for } i,j = 1,...,n$$

$$\Lambda_{i,n+1} \equiv \frac{w_i L_i}{p_i Q_i},\ \text{for } i = 1,...,n$$

$$\Lambda_{0j} \equiv \frac{p_j C_j}{Y},\ \text{for } j = 1,...,n$$

$$\Lambda_{i,j} = 0 \text{ elsewhere.}$$

Here we use the price notations ($p_i$, $w$) instead of the multipliers ($\lambda_i$, $\eta$) to facilitate the connection to BF's formula.

In equation (A.10), the summation is on $j = 0,...,n$ even if the economy is composed of $n$ sectors. In BF's formulation, each CES aggregation is treated as a separate (artificial if necessary) producer. Applied to the equal-elasticity case, the formula has thus one CES aggregate for each sector (since each sector aggregates primary inputs and intermediate inputs with the same elasticity), and one CES for the final good, which is labeled 0. Therefore, we have $\varepsilon_j = \varepsilon_\sigma = \varepsilon_\rho$, $\forall j = 1,...,n$, and $\varepsilon_0 = \varepsilon_\chi$. Note that the matrix $\hat{\Omega}$ differs in general from our matrix $\Omega$ because the elements of $\hat{\Omega}$ are computed as the revenue shares of the inputs whereas the elements of $\Omega$ are functions of the production function parameters. In fact, comparing equation (A.10) to our equation (14), one difference comes from the facts that the weights $(p_j Q_j / Y, \Lambda_{ij}, \hat{\Omega}_{ik})$ are endogenous (they depend not only on the production function parameters but also on the distortions and the elasticity of substitution) whereas in our formula, the weights $(s_i, s_{ij}, x_i)$ are functions of exogenous parameters $(\alpha, \beta, V)$ only. However, if we apply BF's formula at the undistorted allocation, and with the assumptions $\varepsilon_0 = \varepsilon_\chi$ and $\varepsilon_j = \varepsilon_\sigma$ $\forall j = 1, ..., n$, the two formulas give exactly the same result.

### Relation between $s_i$ and the cost-based Domar weights

In their first-order local approximation around the distorted allocation, BF introduce the notion of cost-based Domar weight. How is the cost-based Domar weight ($\tilde{\lambda}$ in BF's notation) related to the importance of a sector $(s_i)$?

Expressed in our notations, the cost-based input-output matrix $\bar{F}$ introduced by BF is given by $\bar{F}_{ij} = p_i X_{ij} / [\sum_k p_k X_{ik} + w L_k]$. Then $\tilde{\lambda} = b'(I - F')^{-1}$, where $b$ is the vector of final-good expenditure shares. In our paper, the importance of the sectors is given by the vector $s = \beta'(I - (\tilde{\alpha} V)')^{-1}$.

In the absence of distortions, the two measures are identical. However, in the presence of markups, the cost-based input-output matrix eliminates the effect of markups in the purchasing sector, but it does not eliminate the effect of distortions on the prices and on the choice of intermediate inputs. In contrast, our variables $s_i$ are constructed from the input-output table of an economy without any distortions. In the special case of Cobb-Douglas production functions, the cost shares depend only on the production function parameters. In this special case, the cost-based Domar weights $\tilde{\lambda}_i$ and the importance of the sector $s_i$ will coincide. In any other case, the two notions are distinct.

# B Proofs

In this section, we present the proofs of all the propositions derived in the main text as well as that of the additional propositions of Appendix A.

## B.1 Model solution and normalization

**Proof of Proposition 1 (General CES case).** Part (i), (ii) and (iii) of the proposition, which give the expressions of $d_i$, $m_{ij}$ and $c_i$, are obtained from the first-order conditions. Equation (6) implies expression (iii). Equation (7) implies:

$$X_{ij} = A_i^{\frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}\varepsilon_\rho} v_{ij} X_i \left[\frac{\alpha_i Q_i}{X_i}\right]^{\frac{\varepsilon_\rho}{\varepsilon_\sigma}} \left(\frac{\lambda_i}{(1+\tau_{Xij})\lambda_j}\right)^{\varepsilon_\rho}, \tag{B.1}$$

which then gives

$$X_i = A_i^{\varepsilon_\sigma - 1} \alpha_i Q_i \left[\sum_j v_{ij} \left(\frac{\lambda_i}{(1+\tau_{Xij})\lambda_j}\right)^{\varepsilon_\rho - 1}\right]^{\frac{1}{\varepsilon_\rho - 1}\varepsilon_\sigma}. \tag{B.2}$$

Combining the two equations and the definition of $\lambda_{Xi}$ (equation (9)) gives expression (ii). Expression (i) is obtained from equation (8):

$$L_i = (A_i B_i)^{\varepsilon_\sigma - 1}(1-\alpha_i) Q_i \left( \frac{\lambda_i}{(1+\tau_{Li})\eta} \right)^{\varepsilon_\sigma} . \tag{B.3}$$

Then plugging equations (B.2) and (B.3) in the production function (1) yields

$$1 = A_i^{\varepsilon_\sigma} \lambda_i^{\varepsilon_\sigma} \left[ (1-\alpha_i) B_i^{\varepsilon_\sigma - 1} [(1+\tau_{Li})\eta]^{1-\varepsilon_\sigma} + \alpha_i \left[ \sum_j v_{ij} \left( \frac{1}{(1+\tau_{Xij})\lambda_j} \right)^{\varepsilon_\rho - 1} \right]^{\frac{1}{\varepsilon_\rho - 1}(\varepsilon_\sigma - 1)} \right]^{\frac{\varepsilon_\sigma}{\varepsilon_\sigma - 1}}, \tag{B.4}$$

which gives expression (v) of the proposition, after substituting the definition of $\lambda_{Xi}$.

From equation (6), $C_i = \beta_i \lambda_i^{-\varepsilon_X} Y$. Plugging in the production function for the final good (3), we obtain $\left( \sum_i \beta_i \lambda_i^{1-\varepsilon_X} \right)^{\frac{1}{1-\varepsilon_X}} = 1$ and hence $\eta = \left( \sum_i \beta_i (\lambda_i/\eta)^{1-\varepsilon_X} \right)^{-\frac{1}{1-\varepsilon_X}}$. Then $\lambda_i = (\lambda_i/\eta)\eta$ yields expression (iv) of the Proposition.

The resource constraint for good $i$ can be written as

$$\sum_j m_{ji} q_j + c_i = q_i, \tag{B.5}$$

where $q_i = Q_i/Y$. In matrix form, this is $M'q + c = q$, so $q = (I - M')^{-1}c$. Since $L_i = d_i Q_i = d_i q_i Y$, the resource constraint on labor can be written as $d'(I - M')^{-1}cY = \bar{L}$, which implies $Y = [d'(I - M')^{-1}c]^{-1}\bar{L}$ as stated in the proposition. ■

**Proof of Proposition 2.** First, we show that we can always reparameterize the production function in such a way to have $B_i = 1$ for all $i$. Let $\{A_i, B_i, \alpha_i, v_{ij}\}_{i,j}$ be some arbitrary parameters of the production function.

Then for all $i$ set

$$B_i' = 1$$

$$\alpha_i' = \frac{\alpha_i}{(1-\alpha_i)B_i^{\varepsilon_\sigma - 1} + \alpha_i}$$

$$A_i' = A_i [(1-\alpha_i) B_i^{\varepsilon_\sigma - 1} + \alpha_i]^{\frac{1}{\varepsilon_\sigma - 1}}.$$

Inspection shows that for all $i$ and $L_i > 0, X_{ij} > 0$,

$$A_i \left[ (1-\alpha_i)^{\frac{1}{\varepsilon_\sigma}} (B_i L_i)^{\frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} + \alpha_i^{\frac{1}{\varepsilon_\sigma}} \left( \sum_j v_{ij}^{\frac{1}{\varepsilon_\rho}} X_{ij}^{\frac{\varepsilon_\rho - 1}{\varepsilon_\rho}} \right)^{\frac{\varepsilon_\rho}{\varepsilon_\rho - 1} \frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} \right]^{\frac{\varepsilon_\sigma}{\varepsilon_\sigma - 1}} =$$

$$A_i' \left[ (1-\alpha_i')^{\frac{1}{\varepsilon_\sigma}} L_i^{\frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} + (\alpha_i')^{\frac{1}{\varepsilon_\sigma}} \left( \sum_j v_{ij}^{\frac{1}{\varepsilon_\rho}} X_{ij}^{\frac{\varepsilon_\rho - 1}{\varepsilon_\rho}} \right)^{\frac{\varepsilon_\rho}{\varepsilon_\rho - 1} \frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} \right]^{\frac{\varepsilon_\sigma}{\varepsilon_\sigma - 1}} .$$

Then $B_i', \alpha_i'$ and $A_i'$ represent the same production function as $B_i, \alpha_i$ and $A_i$. Hence without loss of generality, we can assume $B_i = 1, \forall i$.

Next, we show that by choosing the units appropriately, we can have $A_i = 1$ for all $i$. Let $k \in \mathbf{R}^n_{++}$ be an arbitrary vector of unit changes. Define $Q_i' = k_i Q_i$. Then $X_{ij} = X_{ij}'/k_j$. Substituting in the production function we obtain:

$$Q_i' = k_i A_i \left[ (1-\alpha_i)^{\frac{1}{\varepsilon_\sigma}} L_i^{\frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} + \alpha_i^{\frac{1}{\varepsilon_\sigma}} \left( \sum_j v_{ij}^{\frac{1}{\varepsilon_\rho}} k_j^{-\frac{\varepsilon_\rho - 1}{\varepsilon_\rho}} X_{ij}'^{\frac{\varepsilon_\rho - 1}{\varepsilon_\rho}} \right)^{\frac{\varepsilon_\rho}{\varepsilon_\rho - 1} \frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} \right]^{\frac{\varepsilon_\sigma}{\varepsilon_\sigma - 1}}.$$

The right-hand side of this expression looks the original production function with $v_{ij} k_j^{1-\varepsilon_\rho}$ playing the role of the parameters in the intermediate good aggregator. In general $\sum_j v_{ij} k_j^{1-\varepsilon_\rho} \neq 1$. To get around this problem, define the new parameters $v_{ij}(k) = \frac{v_{ij} k_j^{1-\varepsilon_\rho}}{\sum_j v_{ij} k_j^{1-\varepsilon_\rho}}, m_i(k) = \left( \sum_j v_{ij} k_j^{1-\varepsilon_\rho} \right)^{\frac{1}{1-\varepsilon_\rho}}$, $\alpha_i(k) = 1 - \frac{1-\alpha_i}{1-\alpha_i + \alpha_i m_i(k)^{1-\varepsilon_\sigma}}$ and

$$A_i(k) = k_i A_i \left[ 1 - \alpha_i + \alpha_i m_i(k)^{1-\varepsilon_\sigma} \right]^{\frac{1}{1-\varepsilon_\sigma}}.$$

This parameterization satisfies all the constraints for the production function ($A_i(k) > 0, \alpha_i(k) \in [0,1], v_{ij}(k) \in [0,1], \sum_j v_{ij}(k) = 1$). Again, we can verify that

$$k_i A_i \left[ (1-\alpha_i)^{\frac{1}{\varepsilon_\sigma}} (B_i L_i)^{\frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} + \alpha_i^{\frac{1}{\varepsilon_\sigma}} \left( \sum_j v_{ij}^{\frac{1}{\varepsilon_\rho}} X_{ij}^{\frac{\varepsilon_\rho - 1}{\varepsilon_\rho}} \right)^{\frac{\varepsilon_\rho}{\varepsilon_\rho - 1} \frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} \right]^{\frac{\varepsilon_\sigma}{\varepsilon_\sigma - 1}} =$$

$$A_i(k) \left[ (1-\alpha_i(k))^{\frac{1}{\varepsilon_\sigma}} (B_i L_i)^{\frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} + \alpha_i(k)^{\frac{1}{\varepsilon_\sigma}} \left( \sum_j v_{ij}(k)^{\frac{1}{\varepsilon_\rho}} X_{ij}'^{\frac{\varepsilon_\rho - 1}{\varepsilon_\rho}} \right)^{\frac{\varepsilon_\rho}{\varepsilon_\rho - 1} \frac{\varepsilon_\sigma - 1}{\varepsilon_\sigma}} \right]^{\frac{\varepsilon_\sigma}{\varepsilon_\sigma - 1}}.$$

Hence, the new parameters represent the same production function.

The proposition is equivalent to showing that there exists a vector $k$ such that $A_i(k) = 1 \; \forall i$. Then given the expression of $A_i(k)$ this is:

$$k_i = A_i^{-1} \left[ (1-\alpha_i) + \alpha_i m_i(k)^{1-\varepsilon_\sigma} \right]^{\frac{1}{1-\varepsilon_\sigma}}, \; i = 1, \ldots n. \tag{B.6}$$

Denote the mapping defined by the right-hand side of equation (B.6) as $g(k)$, where $k$ and $g$ are $n \times 1$ vectors. Then (B.6) is simply a fixed point. Since $g$ is monotone and continuous, a sufficient condition for it to have fixed points is that if $k \leq \bar{k}$ then $g(k) \leq \bar{k}$ and if $k \geq \underline{k}$ then $g(k) \geq \underline{k}$ for some positive vectors $\underline{k}, \bar{k}$, $\underline{k} \leq \bar{k}$ and the inequalities are elementwise.

If $k(s) = (s, s, \ldots s)'$, then $g_i(k(s))/s = A_i^{-1} \left[ (1-\alpha_i) s^{\varepsilon_\sigma - 1} + \alpha_i \right]^{\frac{1}{1-\varepsilon_\sigma}}$.

There are two cases to consider.

1. $\varepsilon_\sigma < 1$. Since $\lim_{s \to 0} g_i(k(s))/s = \infty$, $g(k(\underline{s})) > \underline{s}$ for some sufficiently small $\underline{s}$. Set $\underline{k} = (\underline{s}, \ldots \underline{s})'$.

Therefore $g(\underline{k}) > \underline{k}$. Let $k \geq \underline{k}$. Then

$$g(k) \geq g(\min_i k_i, \min_i k_i, \dots) \geq g(\underline{k}) > \underline{k}.$$

On the other hand, $\lim_{s\to\infty} g_i(k(s))/s = A_i^{-1}\alpha_i^{\frac{1}{1-\varepsilon_\sigma}}$, so if $A_i > \alpha_i^{\frac{1}{1-\varepsilon_\sigma}}$ for all $i$, $g(k(s)) < k(s)$ for $s$ sufficiently large. Then set $\bar{k} = (s, \dots s)'$ for $s$ sufficiently large. Showing that $k \leq \bar{k}$ implies $g(k) < \bar{k}$ is the same as the case above.

2. $\varepsilon_\sigma > 1$. We have that $\lim_{s\to 0} g_i(k(s))/s = A_i^{-1}\alpha_i^{\frac{1}{1-\varepsilon_\sigma}}$, so if $A_i < \alpha_i^{\frac{1}{1-\varepsilon_\sigma}}$ for all $i$ it follows that $g(k(\underline{s})) > k(\underline{s})$ for some $\underline{s}$ sufficiently small. Set $\underline{k} = (\underline{s}, \dots, \underline{s})'$.

   On the other hand, $\lim_{s\to\infty} g_i(k(s))/s = 0$, so $g(k(\bar{s})) < k(\bar{s})$ for some $\bar{s}$ sufficiently large. Set $\bar{k} = (\bar{s}, \dots, \bar{s})'$.

   Showing that $\underline{k} \leq k \leq \bar{k}$ implies $\underline{k} < g(k) < \bar{k}$ is the same as in the case of $\varepsilon_\sigma < 1$.

We can unify the conditions for the two cases as $\alpha_i A_i^{\varepsilon_\sigma - 1} < 1,\ i = 1, \dots n$. Let $k^*$ be a fixed point. Then we can set $A_i' = A_i(k^*), \alpha_i'(k^*), v_{ij}' = v_{ij}(k^*)$.

For the redefined variables $A_i' = B_i' = 1$. It is immediate that $\lambda_i/\eta = 1$ satisfy $v$ of Proposition 1. Hence $(iv)$ of proposition 1 implies that $\lambda_i = 1$ for all $i$ ( for any $\{\beta_i\}$). The last two statements of the proposition follow from condition $ii$ of Proposition 1.

Finally, since $C_i = C_i'/k_i$, plugging in the final good production function we obtain

$$Y = \left(\sum_i \beta_i^{\frac{1}{\varepsilon_X}} k_i^{\frac{1-\varepsilon_X}{\varepsilon_X}} C_i^{\frac{\varepsilon_X - 1}{\varepsilon_X}}\right)^{\frac{\varepsilon_X}{\varepsilon_X - 1}}.$$

Then in a similar fashion as above, we define:

$$\beta_i' = \frac{\beta_i k_i^{1-\varepsilon_X}}{\sum_i \beta_i k_i^{1-\varepsilon_X}},$$

$$k_Y = \left(\sum_i \beta_i k_i^{1-\varepsilon_X}\right)^{\frac{1}{1-\varepsilon_X}}.$$

The claim that $\lambda_i C_i'/Y' = \beta_i'$ is proved similarly to above. ■

**Proof of Proposition 3 .** As Proposition 2 allows us to do, we assume that $A_i = B_i = 1\ \forall i$.

First, we consider the case of zero distortions. It is immediate $\lambda/\eta = \mathbf{1}$ is a solution to the system (10). We will show that it is the only solution. Suppose that $\bar{\lambda} \equiv \max_k\{\lambda_k/\eta\} > 1$. Then

$$\frac{\lambda_i}{\eta} \leq \left[(1-\alpha_i)1^{\varepsilon_\sigma - 1} + \alpha_i \bar{\lambda}^{\varepsilon_\sigma - 1}\right]^{\frac{1}{\varepsilon_\sigma - 1}} < \bar{\lambda}.$$

Since this holds for all $i$, we get that $\max_k\{\lambda_k/\eta\} < \max_k\{\lambda_k/\eta\}$, a contradiction, so $\max_k\{\lambda_k/\eta\} \leq 1$. In the same way we get that $\min_k\{\lambda_k/\eta\} \geq 1$, so $\lambda_i/\eta = 1$ for all $i$. Since the rest of the solution is a function of $\lambda/\eta$, this proves uniqueness. The solution will exist if the matrix $(I - M')^{-1}$ exists with nonnegative elements. It is immediate that $M = \tilde{\alpha}V$, so existence is guaranteed by lemma B.1.

Now consider the case with distortions. We use the implicit function theorem to show the existence of solution of system (10) in some neighborhood of the zero vector. Let's rewrite it as $F(\lambda/\eta, \tau) = \mathbf{0}$. This is a continuous function of both sets of parameters. The Jacobian of $F$ with respect to $\lambda/\eta$ evaluated at zero distortions is $\frac{d}{d\lambda/\eta}F(\lambda/\eta, \mathbf{0}) = I - \tilde{\alpha}V$ and lemma B.1 ensures that $\det\left|\frac{d}{d\lambda/\eta}F(\lambda/\eta, \mathbf{0})\right| \neq 0$. So by the implicit function theorem there is a unique continuous solution of $\lambda/\eta$ as a function of the distortions. Lastly the invertibility of the matrix $I - M'$ comes again from lemma B.1 and continuity. ■

**Proof of Proposition 4 (Equal-elasticity case).** With $\varepsilon_\sigma = \varepsilon_\rho$ equation (B.4) becomes:

$$\left(\frac{\lambda_i}{\eta}\right)^{1-\varepsilon_\sigma} = (1-\alpha_i)(A_iB_i)^{\varepsilon_\sigma - 1}(1+\tau_{Li})^{1-\varepsilon_\sigma} + \alpha_i A_i^{\varepsilon_\sigma - 1}\sum_{j=1}^{n} v_{ij}(1+\tau_{Xij})^{1-\varepsilon_\sigma}\left(\frac{\lambda_j}{\eta}\right))^{1-\varepsilon_\sigma}.$$

Expressing this in matrix form,

$$\hat{\lambda} = \hat{A}\hat{B}\Delta^{Lp}(1-\alpha) + \tilde{\alpha}\hat{A}(\Delta^{Xp}\circ V)\hat{\lambda},$$

where $\hat{\lambda}_i = (\lambda_i/\eta)^{1-\varepsilon_\sigma}$, the square matrixes $\hat{A}$, $\hat{B}$, $\Delta^{Lp}$, $\Delta^{Xp}$ and $\tilde{\alpha}$ have diagonal elements equal to $\hat{A}_{ii} = A_i^{\varepsilon_\sigma - 1}$, $\hat{B}_{ii} = B_i^{\varepsilon_\sigma - 1}$, $\Delta^{Lp}_{ii} = (1+\tau_{Li})^{1-\varepsilon_\sigma}$, $\Delta^{Xp}_{ij} = (1+\tau_{Xij})^{1-\varepsilon_\sigma}$, $\tilde{\alpha}_{ii} = \alpha_i$, and nondiagonal elements equal to zero. The equation implies (12).

Now equation (B.1) gives $m_{ij} = \alpha_i v_{ij} A_i^{\varepsilon_\sigma - 1}[(1+\tau_{Xij})\lambda_j/\lambda_i]^{-\varepsilon_\sigma}$. Plugging this expression for $m_{ij}$ in the resource constraint for good i (B.5), we have

$$\sum_j \alpha_j v_{ji} A_j^{\varepsilon_\sigma - 1}(1+\tau_{Xji})^{-\varepsilon_\sigma}\lambda_j^{\varepsilon_\sigma} q_j + \lambda_i^{\varepsilon_\sigma} c_i = \lambda_i^{\varepsilon_\sigma} q_i. \tag{B.7}$$

Using the expression for $c_i$, we get:

$$\begin{aligned}\lambda_i^{\varepsilon_\sigma} c_i &= \beta_i \lambda_i^{\varepsilon_\sigma - \varepsilon_X} \\ &= \beta_i \left(\frac{\lambda_i}{\eta}\right)^{\varepsilon_\sigma - \varepsilon_X} \eta^{\varepsilon_\sigma - \varepsilon_X} \\ &= \beta_i \hat{\lambda}_i^{\frac{1}{1-\varepsilon_\sigma}(\varepsilon_\sigma - \varepsilon_X)}\eta^{\varepsilon_\sigma - \varepsilon_X}.\end{aligned}$$

So denoting $\hat{q}_i \equiv q_i\lambda_i^{\varepsilon_\sigma}$ and $\hat{c}_i \equiv \beta_i\hat{\lambda}_i^{\frac{1}{1-\varepsilon_\sigma}(\varepsilon_\sigma - \varepsilon_X)}$, (B.7) becomes:

$$[\tilde{\alpha}\hat{A}(\Delta^{Xq}\circ V)]'\hat{q} + \hat{c}\eta^{\varepsilon_\sigma - \varepsilon_X} = \hat{q}.$$

Therefore,

$$\hat{q} = [I - (\tilde{\alpha}\hat{A}(\Delta^{Xq}\circ V))']^{-1}\hat{c}\eta^{\varepsilon_\sigma - \varepsilon_X}. \tag{B.8}$$

Finally, using $d_i = (1-\alpha_i)(A_iB_i)^{\varepsilon_\sigma - 1}\lambda_i^{\varepsilon_\sigma}[(1+\tau_{Li})\eta]^{-\varepsilon_\sigma}$ in the resource constraint on labor written as $\sum d_i q_i Y = \bar{L}$, gives $(\mathbf{1}-\alpha)'\hat{A}\hat{B}\Delta^{Lq}\hat{q}\eta^{-\varepsilon_\sigma}Y = \bar{L}$. Then substituting for the expression of $\hat{q}$:

$$\hat{c}'[I - (\tilde{\alpha}\hat{A}(\Delta^{Xq}\circ V))]^{-1}\Delta^{Lq}\hat{A}\hat{B}(\mathbf{1}-\alpha)\eta^{-\varepsilon_X}Y = \bar{L}.$$

Then taking logs and using $\log\eta = \frac{1}{\varepsilon_X - 1}\log\left(\sum_i \beta_i(\lambda_i/\eta)^{1-\varepsilon_X}\right)$, we get equation (11). ■

## B.2 Technical Results

**Lemma B.1** *Suppose* $\max\{\alpha_i\} = \bar{\alpha} < 1$. *Then* $I - \tilde{\alpha}V$ *is invertible, i.e.* $\Omega$ *exists;* $\omega_{ij} \geq 0$ *for all* $i,j$; $\omega_{ii} \geq 1$ *and for all* $i \neq j$, $\omega_{ij} < \omega_{jj}$. *Finally,* $\Omega = I$ *if and only if* $\alpha_i = 0, \forall i$.

**Proof of Lemma B.1.** Let $A = \tilde{\alpha}V$. From the definition it follows that $a_{ij} \geq 0$ and $\sum_j a_{ij} \leq \bar{\alpha}$. Also $\Omega = [I-A]^{-1}$ (if this inverse exists) which can be rewritten $\Omega = I + A\Omega$.

Let $\Omega^{(0)} = I$ and let

$$\Omega^{(p+1)} = I + A\Omega^{(p)}.$$

Note that by induction

$$\Omega^{(p)} = I + A + A^2 + \dots A^p,$$

hence $\omega_{ij}^{(p+1)} \geq \omega_{ij}^{(p)}$ and $\omega_{ij}^{(p)} \geq 0$. The fact that $\sum_k a_{ik} \leq \bar{\alpha} < 1$ implies (by induction) that $a_{ij}^p \leq \bar{\alpha}^p$ and $0 \leq \omega_{ij}^{(p)} \leq \frac{1-\bar{\alpha}^{p+1}}{1-\bar{\alpha}}$. So $\Omega \equiv \lim_{p\to\infty}\Omega^{(p)}$ exists and is finite.

$$(I-A)\Omega = (I-A)\lim_{p\to\infty}\Omega^{(p)} = \lim_{p\to\infty}[(I-A)\Omega^{(p)}] = \lim_{p\to\infty}[I - A^{p+1}] = I,$$

which implies that $\Omega = (I-A)^{-1}$.

By construction $\omega_{ij}^{(p)} \geq 0$, so $\omega_{ij} = \lim_{p\to\infty}\omega_{ij}^{(p)} \geq 0$, which proves the first statement. Similarly $\omega_{ii}^{(p)} \geq 1$ implies that $\omega_{ii} \geq 1$.

Next, we show by induction that $\omega_{ij}^{(p)} < \omega_{jj}^{(p)}$ for all $i \neq j$ and $p$. This is true by construction for $p = 0$. Suppose it is true for some $p$. Consider $p+1$. Suppose that $i \neq j$. Then by the definition of $\Omega^{(p+1)}$:

$$\begin{aligned} \omega_{ij}^{(p+1)} &= \sum_k a_{ik}\omega_{kj}^{(p)} \\ &\leq \omega_{jj}^{(p)}\sum_k a_{ik} \\ &< \omega_{jj}^{(p)} \\ &\leq \omega_{jj}^{(p+1)}. \end{aligned}$$

The second line comes from the inductive step; the third line from the fact that $\sum_k a_{ik} < 1$ and $\omega_{jj}^{(p)} \geq 1$; the last one from the fact that the sequence of matrices is monotone.

Then taking limits with respect to $p$, we establish that $\omega_{ij} \leq \omega_{jj}$. Finally, since $\Omega = I + A\Omega$, by the same chain of inequalities as above we establish that if $i \neq j$, $\omega_{ij} < \omega_{jj}$.

If $\alpha_i = 0\ \forall i$, then $A$ is a square matrix of zeros, so $\Omega = (I-A)^{-1} = I^{-1} = I$. Suppose that $\alpha_i > 0$ for some $i$. Then $a_{ij} > 0$ for some $j$. If $i \neq j$ $\omega_{ij} \geq \omega^1_{ij} = a_{ij} > 0$; if $i = j$, then $\omega_{ii} \geq \omega^1_{ii} > 1$. So in either case $\Omega \neq I$. Hence the last statement is proved by contrapositive. ■

**Lemma B.2** $\Omega(\mathbf{1} - \alpha) = \mathbf{1}$.

**Proof of Lemma B.2.** Since $V\mathbf{1} = \mathbf{1}$, and $\alpha = \tilde{\alpha}\mathbf{1} = \tilde{\alpha}V\mathbf{1}$ we have that

$$\mathbf{1} - \alpha = I\mathbf{1} - \tilde{\alpha}V\mathbf{1} = (I - \tilde{\alpha}V)\mathbf{1}.$$

Premultiplying the equality above by $(I - \tilde{\alpha}V)^{-1}$ implies the result. ■

**Lemma B.3** *Let* $g_{ij} \equiv \sum_r v_{ir}\omega_{rj}$*. Then* $\alpha_i g_{ij} = \omega_{ij}$ *for* $i \neq j$ *and* $\alpha_i g_{ii} = \omega_{ii} - 1$.

**Proof of Lemma B.3.** Let $G$ be a matrix with elements $(i, j)$ equal to $\alpha_i g_{ij}$. Direct inspection shows that $G = \tilde{\alpha}V\Omega$. Since $\Omega = (I - \tilde{\alpha}V)^{-1}$, we have that $I = (I - \tilde{\alpha}V)\Omega = \Omega - G$, so $G = \Omega - I$, which proves the claim. ■

**Lemma B.4** *Suppose that the distortions on intermediate goods are sector-specific, so* $\tau_{Xij} = \tau_{Xik} = \tau_{Xi}, \forall i, j, k = 1, ..., n$*. We will denote the vector of intermediate-good distortions by* $\tau_X$*, the vector of labor distortions by* $\tau_L$*, and the vector of all distortions by* $\tau$*. Suppose that* $A = B = \mathbf{1}$*. Let* $\mathbf{0}$ *be a matrix of zeros with dimensions clear from context. Then,*

$$\begin{aligned}
\frac{d^2}{d\tau_{Li}\tau_{Lj}} \log TFP(\mathbf{0}) &= \left((\varepsilon_\sigma - \varepsilon_\chi)s_{ij} + \varepsilon_\chi s_i s_j\right)(1-\alpha_i)(1-\alpha_j), \quad \textit{if } i \neq j \\
\frac{d^2}{d\tau_{Li}\tau_{Li}} \log TFP(\mathbf{0}) &= \left((\varepsilon_\sigma - \varepsilon_\chi)s_{ii} + \varepsilon_\chi s_i^2\right)(1-\alpha_i)^2 - \varepsilon_\sigma s_i(1-\alpha_i) \\
\frac{d^2}{d\tau_{Xi}\tau_{Lj}} \log TFP(\mathbf{0}) &= \left[-\varepsilon_\sigma s_i \left(\sum_s v_{is}\omega_{sj}\right) + \varepsilon_\chi s_i s_j + (\varepsilon_\sigma - \varepsilon_\chi)s_{ij}\right]\alpha_i(1-\alpha_j) \\
\frac{d^2}{d\tau_{Xi}\tau_{Xj}} \log TFP(\mathbf{0}) &= \left[(\varepsilon_\sigma - \varepsilon_\chi)s_{ij} + \varepsilon_\chi s_i s_j - \varepsilon_\sigma\left(s_i \sum_s v_{is}\omega_{sj} + s_j \sum_s v_{js}\omega_{si}\right)\right]\alpha_i\alpha_j \quad \textit{if } i \neq j \\
\frac{d^2}{d\tau_{Xi}\tau_{Xi}} \log TFP(\mathbf{0}) &= \left[(\varepsilon_\sigma - \varepsilon_\chi)s_{ii} + \varepsilon_\chi s_i^2 - 2\varepsilon_\sigma s_i \sum_s v_{is}\omega_{si}\right]\alpha_i^2 - \varepsilon_\sigma s_i\alpha_i.
\end{aligned}$$

**Proof of Lemma B.4.** If there is no superscript on the distortion, we will mean that it can be either on labor or on intermediates.

Using proposition 4 with the assumption $A_i = B_i = 1 \; \forall i = 1, ..., n$, we can write aggregate productivity as

$$\log TFP = \frac{\varepsilon_\chi}{\varepsilon_\chi - 1} \log \left( \sum_{i=1}^{n} \beta_i \hat{\lambda}_i^{\frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1}} \right) - \log(\hat{c}'[I - \tilde{\alpha}\hat{A}(\Delta^{Xq} \circ V)]^{-1} \Delta^{Lq}(\mathbf{1} - \alpha))$$

with

$$\hat{\lambda} = [I - \tilde{\alpha}(\Delta^{Xp} \circ V)]^{-1} \Delta^{Lp}(\mathbf{1} - \alpha).$$

Denote the first term $N^p(\tau) \equiv \frac{\varepsilon_\chi}{\varepsilon_\chi - 1} \log \left( \sum_{i=1}^{n} \beta_i \hat{\lambda}_i^{\frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1}} \right)$ and the second term $N^q(\tau) \equiv -\log(\hat{c}'[I - \tilde{\alpha}(\Delta^{Xq} \circ V)]^{-1} \Delta^{Lq}(\mathbf{1} - \alpha))$. Let us define $\Omega^p(\tau) \equiv [I - \tilde{\alpha}(\Delta^{Xp} \circ V)]^{-1}$ and $\Omega^q(\tau) \equiv [I - \tilde{\alpha}(\Delta^{Xq} \circ V)]^{-1}$. Note that $\Omega^p(\mathbf{0}) = \Omega^q(\mathbf{0}) = \Omega$.

The assumption that the intermediate goods distortions are purchaser specific implies that the matrices $\Delta^{Xp} \circ V, \Delta^{Xq} \circ V$ can be written as:

$$\Delta^{Xp} \circ V = \begin{bmatrix} \tau_{X1}^{1-\varepsilon_\sigma} & 0 & \dots & 0 \\ 0 & \tau_{X2}^{1-\varepsilon_\sigma} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tau_{Xn}^{1-\varepsilon_\sigma} \end{bmatrix} V, \;\; \Delta^{Xq} \circ V = \begin{bmatrix} \tau_{X1}^{-\varepsilon_\sigma} & 0 & \dots & 0 \\ 0 & \tau_{X2}^{-\varepsilon_\sigma} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tau_{Xn}^{-\varepsilon_\sigma} \end{bmatrix} V.$$

For ease of exposition, we derive the expressions in a sequence of 13 steps. To simplify the expressions, we will use the notation $f_{\tau_i}$ to denote the partial derivative of $f$ with respect to $\tau_i$ at point $\tau$ for any real-valued or matrix function $f$. Similarly we will use the notation $f_{\tau_i \tau_j}$ to denote the partial derivative of $f_{\tau_i}$ with respect to $\tau_j$.

1. For any distortions, $\tau_i, \tau_j$, we have that:

$$\begin{aligned} N^p_{\tau_i} =& \frac{\varepsilon_\chi}{\varepsilon_\sigma - 1} \exp(-N^p) \sum_k \beta_k \hat{\lambda}_k^{\frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1} - 1} \hat{\lambda}_{k\tau_i} \\ N^p_{\tau_i \tau_j} =& -\frac{\varepsilon_\chi}{\varepsilon_\sigma - 1} \frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1} \exp(-2N^p) \left( \sum_k \beta_k \hat{\lambda}_k^{\frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1} - 1} \hat{\lambda}_{k\tau_i} \right) \left( \sum_k \beta_k \hat{\lambda}_k^{\frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1} - 1} \hat{\lambda}_{k\tau_j} \right) \\ &+ \frac{\varepsilon_\chi}{\varepsilon_\sigma - 1} \left( \frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1} - 1 \right) \exp(-N^p) \sum_k \beta_k \hat{\lambda}_k^{\frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1} - 2} \hat{\lambda}_{k\tau_i} \hat{\lambda}_{k\tau_j} \\ &+ \frac{\varepsilon_\chi}{\varepsilon_\sigma - 1} \exp(-N^p) \sum_k \beta_k \hat{\lambda}_k^{\frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1} - 1} \hat{\lambda}_{k\tau_i \tau_j}. \end{aligned}$$

   **Proof**. This follows from the definition of $N^p$.

2. Denote $G = \exp(N^q) = \hat{c}' \Omega^q \Delta^{Lq}(\mathbf{1} - \alpha)$. For any distortions $\tau_i, \tau_j$:

$$
\begin{aligned}
N^q_{\tau_i} &= -\frac{\hat{c}'_{\tau_i}\Omega^q\Delta^{Lq} + \hat{c}'\Omega^q_{\tau_i}\Delta^{Lq} + \hat{c}'\Omega^q\Delta^{Lq}_{\tau_i}}{G}(\mathbf{1}-\alpha) \\
N^q_{\tau_i\tau_j} &= -\frac{\hat{c}'_{\tau_i\tau_j}\Omega^q\Delta^{Lq} + \hat{c}'\Omega^q_{\tau_i\tau_j}\Delta^{Lq} + \hat{c}'\Omega^q\Delta^{Lq}_{\tau_i\tau_j}}{G}(\mathbf{1}-\alpha) \\
&\quad -\frac{\hat{c}'_{\tau_i}\Omega^q_{\tau_j}\Delta^{Lq} + \hat{c}'_{\tau_i}\Omega^q\Delta^{Lq}_{\tau_j} + \hat{c}'_{\tau_j}\Omega^q_{\tau_i}\Delta^{Lq}}{G}(\mathbf{1}-\alpha) \\
&\quad -\frac{\hat{c}'\Omega^q_{\tau_i}\Delta^{Lq}_{\tau_j} + \hat{c}'_{\tau_j}\Omega^q\Delta^{Lq}_{\tau_i} + \hat{c}'\Omega^q_{\tau_j}\Delta^{Lq}_{\tau_i}}{G}(\mathbf{1}-\alpha) + N^q_{\tau_i}N^q_{\tau_j}.
\end{aligned}
$$

**Proof**. This follows from the fact

$$N^q_{\tau_i} = -\frac{G_{\tau_i}}{G}$$

$$N^q_{\tau_i\tau_j} = -\frac{G_{\tau_i\tau_j}}{G} + \frac{G_{\tau_i}G_{\tau_j}}{G^2}.$$

3. Denote $Z^i = \tilde{\alpha}I^iV$, where $I^i$ is a square matrix with $I^i_{ii} = 1$ and 0 elsewhere. Then

$$\Omega^p_{\tau_{Xi}} = (1-\varepsilon_\sigma)(1+\tau_{Xi})^{-\varepsilon_\sigma}\Omega^p Z^i \Omega^p,$$

$$\Omega^p_{\tau_{Xi}\tau_{Xj}} = (1-\varepsilon_\sigma)^2[(1+\tau_{Xi})(1+\tau_{Xj})]^{-\varepsilon_\sigma}[\Omega^p Z^i\Omega^p Z^j\Omega^p + \Omega^p Z^j\Omega^p Z^i\Omega^p] \text{ if i} \neq \text{j},$$

$$\Omega^p_{\tau_{Xi}\tau_{Xi}} = \varepsilon_\sigma(\varepsilon_\sigma - 1)(1+\tau_{Xi})^{-\varepsilon_\sigma-1}\Omega^p Z^i\Omega^p + 2(1-\varepsilon_\sigma)^2(1+\tau_{Xi})^{-2\varepsilon_\sigma}\Omega^p Z^i\Omega^p Z^i\Omega^p.$$

Similarly,

$$\Omega^q_{\tau_{Xi}} = -\varepsilon_\sigma(1+\tau_{Xi})^{-\varepsilon_\sigma-1}\Omega^q Z^i\Omega^q,$$

$$\Omega^q_{\tau_{Xi}\tau_{Xj}} = \varepsilon_\sigma^2[(1+\tau_{Xi})(1+\tau_{Xj})]^{-\varepsilon_\sigma-1}[\Omega^q Z^i\Omega^q Z^j\Omega^q + \Omega^q Z^j\Omega^q Z^i\Omega^q] \text{ if i} \neq \text{j},$$

$$\Omega^q_{\tau_{Xi}\tau_{Xi}} = \varepsilon_\sigma(\varepsilon_\sigma + 1)(1+\tau_{Xi})^{-\varepsilon_\sigma-2}\Omega^q Z^i\Omega^q + 2\varepsilon_\sigma^2(1+\tau_{Xi})^{-2\varepsilon_\sigma^2-2}\Omega^q Z^i\Omega^q Z^i\Omega^q.$$

**Proof**. Let $D(x)$ be a square matrix that depends on the parameter $x$ and is differentiable in $x$. Then standard linear algebra implies that on the interior of the set where $D(x)$ is invertible, we have that

$$\frac{d}{dx}D^{-1}(x) = -D^{-1}(x)\frac{d}{dx}D(x)D^{-1}(x).$$

This fact and the definitions of $\Omega^p, \Omega^q$ implies the results.

4. Let $(\mathbf{1}-\alpha)^i \equiv I^i(\mathbf{1}-\alpha)$, which is a column vector that is zero everywhere, except for the i-th row which is $1-\alpha_i$. We have

$$\hat{\lambda}_{\tau_{Xi}} = (1-\varepsilon_\sigma)(1+\tau_{Xi})^{-\varepsilon_\sigma}\Omega^p Z^i\Omega^p\Delta^{Lp}(\mathbf{1}-\alpha),$$

$$\hat{\lambda}_{\tau_{Lj}} = (1-\varepsilon_\sigma)(1+\tau_{Lj})^{-\varepsilon_\sigma}\Omega^p(\mathbf{1}-\alpha)^j,$$

$$\hat{\lambda}_{\tau_{Lj}\tau^L_s} = \mathbf{0}, \text{ if } s \neq j,$$

$$\hat{\lambda}_{\tau_{Lj}\tau_{Lj}} = \varepsilon_\sigma(\varepsilon_\sigma - 1)(1+\tau_{Lj})^{-\varepsilon_\sigma - 1}\Omega^p(\mathbf{1}-\alpha)^j,$$

$$\hat{\lambda}_{\tau_{Xi}\tau_{Lj}} = (1-\varepsilon_\sigma)^2(1+\tau_{Lj})^{-\varepsilon_\sigma}(1+\tau_{Xi})^{-\varepsilon_\sigma}\Omega^p Z^i \Omega^p(\mathbf{1}-\alpha)^j,$$

$$\hat{\lambda}_{\tau_{Xi}\tau_{Xj}} = (1-\varepsilon_\sigma)^2[(1+\tau_{Xj})(1+\tau_{Xi})]^{-\varepsilon_\sigma}[\Omega^p Z^i \Omega^p Z^j + \Omega^p Z^j \Omega^p Z^i]\Omega^p \Delta^{Lp}(\mathbf{1}-\alpha), \text{ if } i \neq j,$$

$$\begin{aligned}\hat{\lambda}_{\tau_{Xi}\tau_{Xi}} &= \varepsilon_\sigma(\varepsilon_\sigma - 1)(1+\tau_{Xi})^{-\varepsilon_\sigma - 1}\Omega^p Z^i \Omega^p \Delta^{Lp}(\mathbf{1}-\alpha) \\ &\quad + 2(1-\varepsilon_\sigma)^2(1+\tau_{Xi})^{-2\varepsilon_\sigma}\Omega^p Z^i \Omega^p Z^i \Omega^p \Delta^{Lp}(\mathbf{1}-\alpha).\end{aligned}$$

**Proof**. These expressions follow from the fact that $\hat{\lambda} = \Omega^p \Delta^{Lp}(\mathbf{1}-\alpha)$ and the derivatives of the matrix $\Omega^p$ derived in Step 3.

5. Define $C^i = \Omega Z^i \Omega(\mathbf{1}-\alpha), D^{ij} = \Omega Z^i \Omega Z^j \Omega(\mathbf{1}-\alpha), E^i = \Omega(\mathbf{1}-\alpha)^i, F^{ij} = \Omega Z^i \Omega(\mathbf{1}-\alpha)^j$. (Note that they are all column vectors.) Then:

$$\hat{\lambda}_{\tau_{Xi}}(\mathbf{0}) = (1-\varepsilon_\sigma)C^i,$$

$$\hat{\lambda}_{\tau_{Lj}}(\mathbf{0}) = (1-\varepsilon_\sigma)E^j,$$

$$\hat{\lambda}_{\tau_{Lj}\tau_s^L}(\mathbf{0}) = \mathbf{0}, \text{ if } s \neq j,$$

$$\hat{\lambda}_{\tau_{Lj}\tau_{Lj}}(\mathbf{0}) = \varepsilon_\sigma(\varepsilon_\sigma - 1)E^j,$$

$$\hat{\lambda}_{\tau_{Xi}\tau_{Lj}}(\mathbf{0}) = (1-\varepsilon_\sigma)^2 F^{ij},$$

$$\hat{\lambda}_{\tau_{Xi}\tau_{Xj}}(\mathbf{0}) = (1-\varepsilon_\sigma)^2[D^{ij} + D^{ji}] \text{ if } i \neq j,$$

$$\hat{\lambda}_{\tau_{Xi}\tau_{Xi}}(\mathbf{0}) = \varepsilon_\sigma(\varepsilon_\sigma - 1)C^i + 2(1-\varepsilon_\sigma)^2 D^{ii}.$$

**Proof**. This follows from the fact that for $\tau = \mathbf{0}$, $\Delta^{Lp} = I$, $\Omega^p = \Omega$ and the expressions found in Step 4.

6. We have:

$$\hat{c}_{\tau_{Xi}}(\mathbf{0}) = (\varepsilon_\sigma - \varepsilon_\chi)\beta \circ C^i,$$

$$\hat{c}_{\tau_{Lj}}(\mathbf{0}) = (\varepsilon_\sigma - \varepsilon_\chi)\beta \circ E^j,$$

$$\hat{c}_{\tau_{Li}\tau_{Lj}}(\mathbf{0}) = (\varepsilon_\sigma - \varepsilon_\chi)(2\varepsilon_\sigma - \varepsilon_\chi - 1)\beta \circ E^i \circ E^j, \text{ if } i \neq j,$$

$$\hat{c}_{\tau_{Lj}\tau_{Lj}}(\mathbf{0}) = (\varepsilon_\chi - \varepsilon_\sigma)\varepsilon_\sigma \beta \circ E^j + (\varepsilon_\sigma - \varepsilon_\chi)(2\varepsilon_\sigma - \varepsilon_\chi - 1)\beta \circ E^j \circ E^j,$$

$$\hat{c}_{\tau_{Xi}\tau_{Lj}}(\mathbf{0}) = (\varepsilon_\sigma - \varepsilon_\chi)(1-\varepsilon_\sigma)\beta \circ F^{ij} + (\varepsilon_\sigma - \varepsilon_\chi)(2\varepsilon_\sigma - \varepsilon_\chi - 1)\beta \circ C^i \circ E^j,$$

$$\hat{c}_{\tau_{Xi}\tau_{Xj}}(\mathbf{0}) = (\varepsilon_\sigma - \varepsilon_\chi)(1-\varepsilon_\sigma)\beta \circ [D^{ij} + D^{ji}] + (\varepsilon_\sigma - \varepsilon_\chi)(2\varepsilon_\sigma - \varepsilon_\chi - 1)\beta \circ C^i \circ C^j \text{ if } i \neq j,$$

$$\hat{c}_{\tau_{Xi}\tau_{Xi}}(\mathbf{0}) = (\varepsilon_\chi - \varepsilon_\sigma)\varepsilon_\sigma \beta \circ C^i + 2(\varepsilon_\sigma - \varepsilon_\chi)(1-\varepsilon_\sigma)\beta \circ D^{ii} + (\varepsilon_\sigma - \varepsilon_\chi)(2\varepsilon_\sigma - \varepsilon_\chi - 1)\beta \circ C^i \circ C^i.$$

**Proof**. By definition, $\hat{c}_k = \beta_k \hat{\lambda}_k^r$, where $r \equiv \frac{\varepsilon_\chi - \varepsilon_\sigma}{\varepsilon_\sigma - 1}$. Then $\hat{c}_{k,\tau_i} = r\beta_k \hat{\lambda}_{k,\tau_i}\hat{\lambda}_k^{r-1}$; $\hat{c}_{k,\tau_i\tau_j} =$

$r\beta_k\hat{\lambda}_{k,\tau_i\tau_j}\hat{\lambda}_k^{r-1} + r(r-1)\beta_k\hat{\lambda}_{k,\tau_i}\hat{\lambda}_{k,\tau_j}\hat{\lambda}_k^{r-2}$. Then the fact that $\hat{\lambda}(\mathbf{0}) = \mathbf{1}$ and the expressions in Step 5 imply the result.

7.
$$\Delta^{Lq}_{\tau_{Li}}(\mathbf{0}) = -\varepsilon_\sigma I^i,$$
$$\Delta^{Lq}_{\tau_{Li}\tau_{Lj}}(\mathbf{0}) = \mathbf{0}, \text{ if } i \neq j,$$
$$\Delta^{Lq}_{\tau_{Li}\tau_{Li}}(\mathbf{0}) = \varepsilon_\sigma(\varepsilon_\sigma + 1)I^i,$$

and all derivatives involving $\tau_{Xi}$ are zero matrices.

**Proof**. Follows directly from the definition.

8. $C_k^i = \alpha_i\omega_{ki}$; $D_k^{ij} = \omega_{ki}\sum_s v_{is}\omega_{sj}\alpha_i\alpha_j$; $E_k^i = (1-\alpha_i)\omega_{ki}$; $F_k^{ij} = \omega_{ki}\sum_s v_{is}\omega_{sj}\alpha_i(1-\alpha_j)$.

**Proof**. This follows from the definition of these vectors and the facts that $\Omega(\mathbf{1}-\alpha) = \mathbf{1}, V\mathbf{1} = \mathbf{1}$.

9.
$$\sum_k \beta_k\hat{\lambda}_{k\tau_{Li}} = (1-\varepsilon_\sigma)(1-\alpha_i)s_i$$
$$\sum_k \beta_k\hat{\lambda}_{k\tau_{Xi}} = (1-\varepsilon_\sigma)\alpha_i s_i$$
$$\sum_k \beta_k\hat{\lambda}_{k\tau_{Li}\tau_{Lj}} = 0$$
$$\sum_k \beta_k\hat{\lambda}_{k\tau_{Li}\tau_{Li}} = (\varepsilon_\sigma - 1)\varepsilon_\sigma(1-\alpha_i)s_i$$

$$\sum_k \beta_k\hat{\lambda}_{k\tau_{Xi}\tau_{Lj}} = (1-\varepsilon_\sigma)^2\alpha_i(1-\alpha_j)s_i\sum_s v_{is}\omega_{sj}$$
$$\sum_k \beta_k\hat{\lambda}_{k\tau_{Xi}\tau_{Xj}} = (1-\varepsilon_\sigma)^2\left(s_i\sum_s v_{is}\omega_{sj} + s_j\sum_s v_{js}\omega_{si}\right)\alpha_i\alpha_j$$
$$\sum_k \beta_k\hat{\lambda}_{k\tau_{Xi}\tau_{Xi}} = 2(1-\varepsilon_\sigma)^2 s_i\sum_s v_{is}\omega_{si}\alpha_i^2 + (\varepsilon_\sigma - 1)(\varepsilon_\sigma)\frac{\sigma}{(1-\sigma)^2}\alpha_i s_i$$

$$\sum_k \beta_k\hat{\lambda}_{k\tau_{Li}}\hat{\lambda}_{k,\tau_{Lj}} = (1-\varepsilon_\sigma)^2 s_{ij}(1-\alpha_i)(1-\alpha_j)$$
$$\sum_k \beta_k\hat{\lambda}_{k\tau_{Xi}}\hat{\lambda}_{k,\tau_{Xj}} = (1-\varepsilon_\sigma)^2 s_{ij}\alpha_i\alpha_j$$
$$\sum_k \beta_k\hat{\lambda}_{k\tau_{Xi}}\hat{\lambda}_{k,\tau_{Lj}} = (1-\varepsilon_\sigma)^2 s_{ij}\alpha_i(1-\alpha_j).$$

**Proof**. This follows from combining Step 5, Step 7 and the definition of $s_i, s_{ij}$.

10. We can now derive the first expression of Lemma B.4 $\frac{d}{d\tau_{Li}\tau_{Lj}}\log Y(\mathbf{0}),\ i \neq j$.

From Steps 1 and 9, we have

$$N^p_{\tau_{Li}\tau_{Lj}}(\mathbf{0}) = -\varepsilon_\chi(\varepsilon_\chi - 1)(1-\alpha_i)(1-\alpha_j)s_i s_j + \varepsilon_\chi(\varepsilon_\chi - \varepsilon_\sigma)(1-\alpha_i)(1-\alpha_j)s_{ij} \text{for } i \neq j.$$

Since $\Omega^q(\mathbf{0}) = \Omega, G(\mathbf{0}) = 1, \Omega(\mathbf{1}-\alpha) = \mathbf{1}$ and some of the terms in the expression for $\frac{d^2}{d\tau_i d\tau_j}N$ are zero matrices, Step 2 implies

$$\begin{aligned} N^q_{\tau_{Li}\tau_{Lj}}(\mathbf{0}) &= -\left(\hat{c}'_{\tau_{Li}\tau_{Lj}}\Omega + \hat{c}'_{\tau_{Li}}\Omega\Delta^{Lq}_{\tau_{Lj}} + \hat{c}'_{\tau_{Lj}}\Omega\Delta^{Lq}_{\tau_{Li}}\right)(\mathbf{1}-\alpha) \\ &\quad + \left[\left(\hat{c}'_{\tau_{Li}}\Omega + \hat{c}'\Omega\Delta^{Lq}_{\tau_{Li}}\right)(\mathbf{1}-\alpha)\right]\left[\left(\hat{c}'_{\tau_{Lj}}\Omega + \hat{c}'\Omega\Delta^{Lq}_{\tau_{Lj}}\right)(\mathbf{1}-\alpha)\right] \text{ for } i \neq j. \end{aligned}$$

Then substituting the various expressions, we get

$$\begin{aligned} N^q_{\tau_{Li}\tau_{Lj}}(\mathbf{0}) &= -(\varepsilon_\sigma - \varepsilon_\chi)(2\varepsilon_\sigma - \varepsilon_\chi - 1)\sum_k \beta_k E^i_k E^j_k + 2\varepsilon_\sigma(\varepsilon_\sigma - \varepsilon_\chi)\sum_k \beta_k E^i_k E^j_k + \varepsilon_\chi^2\left(\sum_k \beta_k E^i_k\right)\left(\sum_k \beta_k E^j_k\right) \\ &= (\varepsilon_\chi + 1)(\varepsilon_\sigma - \varepsilon_\chi)s_{ij}(1-\alpha_i)(1-\alpha_j) + \varepsilon_\chi^2 s_i s_j (1-\alpha_i)(1-\alpha_j) \text{for } i \neq j. \end{aligned}$$

Then summing the derivatives of $N^p$ and $N^q$, we get that

$$\frac{d^2}{d\tau_{Li}\tau_{Lj}}\log Y(\mathbf{0}) = \left((\varepsilon_\sigma - \varepsilon_\chi)s_{ij} + \varepsilon_\chi s_i s_j\right)(1-\alpha_i)(1-\alpha_j) \text{for } i \neq j.$$

This proves the first statement of the lemma.

11. Similarly, we can derive the second expression of Lemma B.4 $\frac{d}{d\tau_{Li}\tau_{Li}}\log Y(\mathbf{0})$.

$$\begin{aligned} N^p_{\tau_{Li}\tau_{Li}} &= -\varepsilon_\chi(\varepsilon_\chi - 1)(1-\alpha_i)^2 s_i^2 + \varepsilon_\chi(\varepsilon_\chi - \varepsilon_\sigma)s_{ii}(1-\alpha_i)^2 + \varepsilon_\sigma\varepsilon_\chi(1-\alpha_i)s_i, \\ N^q_{\tau_{Li}\tau_{Li}}(\mathbf{0}) &= -\left(\hat{c}'_{\tau_{Li}\tau_{Li}}\Omega + 2\hat{c}'_{\tau_{Li}}\Omega\Delta^{Lq}_{\tau_{Li}} + \hat{c}'\Omega\Delta^{Lq}_{\tau^L_i\tau^L_i}\right)(\mathbf{1}-\alpha) + \left[\left(\hat{c}'_{\tau_{Li}}\Omega + \hat{c}'\Omega\Delta^{Lq}_{\tau_{Li}}\right)(\mathbf{1}-\alpha)\right]^2, \\ &= (\varepsilon_\sigma - \varepsilon_\chi)\varepsilon_\sigma\sum_k \beta_k E^i_k - (\varepsilon_\sigma - \varepsilon_\chi)(2\varepsilon_\sigma - \varepsilon_\chi - 1)\sum_k \beta_k E^i_k E^i_k + 2(\varepsilon_\sigma - \varepsilon_\chi)\varepsilon_\sigma\sum_k \beta_k E^i_k E^i_k, \\ &\quad -\varepsilon_\sigma(\varepsilon_\sigma + 1)\sum_k \beta_k E^i_k + \varepsilon_\chi^2 s_i^2(1-\alpha_i)^2, \\ &= (\varepsilon_\sigma - \varepsilon_\chi)(\varepsilon_\chi + 1)s_{ii}(1-\alpha_i)^2 - \varepsilon_\sigma(\varepsilon_\chi + 1)s_i(1-\alpha_i) + \varepsilon_\chi^2 s_i^2(1-\alpha_i)^2. \end{aligned}$$

Then summing the derivatives of $N^p$ and $N^q$, we get that

$$\frac{d^2}{d\tau_{Li}\tau_{Li}}\log Y(\mathbf{0}) = \left((\varepsilon_\sigma - \varepsilon_\chi)s_{ii} + \varepsilon_\chi s_i^2\right)(1-\alpha_i)^2 - \varepsilon_\sigma s_i(1-\alpha_i).$$

This proves the second statement of the lemma.

12. Find $\frac{d^2}{d\tau_{Xi}d\tau_{Lj}}\log Y(\mathbf{0})$.

In the same way as Step 10, we see that

$$N^p_{\tau_{Xi}\tau_{Lj}}(\mathbf{0}) = \left[-\varepsilon_\chi(\varepsilon_\chi - 1)s_i s_j + \varepsilon_\chi(\varepsilon_\chi - \varepsilon_\sigma)s_{ij} + \varepsilon_\chi(\varepsilon_\sigma - 1)s_i \sum_s v_{is}\omega_{sj}\right] \alpha_i(1-\alpha_j).$$

Taking into account the zero terms,

$$\begin{aligned}
N^q_{\tau_{Xi}\tau_{Lj}}(\mathbf{0}) &= -\left[\hat{c}'_{\tau_{Xi}\tau_{Lj}}\Omega + \hat{c}'_{\tau_{Xi}}\Omega\Delta^{Lq}_{\tau_{Lj}} + \hat{c}'_{\tau_{Lj}}\Omega^q_{\tau_{Xi}} + \hat{c}'\Omega^q_{\tau_{Xi}}\Delta^{Lq}_{\tau_{Lj}}\right](\mathbf{1}-\alpha) \\
&\quad + \left[\hat{c}'_{\tau_{Xi}}\Omega + \hat{c}'\Omega^q_{\tau_{Xi}}\right](\mathbf{1}-\alpha)\left[\hat{c}'_{\tau_{Lj}}\Omega + \hat{c}'\Omega\Delta^{Lq}_{\tau_{Lj}}\right](\mathbf{1}-\alpha) \\
&= -(\varepsilon_\sigma - \varepsilon_\chi)(1-\varepsilon_\sigma)(\beta \circ F^{ij})'\mathbf{1} - (\varepsilon_\sigma - \varepsilon_\chi)(2\varepsilon_\sigma - \varepsilon_\chi - 1)(\beta \circ C^i \circ E^j)'\mathbf{1} \\
&\quad + \varepsilon_\sigma(\varepsilon_\sigma - \varepsilon_\chi)(\beta \circ C^i)'E^j + \varepsilon_\sigma(\varepsilon_\sigma - \varepsilon_\chi)(\beta \circ E^j)'C^i - \varepsilon_\sigma^2\beta' F^{ij} \\
&\quad + [(\varepsilon_\sigma - \varepsilon_\chi)(\beta \circ C^i)'\mathbf{1} - \varepsilon_\sigma\beta' C^i][(\varepsilon_\sigma - \varepsilon_\chi)(\beta \circ E^j)'\mathbf{1} - \varepsilon_\sigma\beta' E^j] \\
&= \left[(\varepsilon_\chi - \varepsilon_\sigma - \varepsilon_\sigma\varepsilon_\chi)s_i\left(\sum_s v_{is}\omega_{sj}\right) + \varepsilon_\chi^2 s_i s_j + (\varepsilon_\sigma - \varepsilon_\chi)(\varepsilon_\chi + 1)s_{ij}\right]\alpha_i(1-\alpha_j).
\end{aligned}$$

Then summing the derivatives of $N^p$ and $N^q$, we get that

$$\frac{d^2}{d\tau_{Xi}\tau_{Lj}}\log Y(\mathbf{0}) = \left[-\varepsilon_\sigma s_i\left(\sum_s v_{is}\omega_{sj}\right) + \varepsilon_\chi s_i s_j + (\varepsilon_\sigma - \varepsilon_\chi)s_{ij}\right]\alpha_i(1-\alpha_j).$$

This proves the third statement of the lemma.

13. Find $\frac{d^2}{d\tau_{Xi}\tau_{Xj}}\log Y(\mathbf{0})$ for $i \neq j$. In the same way as Step 10, we see that

$$N^p_{\tau_{Xi}\tau_{Xj}}(\mathbf{0}) = \left[-\varepsilon_\chi(\varepsilon_\chi - 1)s_i s_j + \varepsilon_\chi(\varepsilon_\chi - \varepsilon_\sigma)s_{ij} + \varepsilon_\chi(\varepsilon_\sigma - 1)\left(s_i\sum_s v_{is}\omega_{sj} + s_i\sum_s v_{is}\omega_{sj}\right)\right]\alpha_i\alpha_j.$$

Taking into account the zero terms,

$$\begin{aligned}
N^q_{\tau_{Xi}\tau_{Xj}}(\mathbf{0}) &= -\left[\hat{c}'_{\tau_{Xi}\tau_{Xj}}\Omega + \hat{c}'\Omega^q_{\tau_{Xi}\tau_{Xj}} + \hat{c}'_{\tau_{Xi}}\Omega^q_{\tau_{Xj}} + \hat{c}'_{\tau_{Xj}}\Omega^q_{\tau_{Xi}}\right](\mathbf{1}-\alpha) \\
&\quad + \left(\hat{c}'_{\tau_{Xi}}\Omega + \hat{c}'\Omega^q_{\tau_{Xi}}\right)(\mathbf{1}-\alpha)\left(\hat{c}'_{\tau_{Xj}}\Omega + \hat{c}'\Omega^q_{\tau_{Xj}}\right)(\mathbf{1}-\alpha) \\
&= -(\varepsilon_\sigma - \varepsilon_\chi)(1-\varepsilon_\sigma)(\beta \circ (D^{ij} + D^{ji}))'\mathbf{1} - (\varepsilon_\sigma - \varepsilon_\chi)(2\varepsilon_\sigma - \varepsilon_\chi - 1)(\beta \circ C^i \circ C^j)'\mathbf{1} \\
&\quad - \varepsilon_\sigma^2\beta'(D^{ij} + D^{ji}) + \varepsilon_\sigma(\varepsilon_\sigma - \varepsilon_\chi)[(\beta \circ C^i)'C^j + (\beta \circ C^j)'C^i] \\
&\quad + \left((\varepsilon_\sigma - \varepsilon_\chi)(\beta \circ C^i)'\mathbf{1} - \varepsilon_\sigma\beta' C^i\right)\left((\varepsilon_\sigma - \varepsilon_\chi)(\beta \circ C^j)'\mathbf{1} - \varepsilon_\sigma\beta' C^j\right) \\
&= \left[(\varepsilon_\chi - \varepsilon_\chi\varepsilon_\sigma - \varepsilon_\sigma)\left(s_i\sum_s v_{is}\omega_{sj} + s_j\sum_s v_{js}\omega_{si}\right) + (\varepsilon_\sigma - \varepsilon_\chi)(\varepsilon_\chi + 1)s_{ij} + \varepsilon_\chi^2 s_i s_j\right]\alpha_i\alpha_j.
\end{aligned}$$

Then summing the derivatives of $N^p$ and $N^q$, we get that

$$\frac{d^2}{d\tau_{Xi}\tau_{Xj}}\log Y(\mathbf{0}) = \left[(\varepsilon_\sigma - \varepsilon_\chi)s_{ij} + \varepsilon_\chi s_i s_j - \varepsilon_\sigma\left(s_i\sum_s v_{is}\omega_{sj} + s_j\sum_s v_{js}\omega_{si}\right)\right]\alpha_i\alpha_j.$$

This proves the fourth statement of the lemma.

14. Find $\frac{d^2}{d\tau_{Xi}d\tau_{Xi}} \log Y(\mathbf{0})$. In the same way as Step 13, we see that

$$N^p_{\tau_{Xi}\tau_{Xi}}(\mathbf{0}) \quad = \quad \left[-\varepsilon_\chi(\varepsilon_\chi - 1)s_i^2 + \varepsilon_\chi(\varepsilon_\chi - \varepsilon_\sigma)s_{ii} + 2\varepsilon_\chi(\varepsilon_\sigma - 1)s_i \sum_s v_{is}\omega_{si}\right] \alpha_i^2 + \varepsilon_\chi\varepsilon_\sigma s_i \alpha_i.$$

Taking into account the zero terms,

$$\begin{aligned}
N^q_{\tau_{Xi}\tau_{Xi}}(\mathbf{0}) &= -\left[\hat{c}'_{\tau_{Xi}\tau_{Xi}}\Omega + \hat{c}'\Omega^q_{\tau_{Xi}\tau_{Xi}} + 2\hat{c}'_{\tau_{Xi}}\Omega^q_{\tau_{Xi}}\right](\mathbf{1}-\alpha) + \left[\left(\hat{c}'_{\tau_{Xi}}\Omega + \hat{c}'\Omega_{q,\tau_{Xi}}\right)(\mathbf{1}-\alpha)\right]^2 \\
&= (\varepsilon_\sigma - \varepsilon_\chi)\varepsilon_\sigma(\beta \circ C^i)'\mathbf{1} - 2(\varepsilon_\sigma - \varepsilon_\chi)(1-\varepsilon_\sigma)(\beta \circ D^{ii})'\mathbf{1} - (\varepsilon_\sigma - \varepsilon_\chi)(2\varepsilon_\sigma - \varepsilon_\chi - 1)(\beta \circ C^i \circ C^i)'\mathbf{1} \\
&\quad -\varepsilon_\sigma(\varepsilon_\sigma + 1)\beta' C^i - 2\varepsilon_\sigma^2 \beta' D^{ii} \\
&\quad +2\varepsilon_\sigma(\varepsilon_\sigma - \varepsilon_\chi)(\beta \circ C^i)'C^i + \left((\varepsilon_\sigma - \varepsilon_\chi)(\beta \circ C^i)'\mathbf{1} - \varepsilon_\sigma \beta' C^i\right)^2 \\
&= \left[2(\varepsilon_\chi - \varepsilon_\chi\varepsilon_\sigma - \varepsilon_\sigma)s_i \sum_s v_{is}\omega_{si} + (\varepsilon_\sigma - \varepsilon_\chi)(\varepsilon_\chi + 1)s_{ii} + \varepsilon_\chi^2 s_i^2\right]\alpha_i^2 - \varepsilon_\sigma(\varepsilon_\chi + 1)s_i\alpha_i.
\end{aligned}$$

Then summing the derivatives of $N^p$ and $N^q$, we get that

$$\frac{d^2}{d\tau_{Xi}\tau_{Xi}} \log Y(\mathbf{0}) = \left[(\varepsilon_\sigma - \varepsilon_\chi)s_{ii} + \varepsilon_\chi s_i^2 - 2\varepsilon_\sigma s_i \sum_s v_{is}\omega_{si}\right]\alpha_i^2 - \varepsilon_\sigma s_i \alpha_i.$$

This concludes the proof. ■

**Lemma B.5** *Suppose* $\max\{\alpha_i\} = \bar{\alpha} < 1$. *Let $z$ be some $n \times 1$ vector such that $\mathbf{0} \le z \le \mathbf{1}$. Let $a = \Omega z$ and $c = \Omega\Omega z$. Then,* $2c_i - a_i - a_i^2 \ge 0$.

**Proof of Lemma B.5.** Let $\bar{V} = \tilde{\alpha}V$. Using $\Omega = (I + \bar{V} + \bar{V}^2 + .....)$, we can write

$$\begin{aligned}
a &= \left(\sum_{\ell=0}^{\infty} \bar{V}^\ell\right) z \\
a_i &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{n} \bar{V}^\ell_{ik} z_k = \sum_{\ell=0}^{\infty} \hat{a}_i^\ell,
\end{aligned}$$

where $\hat{a}_i^\ell \equiv \sum_{k=1}^n \bar{V}^\ell_{ik} z_k$.

$$\begin{aligned}
c &= \left(\sum_{\ell=0}^{\infty} \bar{V}^\ell\right)^2 z = \left(\sum_{\ell=1}^{\infty} \ell \bar{V}^{\ell-1}\right) z \\
c_i &= \sum_{\ell=1}^{\infty} \ell \hat{a}_i^{\ell-1}
\end{aligned}$$

We then have

$$
\begin{aligned}
2c_i - a_i - a_i^2 &= 2\sum_{\ell=1}^{\infty} \ell \hat{a}_i^{\ell-1} - \sum_{\ell=0}^{\infty} \hat{a}_i^{\ell} - \left(\sum_{\ell=0}^{\infty} \hat{a}_i^{\ell}\right)^2 \\
&= 2\sum_{\ell=0}^{\infty} (\ell+1)\hat{a}_i^{\ell} - \sum_{\ell=0}^{\infty} \hat{a}_i^{\ell} - \sum_{\ell=0}^{\infty} (\hat{a}_i^{\ell})^2 - 2\sum_{\ell=1}^{\infty}\sum_{s=0}^{\ell-1} \hat{a}_i^{\ell} a_i^{s} \\
&= 2\sum_{\ell=1}^{\infty} \ell \hat{a}_i^{\ell} + \sum_{\ell=0}^{\infty} \hat{a}_i^{\ell} - \sum_{\ell=0}^{\infty} (\hat{a}_i^{\ell})^2 - 2\sum_{\ell=1}^{\infty}\sum_{s=0}^{\ell-1} \hat{a}_i^{\ell} \hat{a}_i^{s}
\end{aligned}
$$

Given the equation above, to establish the inequality it will be sufficient to show that $0 \leq \hat{a}_i^{\ell} < 1$, if $\ell \geq 1$. We will show by induction that $\mathbf{0} \leq \hat{a}^{\ell} \leq \mathbf{1}\bar{\alpha}^{\ell} < \mathbf{1}$. From the definition, $\hat{a}_i^0 = z_i \in [0, \bar{\alpha}^0]$. Then $\hat{a}^{\ell} = \bar{V}\hat{a}^{\ell-1} \leq \bar{V}\mathbf{1}\bar{\alpha}^{\ell-1} \leq \bar{\alpha}^{\ell}\mathbf{1}$. (We used the fact that $\bar{V}\mathbf{1} \leq \alpha$.) Showing that $\hat{a}^{\ell} \geq \mathbf{0}$ is similar. ■

**Lemma B.6** *Suppose that the hypothesis of lemma B.5 holds, $z = \alpha$, all sectors are nontrivial and there are positive input-output linkages in the economy. Then for some $i$, $\beta_i[2c_i - a_i - a_i^2] > 0$.*

**Proof of Lemma B.6.** We will use the same construction as the proof of lemma B.5. Let $i$ be such that $\alpha_i > 0$. Since $\hat{a}_i^0 = \alpha_i \in (0,1)$, $\hat{a}_i^0 - (\hat{a}_i^0)^2 > 0$, so $2c_i - a_i - a_i^2 > 0$. Then if $\beta_i > 0$, we are done.

On the other hand, suppose that $\beta_i = 0$. Since $s_i > 0$, there is some $k \neq i$ such that $\omega_{ki} > 0$ and $\beta_k > 0$. If $\alpha_k = 0$, then $\omega_{ki} = 0$ for all $i \neq k$, which is a contradiction. Therefore $\alpha_k > 0$. Now $k$ is such that $\beta_k > 0, \alpha_k > 0$. Then by the same logic as above, $\beta_k[2c_k - a_k - a_k^2] > 0$. ■

**Lemma B.7** *Suppose that the hypothesis of lemma B.5 holds, $\mathbf{0} < z \leq \mathbf{1}$, all sectors are nontrivial and there are positive input-output linkages in the economy. Then for some $i$, $\beta_i[2c_i - a_i - a_i^2] > 0$.*

**Proof of Lemma B.7.** Using the same construction as the proof of lemma B.5, it is sufficient to show that for some $i$ it holds that $\beta_i > 0$ and $\hat{a}_i^{\ell} > 0$, $\ell \geq 1$.

Since there are positive sectoral linkages, $\bar{V}_{ij} > 0$ for some $i, j$. Note that $\hat{a}_i^1 \geq \bar{V}_{ij} z_j > 0$, so if $\beta_i > 0$, we are done. Suppose not. Since $s_i > 0$, there exists some $k \neq i$ with $\beta_k > 0$ and $\omega_{ki} > 0$. We have that $\omega_{ki} = \sum_{\ell=1}^{\infty} \bar{V}_{ki}^{\ell}$, so $\bar{V}_{ki}^{\ell} > 0$ for some $\ell$. Then now $\hat{a}_k^{\ell} \geq \bar{V}_{ki}^{\ell} z_i > 0$, so we are done. ■

**Lemma B.8** *The square symmetric matrix $(s_{ij} r_i r_j - s_i s_j r_i r_j)_{i,j}$, where $r_i \neq 0$ are arbitrary, is positive semi-definite.*

**Proof of Lemma B.8.** Let $\xi^i$ be a random variable that takes the values $\omega_{ki}r_i$, $k = 1, \dots n$ with probabilities $\beta_k$. Then $s_{ij}r_ir_j = E\{\xi^i\xi^j\}$, $s_ir_i = E\xi^i$ and $s_{ij}r_ir_j - s_is_jr_ir_j = E\left\{(\xi^i - E\xi^i)(\xi^j - E\xi^j)\right\} = cov(\xi^i, \xi^j)$. Let $d_{ij} = (s_{ij} - s_is_j)r_ir_j$ and $D = (d_{ij})$. Then $D$ is a variance-covariance matrix and hence positive semi-definite. ■

**Lemma B.9** *For any vector* $\mathbf{r}$, $\sum_i s_i(1-\alpha_i)r_i^2 - \sum_i \sum_j s_{ij}(1-\alpha_i)(1-\alpha_j)r_ir_j \geq 0$.

**Proof of Lemma B.9.**

$$\begin{aligned}\sum_i s_i(1-\alpha_i)r_i^2 - \sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j)r_ir_j &= \sum_k \beta_k \left[\sum_i \omega_{ki}(1-\alpha_i)r_i - \sum_i\sum_j \omega_{ki}(1-\alpha_i)\omega_{kj}(1-\alpha_j)r_ir_j\right] \\ &= \sum_k \beta_k \left[\sum_i \omega_{ki}(1-\alpha_i)r_i^2 - \left(\sum_i \omega_{ki}(1-\alpha_i)r_i\right)^2\right] \geq 0.\end{aligned}$$

The last inequality follows from the fact that $\sum_i \omega_{ki}(1-\alpha_i) = 1$, so this is a sum of variances. ■

## B.3 Proofs of main results

**Proof of Proposition 5.** From lemma B.4 and the fact that $Y = 1$ when there are no distortions, a Taylor expansion around the non distorted allocation gives

$$\log TFP \approx \frac{1}{2}\varepsilon_\chi \left(\sum_i s_i(1-\alpha_i)\tau_{Li}\right)^2 + \frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi)\sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j)\tau_{Li}\tau_{jL} - \frac{1}{2}\varepsilon_\sigma \sum_i s_i(1-\alpha_i)\tau_{Li}^2.$$

All the first-order terms are zero. Then, rearranging terms yields

$$\begin{aligned}\log TFP &\approx -\frac{1}{2}\varepsilon_\chi \left[\sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j)\tau_{Li}\tau_{Lj} - \left(\sum_i s_i(1-\alpha_i)\tau_{Li}\right)^2\right] \\ &\quad -\frac{1}{2}\varepsilon_\sigma \left[\sum_i s_i(1-\alpha_i)\tau_{Li}^2 - \sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j)\tau_{Li}\tau_{Lj}\right].\end{aligned}$$

Alternatively, we can rewrite the Proposition as

$$\log TFP \approx -\frac{1}{2}\varepsilon_\chi \,\mathrm{var}_{\beta_i}\left(\sum_j \omega_{ij}(1-\alpha_j)\tau_j\right) - \frac{1}{2}\varepsilon_\sigma \sum_i \beta_i\big(\mathrm{var}_{\omega_{ij}(1-\alpha_j)}(\tau_j)\big),$$

where we used $\sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j)\tau_{Li}\tau_{Lj} = \sum_k \beta_k \left(\sum_i \omega_{ki}(1-\alpha_i)\tau_{Li}\right)^2$, $\sum_i s_i(1-\alpha_i)\tau_{Li} = \sum_k\sum_i \beta_k\omega_{ki}(1-\alpha_i)\tau_{Li}$ and $\sum_i s_i(1-\alpha_i)\tau_{Li}^2 = \sum_k\sum_i \beta_k\omega_{ki}(1-\alpha_i)\tau_{Li}^2$.

■

**Proof of Proposition 6.** Since $\tau_{Xi} = \tau_{Li} = \tau_i$, we have that for any $i, j$ (including $i = j$)

$$\frac{d^2}{d\tau_i d\tau_j} \log TFP = \frac{d^2}{d\tau_{Xi} d\tau_{Xj}} \log TFP + \frac{d^2}{d\tau_{Xi} d\tau_{Lj}} \log TFP + \frac{d^2}{d\tau_{Li} d\tau_{Xj}} \log TFP + \frac{d^2}{d\tau_{Li} d\tau_{Lj}} \log TFP.$$

Then applying lemma B.4 and grouping terms, we obtain for $i \neq j$:

$$\begin{aligned} \frac{d^2}{d\tau_i d\tau_j} \log TFP(\mathbf{0}) &= (\varepsilon_\sigma - \varepsilon_\chi) s_{ij} + \varepsilon_\chi s_i s_j - \varepsilon_\sigma [s_i \alpha_i g_{ij} + s_j \alpha_j g_{ji}] \\ &= (\varepsilon_\sigma - \varepsilon_\chi) s_{ij} + \varepsilon_\chi s_i s_j - \varepsilon_\sigma [s_i x_{ij} + s_j x_{ji}], \end{aligned}$$

where we used lemma B.3 in the second line.

Similarly, if $i = j$

$$\begin{aligned} \frac{d^2}{d\tau_i d\tau_i} \log TFP(\mathbf{0}) &= (\varepsilon_\sigma - \varepsilon_\chi) s_{ij} + \varepsilon_\chi s_i s_j - 2\varepsilon_\sigma s_i \alpha_i g_{ii} - \varepsilon_\sigma s_i \\ &= (\varepsilon_\sigma - \varepsilon_\chi) s_{ij} + \varepsilon_\chi s_i s_j - 2\varepsilon_\sigma s_i x_{ii} - \varepsilon_\sigma s_i. \end{aligned}$$

Then Taylor's expansion around $\tau_i = 0, \forall i$ implies

$$\log TFP \approx \frac{1}{2}\varepsilon_\chi \left(\sum_i s_i \tau_i\right)^2 + \frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi) \sum_i \sum_j s_{ij} \tau_i \tau_j - \frac{1}{2}\varepsilon_\sigma \sum_i s_i \tau_i^2 + 2\varepsilon_\sigma \sum_i \sum_j s_i x_{ij} \tau_i \tau_j.$$

Then noting that $\sum_j x_{ij}\tau_j = x_i \sum_j (x_{ij}/x_j)\tau_j = x_i \bar{\tau}^i_{\text{suppliers}}$ and rearranging, we obtain

$$\begin{aligned} \log TFP \approx -\frac{1}{2}\varepsilon_\chi & \left[\sum_i \sum_j s_{ij}\tau_i\tau_j - \left(\sum_i s_i \tau_i\right)^2\right] \\ & - \frac{1}{2}\varepsilon_\sigma \left[\sum_i s_i \tau_i^2 - \sum_i \sum_j s_{ij}\tau_i\tau_j + 2\sum_i s_i x_i \tau_i \bar{\tau}^i_{\text{suppliers}}\right]. \end{aligned}$$

Alternatively, we can express TFP as

$$\begin{aligned} \log TFP \approx -\frac{1}{2}\varepsilon_\chi & \left[\text{var}_\beta(\tau_i + x_i \tau^i_{supplier})\right] \\ & - \frac{1}{2}\varepsilon_\sigma \left[\sum_i \gamma_i (\tau_i + x_i \tau^i_{supplier})^2 - \sum_i s_i (x_i \tau^i_{supplier})^2\right], \end{aligned}$$

where we used

$$\sum_i \sum_j s_{ij}\tau_i\tau_j = \sum_i \sum_j \sum_k \beta_k \omega_{ki}\omega_{kj}\tau_i\tau_j = \sum_k \beta_k \left(\sum_i \omega_{ki}\tau_i\right)^2 = \sum_k \beta_k \left(\tau_k + x_k \tau^k_{supplier}\right)^2,$$

$$\sum_i \beta_i \left(\tau_i + x_i \tau^i_{supplier}\right)^2 = \sum_i \beta_i \left(\tau_i^2 + 2\tau_i x_i \tau^i_{supplier} + (x_i \tau^i_{supplier})^2\right),$$

and

$$\sum_i s_i\tau_i^2 + 2\sum_i s_i\tau_i x_i\tau^i_{supplier} - \sum_i\sum_j s_{ij}\tau_i\tau_j = \sum_i \gamma_i\tau_i^2 + 2\sum_i \gamma_i\tau_i x_i\tau^i_{supplier} - \sum_i \beta_i(x_i\tau^i_{supplier})^2$$
$$= \sum_i \gamma_i(\tau_i + x_i\tau^i_{supplier})^2 - \sum_i(\beta_i+\gamma_i)(x_i\tau^i_{supplier})^2.$$

■

**Proof of Result 1.**

Propositions 5 and 6 imply that we can represent the Hessian (in both cases) as

$$\frac{d^2}{d\tau d\tau}\log TFP(\mathbf{0}) = -\varepsilon_\chi M - \varepsilon_\sigma N,$$

where $M$ and $N$ are some matrices that are functions of the undistorted allocation, but not the elasticities $\varepsilon_\chi, \varepsilon_\sigma$. Proving the first claim is equivalent to showing that $M$ and $N$ are positive semi-definite.

Suppose that $N$ is not positive semi-definite, so for some vector $\tau$ we have $\tau' N\tau < 0$. Then for some $\varepsilon_\sigma$ large enough, $\frac{1}{2}\tau'\frac{d^2}{d\tau d\tau}\log TFP(\mathbf{0})\tau = -\frac{1}{2}\varepsilon_\chi\tau' M\tau - \frac{1}{2}\varepsilon_\sigma\tau' N\tau > 0$. Since $\frac{d\log TFP}{d\tau} = \mathbf{0}$, from Taylor's theorem we know that

$$\log TFP(z\tau) = -z^2\frac{1}{2}\tau'\frac{d^2\log TFP}{d\tau d\tau}(\mathbf{0} + \xi(z)z\tau)\tau,$$

where $z$ is a scalar and $\xi(z) \in [0,1]$. Since $\tau'\frac{d^2\log TFP}{d\tau d\tau}(\mathbf{0})\tau < 0$ continuity implies that for $z$ small enough $\tau'\frac{d^2\log TFP}{d\tau d\tau}(\mathbf{0}+\xi(z)z\tau)\tau < 0$ and hence $\log TFP(z\tau) > 0$, which is a contradiction. Hence $N$ is positive semi-definite.

The proof that $M$ is positive semi-definite is identical.

■

**Proof of Result 2, part (i).**

From proposition 1 it follows that $d_i = 1 - \alpha_i, M_{ij} = \alpha_i v_{ij}$ and $c_i = \beta_i$. Then applying the expression for TFP we get that $TFP = [\beta'(I - \tilde{\alpha}V)^{-1}(\mathbf{1} - \alpha)]^{-1} = (\beta'\mathbf{1})^{-1} = 1$, where we used lemma B.2. ■

**Proof of result 2, part (ii).** Consider a function $f(\varepsilon_\sigma, \varepsilon_\chi, t)$, where $t$ is a scalar. Suppose that $f(\varepsilon_\chi, \varepsilon_\sigma, 0) = 0, f_t(\varepsilon_\chi, \varepsilon_\sigma, 0) = 0, f_{tt}(\varepsilon_\chi, \varepsilon_\sigma, 0) = -M\varepsilon_\chi - N\varepsilon_\sigma$, where $M \geq 0, N > 0$. Then for any $t$,

$$f(\varepsilon_{\sigma 2}, \varepsilon_\chi, t) - f(\varepsilon_{\sigma 1}, \varepsilon_\chi, t) =$$
$$\frac{1}{2}t^2[f_{tt}(\varepsilon_\chi, \varepsilon_{\sigma 2}, 0 + \xi_2(t)t) - f_{tt}(\varepsilon_\chi, \varepsilon_{\sigma 2}, 0 + \xi_1(t)t)] =$$
$$\frac{1}{2}t^2\left\{\underbrace{[(f_{tt}(\varepsilon_\chi, \varepsilon_{\sigma 2}, 0 + \xi_2(t)t) + \varepsilon_\chi M + \varepsilon_{\sigma 2}N]}_{A} + \underbrace{[-f_{tt}(\varepsilon_\chi, \varepsilon_{\sigma 1}, 0 + \xi_2(t)t) - \varepsilon_\chi M - \varepsilon_{\sigma 1}N]}_{B}\underbrace{-[\varepsilon_{\sigma 2} - \varepsilon_{\sigma 1}]N}_{C}\right\},$$

where $\xi_1(t) \in (0,1), \xi_2(t) \in (0,1)$. We have that $C < 0$ and is independent of $t$ and $A$ and $B$ converge to 0 as $t$ converges to 0 due to continuity. Therefore for all $t$ such that $|t| \leq \delta$, $f\varepsilon_{\sigma 2}, \varepsilon_\chi, t) < f(\varepsilon_{\sigma 1}, \varepsilon_\chi, t)$. Then to prove the statements it is sufficient to show that $\frac{d}{d\varepsilon_\sigma}\left(\frac{d}{d\tau^2}\right)\log TFP(\mathbf{0}) < 0$ and is independent of $\varepsilon_\sigma$.

The strict inequality in the case of random distortions follows directly from Proposition A.4.

Consider uniform distortions. Since uniform labor distortions have no effect on TFP (proposition A.5), without loss of generality we can consider uniform distortions on intermediates: $\tau_{Xij} = \bar{\tau}$. With the same kind of derivation as in Proposition 6, we can write

$$\frac{d}{d\varepsilon_\sigma}\left(\frac{d}{d\tau^2}\right)\log TFP(\mathbf{0}) \quad = \quad -\left[2\sum_i\sum_j s_i\omega_{ij}\alpha_j - \sum_i \alpha_i s_i - \sum_i\sum_j \alpha_i\alpha_j s_{ij}\right].$$

To show that $\frac{d}{d\varepsilon_\sigma}\left(\frac{d}{d\tau^2}\right)\log TFP(\mathbf{0}) < 0$, let us rewrite it as

$$\begin{aligned}
\frac{d}{d\varepsilon_\sigma}\left(\frac{d}{d\tau^2}\right)\log TFP(\mathbf{0}) \quad &= \quad -\left[2\sum_k \beta_k \sum_i \omega_{ki}\sum_j \omega_{ij}\alpha_j - \sum_k \beta_k \sum_i \omega_{ki}\alpha_i - \sum_k \beta_k \sum_i \omega_{ki}\alpha_i \sum_j \omega_{kj}\alpha_j\right] \\
&= \quad -\left[\sum_k \beta_k\left(2c_k - a_k - a_k^2\right)\right],
\end{aligned}$$

with $a_k = \sum_j \omega_{kj}\alpha_j$ and $c_k = \sum_i \omega_{ki}a_i$, that is $a = \Omega\alpha$ and $c = \Omega\Omega\alpha$.
Using Lemma B.7, we then have $\beta_k[2c_k - a_k - a_k^2] \geq 0$, with strict inequality for at least one $k$, which implies $\frac{d}{d\varepsilon_\sigma}\left(\frac{d}{d\tau^2}\right)\log TFP(\mathbf{0}) > 0$. ■

**Proof of Result 2, part (iii).** We prove the second result by showing 3 claims.

1. Suppose that distortions are sector-specific and $\tau_i \geq 0$. Define $TL = (I-M)^{-1}d$, which is the vector of inverse sectoral TFP and $Cost_i = \frac{\lambda_i/\eta}{1+\tau_i}$, which is the marginal cost in sector $i$ relative to the wage. Then, $TL_i \leq Cost_i$ for all $i$.

   **Proof**. Since $(I-A)^{-1} = I + A + A^2 + \cdots$ whenever the inverse exists, we have:

   $$TL = d + Md + M^2d + \ldots. \tag{B.9}$$

   Proposition 1 implies that

   $$\lambda/\eta = \text{diag}(\mathbf{1}+\tau)[d + M(\lambda/\eta)],$$

   so

   $$\lambda/\eta = \tilde{d} + \tilde{M}\tilde{d} + \tilde{M}^2\tilde{d} + \cdots,$$

   where $\tilde{d} = \text{diag}(\mathbf{1}+\tau)d$ and $\tilde{M} = \text{diag}(\mathbf{1}+\tau)M$. Then we have:

   $$Cost_i = d + M(\lambda/\eta) = d + (M + M\tilde{M} + M\tilde{M}^2 + \ldots)\tilde{d}. \tag{B.10}$$

Since $M \geq 0$ and $d \geq 0$ elementwise and $\tau_i \geq 0$ for all $i$, we have that $d \leq \tilde{d}$ and $M \leq \tilde{M}$ (element by element). Let $a^{(k)} \equiv M^{k-1}d$ and $b^{(k)} = M\tilde{M}^{k-2}\tilde{d}$ be the k-th terms ($n \times 1$ vectors) in the infinite sums (B.9) and (B.10) respectively. Since $\mathbf{0} \leq a^{(k)} \leq b^{(k)}$ elementary calculus implies that $TL \leq Cost_i$.

2. Suppose that distortions are purchaser-specific with $\tau_i > 0$ for all $i$. Then $\lambda_i \geq (1+\tau_i)$ for all $i$. Also, $\lim_{\varepsilon_\sigma \to \infty} \lambda_i/\eta = 1 + \tau_i$.

   **Proof**. Claim 1 implies that $\lambda_i/\eta \geq (1+\tau_i)TL_i$. The undistorted equilibrium minimizes $TL$, so $TL_i \geq 1$, hence $\lambda_i/\eta \geq (1+\tau_i)$.

   Now we prove the second statement. Using equation (10), we have for all $\varepsilon_\sigma > 1$

$$\begin{aligned}\frac{\lambda_i}{\eta} &\leq (1+\tau_i)\left[1-\alpha_i+\alpha_i\bar{\lambda}^{1-\varepsilon_\sigma}\right]^{\frac{1}{1-\varepsilon_\sigma}} \\ &\leq (1+\tau_i)\left[1-\alpha_i\right]^{\frac{1}{1-\varepsilon_\sigma}},\end{aligned}$$

   where $\hat{\lambda} = \max\{\lambda_i/\eta\}$ and the inequality follows from the fact that $\hat{\lambda} \geq 0$.

   From the first statement $1+\tau_i \leq \frac{\lambda_i}{\eta}$. So since $\lim_{\varepsilon_\sigma \to \infty}(1+\tau_i) = \lim_{\varepsilon_\sigma \to \infty}(1+\tau_i)\left[1-\bar{\alpha}\right]^{\frac{1}{1-\varepsilon_\sigma}} = 1+\tau_i$, it follows that $\lim_{\varepsilon_\sigma \to \infty} \lambda_i/\eta = 1+\tau_i$.

3. Let $\varepsilon_\chi^{(k)} \to \infty$ as $k \to \infty$. Let $\lambda^{(k)}/\eta^{(k)}$ be an arbitrary sequence of strictly positive $n \times 1$ vectors. Then $lim_{k\to\infty}\sum_{i=1}^n c_i^{(k)} = 1$, where $c_i^{(k)}$ are given by Proposition 1 for $\varepsilon_\chi^{(k)}, \lambda^{(k)}/\eta^{(k)}$.

   **Proof** Define

$$g(\varepsilon_\chi) = \min\left(\sum_i \beta_i^{\frac{1}{\varepsilon_\chi}} r_i^{\frac{\varepsilon_\chi-1}{\varepsilon_\chi}}\right)^{\frac{\varepsilon_\chi}{\varepsilon_\chi-1}}$$

$$\text{subject to } r_i \in [0,1], \sum_i r_i = 1.$$

   The objective function is strictly concave and the constraint set is convex. Then by Corollary 32.3.2. in Rockafellar (1970) (page 344), the minimum is attained at an extreme point, which in our case is a vertex with $r_i = 1$ for some $i$ and $r_j = 0$ for all $j \neq i$. So

$$g(\varepsilon_\chi) = \min\{\beta_1^{\frac{1}{\varepsilon_\chi-1}}, \beta_2^{\frac{1}{\varepsilon_\chi-1}} \dots, \beta_n^{\frac{1}{\varepsilon_\chi-1}}\}$$

   and for $\varepsilon_\chi > 1$, $g(\varepsilon_\chi) = (\min \beta_i)^{\frac{1}{\varepsilon_\chi-1}}$. Then clearly $\lim_{\varepsilon_\chi \to \infty} g(\varepsilon_\chi) = 1$.

   Then for any $r \in \mathbf{R}_+^n$,

$$\left(\sum_i \beta_i^{\frac{1}{\varepsilon_\chi}} r_i^{\frac{\varepsilon_\chi-1}{\varepsilon_\chi}}\right)^{\frac{\varepsilon_\chi}{\varepsilon_\chi-1}} \geq \left(\sum_i r_i\right) g(\varepsilon_\chi).$$

   So,

$$1 = \left(\sum_i \beta_i^{\frac{1}{\varepsilon_\chi}} c_i^{\frac{\varepsilon_\chi-1}{\varepsilon_\chi}}\right)^{\frac{\varepsilon_\chi}{\varepsilon_\chi-1}} \geq \left(\sum_i c_i\right) g(\varepsilon_\chi).$$

So, $1 \leq \sum_i c_i \leq (\min_i \beta_i)^{\frac{1}{1-\varepsilon_\chi}}$. Then taking limits and using the fact that $\varepsilon_\chi^{(k)} \to \infty$ establishes the claim.

Now with claims above we prove the result. Define $TL = (I - M)^{-1} d$. From Claim 1,

$$TL_i \leq \frac{\lambda_i/\eta}{1+\tau_i}.$$

Also $1 \leq TL_i$. Then using claim 2 and taking limits, we obtain that

$$\lim_{\varepsilon_\sigma \to \infty} TL_i = 1.$$

Then Proposition 1 implies that $TFP = (\sum_i c_i TL_i)^{-1}$. Claim 3 shows that $\sum_i c_i \to 1$ as $\varepsilon_\chi \to \infty$ and moreover this convergence is independent of $\varepsilon_\sigma, \varepsilon_\rho$. So using the two limits above and continuity establishes the result. ■

**Proof of Result 3.** Consider the frictionless economy with $\varepsilon_\sigma = \varepsilon_\rho$, and $A = \mathbf{1}$. In that economy, aggregate productivity is

$$\log TFP = \frac{\varepsilon_\chi}{\varepsilon_\chi - 1} \log \left( \sum_{i=1}^{n} \beta_i \hat{\lambda}_i^{\frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1}} \right) - \log(\hat{c}' \Omega \hat{B} (\mathbf{1} - \alpha))$$

with

$$\hat{\lambda} = \Omega \hat{B} (\mathbf{1} - \alpha).$$

Using the definition of $\hat{c}$, we get that $\hat{c}' \Omega \hat{B}(\mathbf{1} - \alpha) = \hat{c}' \hat{\lambda} = \sum_i \beta_i \hat{\lambda}_i^{\frac{1}{1-\varepsilon_\sigma}(\varepsilon_\sigma - \varepsilon_\chi)} \hat{\lambda}_i = \sum_i \beta_i \hat{\lambda}_i^{\frac{1-\varepsilon_\chi}{1-\varepsilon_\sigma}}$. Therefore,

$$\log TFP = \frac{1}{\varepsilon_\chi - 1} \log \left( \sum_{i=1}^{n} \beta_i \hat{\lambda}_i^{\frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1}} \right),$$

which implies that $\log TFP$ is increasing in $\hat{\lambda}_i^{\frac{1}{\varepsilon_\sigma - 1}}$. We have:

$$\hat{\lambda}_k^{\frac{1}{\varepsilon_\sigma - 1}} = \left( \sum_j (1 - \alpha_j) \omega_{kj} B_j^{\varepsilon_\sigma - 1} \right)^{\frac{1}{\varepsilon_\sigma - 1}}.$$

Since $\sum_j (1 - \alpha_j) \omega_{kj} = 1$, the expression above is a power mean with parameter $\varepsilon_\sigma - 1$. Since power means are increasing in the exponent, it follows that $\hat{\lambda}_i^{\frac{1}{\varepsilon_\sigma - 1}}$ is increasing in $\varepsilon_\sigma$ and hence $Y$ is increasing in $\varepsilon_\sigma$.

We have already established that

$$TFP = \left( \sum_{i=1}^{n} \beta_i \hat{\lambda}_i^{\frac{\varepsilon_\chi - 1}{\varepsilon_\sigma - 1}} \right)^{\frac{1}{\varepsilon_\chi - 1}},$$

which is another power mean of $\hat{\lambda}_i^{\frac{1}{\varepsilon_\sigma - 1}}$, which by the same argument is increasing in $\varepsilon_\chi$.

Finally, we prove the last statement of the result. Since $s_k > 0$, there exists $m$ such that $\beta_m\omega_{mk} > 0$. This implies that $\beta_m\omega_{mj_1} > 0, \beta_m\omega_{mj_2} > 0$. From the properties of generalized means, $\hat{\lambda_m}^{\frac{1}{\varepsilon_\sigma - 1}}$ is strictly increasing in $\varepsilon_\sigma$ and hence $Y$ is also strictly increasing in $\varepsilon_\sigma$. ■

**Proof of Result 4.**

Let $\widehat{}$ denote second order approximation around $\mathbf{0}$. Then Proposition 6 implies that there exists $\bar{\varepsilon}_\sigma$ such that for all $\varepsilon_\sigma < \bar{\varepsilon}_\sigma$, $\widehat{\log TFP} > \log \widehat{TFP}|_{\alpha=0}$ if and only if

$$-\frac{1}{2}\left[\sum_i\sum_j s_{ij}\tau_i\tau_j - \left(\sum_i s_i\tau_i\right)^2\right] > -\frac{1}{2}\left[\sum_i \beta_i\tau_i^2 - \left(\sum_i \beta_i\tau_i\right)^2\right].$$

The term on the right-hand side is simply $-\frac{1}{2}\operatorname{var}_\beta \tau$. The definitions imply that

$$\begin{aligned}\sum_i\sum_j s_{ij}\tau_i\tau_j &= \sum_i\sum_j\sum_k \beta_k\omega_{ki}\omega_{kj}\tau_j\tau_i = \sum_k \beta_k(\sum_i \omega_{ki}\tau_i)^2 \\ \sum_i s_i\tau_i &= \sum_i\sum_k \beta_k\omega_{ki}\tau_i = \sum_k \beta_k\left(\sum_i \omega_{ki}\tau_i\right).\end{aligned}$$

Then the definitions of $x_i$, $\bar{\tau}_{suppliers}$ and the fact that $\omega_{kk} = 1 + x_{kk}$ imply the inequality above can be written as:

$$-\frac{1}{2}\left[\sum_i \beta_i\left(\tau_i + x_i\bar{\tau}^i_{suppliers}\right)^2 - \left(\sum_i \beta_i\left(\tau_i + x_i\bar{\tau}^i_{suppliers}\right)\right)^2\right] > -\frac{1}{2}\operatorname{var}_\beta(\tau),$$

which proves the condition $Cov_\beta(\tau_i, x_i\bar{\tau}_i^{suppliers}) < -(1/2)Var_\beta(x_i\bar{\tau}_i^{suppliers})$. Finally, since the Hessian is continuous, if the vector $\tau$ is not too large, $\widehat{\log TFP} > \log \widehat{TFP}|_{\alpha=0}$ implies that $\log TFP > \log TFP|_{\alpha=0}$. ■

**Proof of Result 5.** Since $\frac{d\log TFP}{d\tau_i}(\mathbf{0}) = 0$ and

$$\frac{d^2}{d\tau_i d\tau_i}\log TFP(\mathbf{0}) = (\varepsilon_\sigma - \varepsilon_\chi)s_{ii} + \varepsilon_\chi s_i^2 - 2\varepsilon_\sigma s_i x_{ii} - \varepsilon_\sigma s_i,$$

(this expression is derived in the proof of Proposition 6), Taylor approximation of $\frac{d\log TFP}{d\tau_i}$ around $\mathbf{0}$ implies the result. ■

## B.4 Proofs of Additional Propositions

**Proof of Proposition A.1.** Define $\lambda_i' = \lambda_i(1 + \tau_{Ci})$. Then the conditions can be rewritten as:

$$\frac{dY}{dC_i} = \lambda_i'$$

$$\lambda_i' \frac{dQ_i}{dX_{ij}} = \lambda_j' \frac{1+\tau_{Ci}}{1+\tau_{Cj}}(1+\tau_{Xij})$$

$$\lambda_i' \frac{dQ_i}{dL_i} = \eta(1+\tau_{Li})(1+\tau_{Ci}).$$

Define $\tau_{Xij}' = \frac{1+\tau_{Ci}}{1+\tau_{Cj}}(1+\tau_{Xij}) - 1$ and $\tau_{Li}' = (1+\tau_{Li})(1+\tau_{Ci}) - 1$. Then $\lambda', \tau_{Xij}', \tau_{Li}'$ and the original allocation satisfy satisfies (6)- (8).

Next, we prove the second statement of the proposition. Define $\lambda_i$ by (6). Define

$$1+\tau_{Xij} = \frac{dQ_i}{dX_{ij}} \frac{dY/dC_i}{dY/dC_j}$$

and

$$\frac{1+\tau_{Li}}{1+\tau_{1L}} = \frac{dY/dC_i \frac{dQ_i}{dL_i}}{dY/dC_1 \frac{dQ_1}{dL_1}}.$$

We can set $\tau_{i1}$ at an arbitrary value greater than $-1$. ■

**Proof of Proposition A.2.** With $\varepsilon_\rho = \varepsilon_\sigma = 1$, we get $m_{ij} = \alpha_i v_{ij}\lambda_i/((1+\tau_{Xij})\lambda_j)$ and $d_i = (1-\alpha_i)\lambda_i/[\eta(1+\tau_{Li})]$. Substituting $X_{ij} = m_{ij}Q_i, L_i = d_iQ_i$ in the production function:

$$Q_i = A_i(1-\alpha_i)^{-(1-\alpha_i)}(\alpha_i)^{-\alpha_i}(B_iL_i)^{1-\alpha_i}\left(\prod_{j=1}^{n} v_{ij}^{-v_{ij}} X_{ij}^{v_{ij}}\right)^{\alpha_i},$$

we obtain

$$\log(\lambda/\eta) = -(I-\tilde{\alpha})\log B - \log A + [(I-\tilde{\alpha})\Delta^{Lp} + \tilde{\alpha}(V \circ \Delta^{Xp})]\mathbf{1} + \tilde{\alpha}V\log(\lambda/\eta),$$

which proves (A.1).

We can use the expressions derived in the proof of proposition 4 (the case $\varepsilon_\rho = \varepsilon_\sigma$) and substitute $\varepsilon_\sigma = 1, \varepsilon_\chi = 1$. Equation (B.8) becomes

$$\hat{q}' = \beta'(I - \tilde{\alpha}(V \circ \Delta^{Xq}))^{-1},$$

where $\hat{q}_i = \lambda_i Q_i/Y$. Using the expression for $d_i$ in the resource constraint on labor written as $\hat{q}' dY = \bar{L}$, gives $(\mathbf{1}-\alpha)'\Delta^{Lq}\hat{q}\eta^{-1}Y = \bar{L}$. Then substituting for the expression of $\hat{q}$, we get

$$\log Y = \log\eta - \log(\beta'(I - \tilde{\alpha}(V \circ \Delta^{Xq}))^{-1}\Delta^{Lq}(\mathbf{1}-\alpha) + \log\bar{L}.$$

Since $\eta = -\beta'\log(\lambda/\eta)$, this yields the expression of output stated in the proposition. ■

**Proof of Proposition A.3.** It is immediate that in this economy the only solution to the system of equations (10) in Proposition 1 is $\mathbf{1}$. Then part (iii) of this proposition implies that $\lambda = \mathbf{1}, \eta = 1$. Plugging in we obtain: $d = \mathbf{1} - \alpha$, $c = \beta$, $M_{ij} = \alpha_i v_{ij}$. Then $[I - M']^{-1} = \Omega'$. So $TFP = [(\mathbf{1}-\alpha)'\Omega'\beta]^{-1} = [\mathbf{1}'\beta]^{-1} = 1$, where we used Lemma B.2.

Next, since $Q = [I - M']^{-1}cY = \Omega'\beta L$, it is immediate that $Q_i = \sum_k \beta_k \omega_{ki} L = s_i L$, which implies the last result. The other results are immediate. ■

**Proof of Proposition A.4.** Approximation (14) can be expressed as:

$$\log TFP \approx \frac{1}{2}\varepsilon_\chi \left(\sum_i s_i\tau_i\right)^2 - \varepsilon_\sigma \sum_i\sum_j s_i\omega_{ij}\tau_i\tau_j + \frac{1}{2}\varepsilon_\sigma \sum_i s_i\tau_i^2 - \frac{1}{2}(\varepsilon_\chi - \varepsilon_\sigma)\sum_i\sum_j s_{ij}\tau_i\tau_j.$$

Then through the law of large numbers we obtain:

$$\begin{aligned}\log TFP \approx\; & \frac{1}{2}\varepsilon_\chi \left(\sum_i s_i\right)^2 \bar{\tau}^2 + \frac{1}{2}\varepsilon_\chi \left(\sum_i s_i^2\right)\left(E(\tau_i^2) - \bar{\tau}^2\right) - \varepsilon_\sigma \left(\sum_i\sum_j s_i\omega_{ij}\right)\bar{\tau}^2 - \varepsilon_\sigma \left(\sum_i s_i\omega_{ii}\right)\left(E(\tau_i^2) - \bar{\tau}^2\right) \\ & + \frac{1}{2}\varepsilon_\sigma \left(\sum_i s_i\right)\left(E(\tau_i^2) - \bar{\tau}^2\right) + \frac{1}{2}\varepsilon_\sigma \left(\sum_i s_i\right)\bar{\tau}^2 \\ & + \frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi)\left(\sum_i\sum_j s_{ij}\right)\bar{\tau}^2 + \frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi)\left(\sum_i s_{ii}\right)\left(E(\tau_i^2) - \bar{\tau}^2\right).\end{aligned}$$

Rearranging, we get

$$\begin{aligned}\log TFP \approx\; & -\frac{1}{2}\varepsilon_\chi \left[\sum_i\sum_j s_{ij} - \sum_i\sum_j s_i s_j\right]\bar{\tau}^2 - \frac{1}{2}\varepsilon_\chi \left[\sum_i s_{ii} - \sum_i s_i^2\right]\text{var}(\tau) \\ & -\frac{1}{2}\varepsilon_\sigma \left[2\sum_i\sum_j s_i\omega_{ij} - \sum_i\sum_j s_{ij} - \sum_i s_i\right]\bar{\tau}^2 - \frac{1}{2}\varepsilon_\sigma \left[2\sum_i s_i\omega_{ii} - \sum_i s_{ii} - \sum_i s_i\right]\text{var}(\tau).\end{aligned}$$

Then this gives us equation (A.2) with the expressions for $\Gamma_\chi, \Xi_\chi, \Gamma_\psi, \Xi_\psi$ clear from the equation above.

Notice that:

$$\begin{aligned}2\Gamma_\sigma &= 2\sum_i\sum_j s_i\omega_{ij} - \sum_i\sum_j s_{ij} - \sum_i s_i \\ &= 2\sum_i\sum_j\sum_k \beta_k\omega_{ki}\omega_{ij} - \sum_i\sum_j\sum_k \beta_k\omega_{ki}\omega_{kj} - \sum_i\sum_k \beta_k\omega_{ki} \\ &= \sum_k \beta_k \left[2\sum_i\sum_j \omega_{ki}\omega_{ij} - \left(\sum_i \omega_{ki}\right)^2 - \sum_i \omega_{ki}\right].\end{aligned}$$

The expression in the square brackets is nonnegative by Lemma B.5 with $z = \mathbf{1}$, so $\Gamma_\sigma \geq 0$. In the same way, when $\alpha_i > 0$ for some $i$, lemma B.7 implies that $\Gamma_\sigma > 0$.

$\Gamma_\chi = \sum_i\sum_j (s_{ij} - s_i s_j)$. Let $d_{ij} = s_{ij} - s_i s_j$ and $D = (d_{ij})$. By lemma B.8 $D$ is positive semi-definite. Then $\Gamma_\chi = \frac{1}{2}\mathbf{1}'D\mathbf{1} \geq 0$.

$$\begin{aligned}
2\Xi_\psi(\alpha, \beta, V) &= \sum_i s_i(\omega_{ii} - 1) + \sum_i \sum_k \beta_k \omega_{ki}\omega_{ii} - \sum_i \sum_k \beta_k \omega_{ki}^2 \\
&= \sum_i s_i(\omega_{ii} - 1) + \sum_i \sum_k \beta_k \omega_{ki}(\omega_{ii} - \omega_{ki}).
\end{aligned}$$

Lemma B.1 states that $\omega_{ii} - 1 \geq 0$ and $\omega_{ii} - \omega_{ki} \geq 0$, which completes the proof that $\Xi_\psi \geq 0$.

Suppose that $\Xi_\psi = 0$. Both terms are nonnegative. Then since the first term must be zero and $s_i > 0$ then $\omega_{ii} = 1$ for all $i$. Since $\omega_{ii} - \omega_{ki} > 0$ for all $k \neq i$, then $\omega_{ki} = 0$ for all $k \neq i$, that is $\Omega = I$. Then lemma B.1 implies that $\alpha_i = 0$ for all $i$. This proves (by contrapositive) that $\chi_1 > 0$ if $\alpha_i > 0$ for some $i$.

Finally, $\Xi_\chi = \sum_i [s_{ii} - s_i^2]$. Lemma B.8 implies that $s_{ii} - s_i^2 \geq 0,\ \forall i$. This proves that $\Xi_\chi \geq 0$. ■

**Proof of Result A.1.** For moderate distortions, we approximate the partial derivatives of $\log TFP$ with their values at the vector of zero distortions. Then the result follow directly from lemma B.4 and from an expression for $\frac{d^2}{d\tau_i d\tau_j} \log TFP(\mathbf{0})$ derived in the proof of Proposition 6. ■

**Proof of Proposition A.5.** Consider the equilibrium variables for economy 1: $\{\lambda_{1i}, \eta_1, Q_{1i}, X_{1ij}, L_{1i}, d_{1i}, c_{1i}, M_1, Y_1, TFP_1\}$. We will show that if we set $\lambda_{2i} = \lambda_{2i}$, $\eta_2 = \eta_1/(1+z)$ and all the other variables identical to economy 1, all the conditions in proposition 1 are satisfied.

First we show that condition (v) is satisfied.

$$\begin{aligned}
\frac{\lambda_{2i}}{\eta_2} &= A_i^{-1}\left[(1-\alpha_i)B_i^{\frac{\sigma}{1-\sigma}}(1+\tau_{2Li})^{1-\varepsilon_\sigma} + \alpha_i\left[\sum_{j=1}^n v_{ij}\left((1+\tau_{X2ij})\lambda_{2j}/\eta_2\right)^{1-\varepsilon_\rho}\right]^{\frac{1}{1-\varepsilon_\rho}1-\varepsilon_\sigma}\right]^{\frac{1}{1-\varepsilon_\sigma}} \\
&= (1+z)A_i^{-1}\left[(1-\alpha_i)B_i^{\frac{\sigma}{1-\sigma}}(1+\tau_{1Li})^{1-\varepsilon_\sigma} + \alpha_i\left[\sum_{j=1}^n v_{ij}\left((1+\tau_{X1ij})\lambda_{1j}/\eta_1\right)^{1-\varepsilon_\rho}\right]^{\frac{1}{1-\varepsilon_\rho}(1-\varepsilon_\sigma)}\right]^{\frac{1}{1-\varepsilon_\sigma}} \\
&= (1+z)\frac{\lambda_{1i}}{\eta_1},
\end{aligned}$$

which proves the claim. Then it is immediate that the constructed $\lambda_2$ and $\eta_2$ satisfy (iv). Plugging in the formulas in parts (i), (ii) and (iii) implies that $d_1 = d_2, M_1 = M_2, c_1 = c_2$ and hence $TFP_1 = TFP_2, Y_1 = Y_2$ and so on. ■

**Proof of Proposition A.6.** We can rewrite the approximation (13)

$$
\begin{aligned}
\log TFP \quad &\approx \quad \frac{1}{2}\varepsilon_\chi \left( n\frac{1}{n}\sum_i s_i(1-\alpha_i)\tau_{Li} \right)^2 \\
&+\frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi) \left( n\frac{1}{n}\sum_i\sum_{j\neq i} s_{ij}(1-\alpha_i)(1-\alpha_j)\tau_{Li}\tau_{jL} + n\frac{1}{n}\sum_i s_{ii}(1-\alpha_i)^2\tau_{Li}^2 \right) \\
&-\frac{1}{2}\varepsilon_\sigma n\frac{1}{n}\sum_i s_i(1-\alpha_i)\tau_{Li}^2.
\end{aligned}
$$

With the assumptions that the distortions are independent of any other sector parameters, and are independently distributed with mean $\bar{\tau}_L$ and cross-sectional variance $V(\tau_{Li})$, and using the Law of large numbers, we have

$$
\begin{aligned}
\log TFP \quad \approx \quad & \frac{1}{2}\varepsilon_\chi \left( (\sum_i s_i(1-\alpha_i))\bar{\tau}_L) \right)^2 + \frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi) \left( \sum_i\sum_{j\neq i} s_{ij}(1-\alpha_i)(1-\alpha_j)\bar{\tau}_L^2 + \sum_i s_{ii}(1-\alpha_i)^2 E(\tau_{Li}^2) \right) \\
&-\frac{1}{2}\varepsilon_\sigma \sum_i s_i(1-\alpha_i)E(\tau_{Li}^2).
\end{aligned}
$$

Which can be rewritten as

$$
\begin{aligned}
\log TFP \quad \approx \quad & \frac{1}{2}\varepsilon_\chi\bar{\tau}_L^2 \left( \sum_i s_i(1-\alpha_i) \right)^2 - \frac{1}{2}\varepsilon_\sigma E(\tau_{Li}^2)\sum_i s_i(1-\alpha_i) \\
&+\frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi) \left( \bar{\tau}_L^2 \sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j) - \bar{\tau}_L^2 \sum_i s_{ii}(1-\alpha_i)^2 + E(\tau_{Li}^2)\sum_i s_{ii}(1-\alpha_i)^2 \right).
\end{aligned}
$$

After re-arranging the terms, we obtain

$$
\begin{aligned}
\log TFP \quad \approx \quad & \frac{1}{2}\varepsilon_\chi\bar{\tau}_L^2 \left( \sum_i s_i(1-\alpha_i) \right)^2 - \frac{1}{2}\varepsilon_\chi \sum_i s_i(1-\alpha_i)E(\tau_{Li}^2) - \frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi)E(\tau_{Li}^2)\sum_i s_i(1-\alpha_i) \\
&+\frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi) \left( \bar{\tau}_L^2 \sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j) - \bar{\tau}_L^2 \sum_i s_{ii}(1-\alpha_i)^2 + E(\tau_{Li}^2)\sum_i s_{ii}(1-\alpha_i)^2 \right).
\end{aligned}
$$

Finally, using $\sum_i s_i(1-\alpha_i) = 1$, and noticing that $\sum_i\sum_j s_{ij}(1-\alpha_i)(1-\alpha_j) = \sum_i s_i(1-\alpha_i) = 1$, we have

$$
\begin{aligned}
\log TFP \quad &\approx \quad -\frac{1}{2}\varepsilon_\chi \operatorname{var}(\tau_{Li}) - \frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi) \left( 1 - \sum_i s_{ii}(1-\alpha_i)^2 \right) \operatorname{var}(\tau_{Li}) \\
&\approx \quad -\frac{1}{2}\varepsilon_\chi \sum_i s_{ii}(1-\alpha_i)^2 \operatorname{var}(\tau_{Li}) - \frac{1}{2}\varepsilon_\sigma \left( 1 - \sum_i s_{ii}(1-\alpha_i)^2 \right) \operatorname{var}(\tau_{Li}).
\end{aligned}
$$

Next, we prove that $\sum_i s_{ii}(1-\alpha_i)^2 > 0$. For every $i$, there exists some $k$ such that $\beta_k\omega_{ki} > 0$, hence $s_{ii} = \sum_k \beta_k \omega_{ki}^2 > 0$ for all $i$. Also, $(1-\alpha_i)^2 \geq (1-\bar{\alpha})^2 > 0$, so $s_{ii}(1-\alpha_i)^2 > 0$ for all $i$.

Next we prove $1 - \sum_i s_{ii}(1-\alpha_i)^2 \geq 0$. To see this, recall that $\sum_i \omega_{ki}(1-\alpha_i) = 1$. Therefore, $\omega_{ki}(1-\alpha_i) \leq 1$ for all $i$, and $\omega_{ki}^2(1-\alpha_i)^2 \leq \omega_{ki}(1-\alpha_i)$ which gives $\sum_i \omega_{ki}^2(1-\alpha_i)^2 \leq 1$, hence $\sum_i s_{ii}(1-\alpha_i)^2 \equiv \sum_k \beta_k \sum_i \omega_{ki}^2(1-\alpha_i)^2 \leq 1$.

Finally, suppose that $\alpha_i v_{ij} > 0$ for some $i \neq j$. Since $i$ is nontrivial, $\omega_{ki} > 0$ for some $k$ with $\beta_k > 0$. This implies that $\omega_{kj} > 0$. Suppose that $k \neq j$. We have that $\omega_{kk} > 1$ and $\omega_{kk}(1-\alpha_k)+\omega_{kj}(1-\alpha_j) \leq 1$, so $\omega_{kj}(1-\alpha_j) < 1$. Also $1-\alpha_j > 1-\bar{\alpha} > 0, \omega_{kj} > 0$, so $\omega_{kj}(1-\alpha_j) \in (0,1)$ and hence $\omega_{kj}^2(1-\alpha_j)^2 < \omega_{kj}(1-\alpha_j)$, so $\sum_i \omega_{ki}^2(1-\alpha_i)^2 < 1$ and by the logic above $1 - \sum_i s_{ii}(1-\alpha_i)^2 > 0$. Now suppose that $k = j$. Then it must be that $\omega_{ki} > 0$ with $\beta_k > 0$ and $k \neq i$. Then the construction above goes through in exactly the same way. ■

**Proof of Result A.2.** (i) If $v_{ii} = 1$, $v_{ij} = 0$ for $i \neq j$, direct inspection shows that $\omega_{ii} = \frac{1}{1-\alpha_i}$ and $\omega_{ij} = 0$ for $i \neq j$. Then $s_i = \beta_i/(1-\alpha_i)$, $s_{ii} = \beta_i/(1-\alpha_i)^2$, $s_{ij} = 0$ if $i \neq j$. So applying Proposition 5 immediately implies that $\log TFP = \log TFP|_{\alpha=0}$.

(ii) Proposition A.6 implies that:

$$\log TFP \approx -\frac{1}{2}\varepsilon_\chi \operatorname{var}(\tau_{Li}) - \frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi)\left(1 - \sum_i s_{ii}(1-\alpha_i)^2\right) \operatorname{var}(\tau_{Li}),$$

with $1 - \sum_i s_{ii}(1-\alpha_i)^2 \geq 0$. With uncorrelated distortions, $\sum_i \beta_i \tau_{Li}^2 - \left(\sum_i \beta_i \tau_{Li}\right)^2 \mathit{plim} \operatorname{var} \tau_L$, which immediately proves the first result. In the proof of Proposition A.6 we establish that if $\alpha_i v_{ij} > 0$ for some $i \neq j$, $1 - \sum_i s_{ii}(1-\alpha_i)^2 > 0$. This completes the proof of this statement.

(iii) Proposition 5 can be rewritten as:

$$\log TFP \approx -\frac{1}{2}\varepsilon_\chi \left[\sum_i s_i(1-\alpha_i)\tau_{Li}^2 - \left(\sum_i s_i(1-\alpha_i)\tau_{Li}\right)^2\right]$$
$$- \frac{1}{2}(\varepsilon_\sigma - \varepsilon_\chi)\left[\sum_i s_i(1-\alpha_i)\tau_{Li}^2 - \sum_i \sum_j s_{ij}(1-\alpha_i)(1-\alpha_j)\tau_{Li}\tau_{Lj}\right].$$

Lemma B.9 shows that $\sum_i s_i(1-\alpha_i)\tau_{Li}^2 - \sum_i \sum_j s_{ij}(1-\alpha_i)(1-\alpha_j)\tau_{Li}\tau_{Lj} \geq 0$. Then if $\varepsilon_\sigma - \varepsilon_\chi \leq 0$, the proposition implies that

$$\log TFP \geq -\frac{1}{2}\varepsilon_\chi \left[\sum_i s_i(1-\alpha_i)\tau_{Li}^2 - \left(\sum_i s_i(1-\alpha_i)\tau_{Li}\right)^2\right],$$

and

$$-\frac{1}{2}\varepsilon_\chi \left[\sum_i s_i(1-\alpha_i)\tau_{Li}^2 - \left(\sum_i s_i(1-\alpha_i)\tau_{Li}\right)^2\right] = -\frac{1}{2}\varepsilon_\chi \left[\sum_i \beta_i \tau_{Li}^2 - \left(\sum_i \beta_i \tau_{Li}\right)^2\right] = \log TFP|_{\alpha=0}.$$

■

**Proof of Result A.3.** Since $\frac{d \log TFP}{d\tau_{Li}}(\mathbf{0}) = 0$ and

$$\frac{d^2}{d\tau_{Li}\tau_{Li}} \log TFP(\mathbf{0}) = \left((\varepsilon_\sigma - \varepsilon_\chi)s_{ii} + \varepsilon_\chi s_i^2\right)(1-\alpha_i)^2 - \varepsilon_\sigma s_i (1-\alpha_i)$$

(lemma B.4), Taylor approximation of $\frac{d \log TFP}{d\tau_{Li}}$ around $\mathbf{0}$ implies the result. ■

**Proof of Result A.4.** The derivation is analogous to the derivation in result A.1. ■

# C Application to markups

This section describes the correspondence between an environment with markups and the model of Section 2 and presents the sources of data and the construction of the variables. The section also provides supplemental tables and figures on the estimated markups and on the parameters of the model, as well as the table of results for the robustness checks and additional exercises summarized in section 4.4.

## C.1 Model with markups

Consider an economy in which firms sell their output at an exogenous markup. The firms sell their output to firms in the different sectors of the economy and to the final-good sector, which is assumed to be perfectly competitive.[1] We do not specify the source of the markups, which could be the result of product differentiation, the market structure, or (the lack of) competition regulation.[2] Given the framework considered, the productivity loss would be the same whatever the sources of the markup.

More precisely, we assume that each firm is committed to selling unlimited quantities at a constant markup over marginal cost. Prices are then

$$p_i = \mu_i mc_i,$$

where $\mu_i$ is the sector-specific markup, and $mc_i$ is the marginal cost of production. Under this assumption, markups are isomorphic to all-input distortions.

First, notice that minimization in the final goods sector implies that (6) holds, with $\lambda_i = p_i$. Next, consider the cost minimization in the various sectors. Given the prices of the different inputs,

[1] The perfect-competition assumption in the final good sector is without loss of generality.

[2] In the case of product differentiation, the markups could depend on the elasticity of substitution across firms; we assume that the markups are exogenous to the elasticity of substitution *between* sectors (but they could depend on the elasticity of substitution *within* sectors).

we have the following optimality condition:

$$\theta_i A_i^\sigma \left(\frac{\alpha_i Q_i}{X_i}\right)^{1-\sigma} \left(\frac{v_{ij} X_i}{X_{ij}}\right)^{1-\rho} = p_j, \tag{C.1}$$

$$\theta_i A_i^\sigma B_i^\sigma \left(\frac{(1-\alpha_i)Q_i}{L_i}\right)^{1-\sigma} = w, \tag{C.2}$$

$$\theta_i \left(A_i \left[(1-\alpha_i)^{1-\sigma}(B_i L_i)^\sigma + \alpha_i^{1-\sigma} X_i^\sigma\right]^{\frac{1}{\sigma}} - 1\right) = 0, \tag{C.3}$$

where $\theta_i$ is the multiplier to the constraint $Q_i \geq 1$. Since $\theta_i = mc_i = p_i/\mu_i = \lambda_i/\mu_i$, the allocation satisfies (7) and (8) with $\tau_{ij} = \tau_{iL} = \mu_i - 1$ and $w = \eta$.

Plugging (C.1) and (C.2) into the complementary slackness condition (C.3) we get

$$p_i = A_i^{-1} \left[(1-\alpha_i) B_i^{\frac{\sigma}{1-\sigma}} w^{-\frac{\sigma}{1-\sigma}} + \alpha_i \left(\sum_j v_{ij} p_j^{-\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}\frac{\sigma}{1-\sigma}}\right]^{-\frac{1-\sigma}{\sigma}} \mu_i. \tag{C.4}$$

Given the expression for $mc_i$, and hence $\lambda_i$, Proposition 1 pins down the equilibrium.

## C.2 Data

Our main source of data is the World Input-Output Database (Timmer et al., 2015). The data are publicly available at: `http://www.wiod.org/home`. The data set gives industry-level data on sales, labor compensation, capital and intermediate input purchases for 40 countries and 35 sectors over the period 1995-2011. We use the 2013 release because 2016 release does not include data on capital. The comprehensive sectoral coverage, the availability of the capital-stock measure at the industry level, the large selection of countries, and the inclusion of several middle-income countries are key advantages of this data set. The countries in the data set are: Australia, Austria, Belgium, Brazil, Bulgaria, Canada, China, Cyprus, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, India Indonesia, Ireland, Italy, Japan, Republic of Korea, Latvia, Lithuania, Luxembourg, Malta, Mexico, Netherlands, Poland, Portugal, Romania, Russia, Slovak Republic, Slovenia, Spain, Sweden, Taiwan, Turkey, United Kingdom, United States. In our quantitative evaluation of the cost of markups, we exclude Latvia, Malta, Cyprus, and Luxembourg. The input-output table of these countries contains too many zeros, which prevents us from solving the model at the same disaggregation level as the other countries. We also exclude Italy because the patterns in the data makes us suspect classification discrepancies relative to other countries.[3]

We restrict our attention to market activities and we therefore conduct our analysis on 30 sectors, after removing the "Public Admin and Defence," "Education," "Health and Social Work," "Other

[3]In Italy, the real-estate sector has an unusually low capital–output ratio and the renting-of-material-and-equipment sector has unusually high capital–output ratio (factor of 0.09 and 21 relative to the industry median).

Community, Social and Personal Services," and "Private Households with Employed Persons"(sectors c31-c35). The 30 sectors are: Agriculture, Hunting, Forestry and Fishing; Mining and Quarrying; Food, Beverages and Tobacco; Textiles and Textile Products; Leather, Leather and Footwear; Wood and Products of Wood and Cork; Pulp, Paper, Paper Products, Printing and Publishing; Coke, Refined Petroleum and Nuclear Fuel; Chemicals and Chemical Products; Rubber and Plastics; Other Non-Metallic Mineral; Basic Metals and Fabricated Metal; Machinery, Nec; Electrical and Optical Equipment; Transport Equipment; Manufacturing Nec, Recycling; Electricity, Gas and Water Supply; Construction; Sale, Maintenance and Repair of Motor Vehicles and Motorcycles, Retail Sale of Fuel; Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles; Retail Trade, Except of Motor Vehicles and Motorcycles, Repair of Household Goods; Hotels and Restaurants; Inland Transport; Water Transport; Air Transport; Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies; Post and Telecommunications; Financial Intermediation; Real Estate Activities; Renting of M&Eq and Other Business Activities. The variables needed for our quantitative exercise are available over the period 1995-2007, for all sectors and all countries, except for China and Indonesia, whose data for the "Sale, maintenance and repair of motor vehicles and motorcycles; retail sale of fuel" sector are missing.[4] We use the average of the wholesale and retail trade sectors' target values for these two countries.

We now describe how the variables used in the calibration are constructed. We compute the intermediate-input shares by dividing the value of intermediate inputs purchased by each sector by the sector's sales net of the value of intermediate inputs imported by the sector.

$$\frac{p_j X_{ij}}{p_i Q_i} = \frac{\mathsf{interm\text{-}input}_{ij}}{\mathsf{sales}_i - \sum_j \mathsf{imported\text{-}interm\text{-}input}_{ij}}$$

This adjustment is done to be consistent with the model in which we abstract from international trade.

We compute final expenditures as follows:

$$p_i C_i = \mathsf{sales}_i - \sum_k \mathsf{interm\text{-}input}_{ki}.$$

Sales ("Output at basic price") and intermediate inputs purchases are available in the National Input-Output Tables (NIOT) of the WIOD. We take the median values of the final expenditure shares and the intermediate-input shares over 1995–2007 as our calibration targets.

For each country-industry pair, we measure the price-cost margin, $\mathsf{pcm}$, as sales minus total costs over sales:

$$\mathsf{pcm} = (\mathsf{sales} - \mathsf{labor\ cost} - \mathsf{intermediate\ input\ cost} - \mathsf{capital\ cost})/\mathsf{sales}.$$

To reduce the influence of outliers, we take the median price-cost margin over the period 1995–2007

[4]The real capital stock variable is not available at all in 2010 and 2011 and not available for many countries in 2008 and 2009.

for each country-industry pair. The price-cost margin is negative in several country-sector pairs. We attribute these negative values to measurement errors and set the lower bound of the pcm to zero.

The sales, labor cost, and intermediate-input cost, are obtained from the Socio-Economic Accounts (SEA) of the WIOD. These variables ("Gross output by industry at current basic prices)", "Labour compensation ", and "Intermediate inputs at current purchasers' prices") are constructed to reflect the sectors' income and costs; the sales are measured at basic prices, the labor cost includes social contribution as well as the compensation of self-employed labor, and the intermediate-input cost includes trade and transportation margins. These variables can therefore be used directly, without any adjustments, to compute the pcm.[5]

We compute capital costs as $(r + \delta)p_I K$, where $r$ is the real interest rate, $\delta$ is the depreciation rate and $K$ is the capital stock ("Real fixed capital stock, 1995 prices") and $p_I$ is the price index for investment goods ("Price levels of gross fixed capital formation, 1995=100").[6] Both the capital stock and the investment price index are obtained from the WIOD SEA data set. Since $p_I$ and $K$ are both benchmarked to the same year, $p_I K$ is the current-year replacement cost of the capital stock, which is the relevant opportunity cost when computing the capital cost.

We set the interest rate to 4%. As explained in the main text, although the productivity cost of markups is sensitive to this assumption, our main conclusion on the role of input substitutability is not. We would reach the same conclusion if we were able to precisely measure the interest rate in each country of our sample.

Because the rate at which different types of asset depreciates varies substantially from one asset to the other, the depreciation rate will vary across sectors and countries depending on the intensity with which the different assets are used. We therefore compute country-sector-specific depreciation rates whenever possible.

We compute the depreciation rate for sector $i$ in country $c$ as follows

$$\delta_i^c = \sum_j \bar{\delta}_{ij}(K_{ij}^c / K_i^c), \tag{C.5}$$

where $\bar{\delta}_{ij}$ is the depreciation rate of asset $j$ used by industry $i$ and $K_{ij}^c$ is the net real capital stock of asset $j$ used in sector $i$ of country $c$, and $K_i^c = \sum_j K_{ij}^c$ is the total net real capital stock of sector $i$ of country $c$.

We use the 2019 release of EUKLEMS data set (Stehrer et al., 2019), available at `https://euklems.eu/download/`, which provides measures of the real capital stock ("capital stock net, volume 2010 ref. prices") by country-sector-asset for 28 European countries, Japan and the United States. The asset types are the following: Computing equipment, Communications equipment, Computer software and databases, Transport Equipment, Other Machinery and Equipment, Total Non-residential investment, Residential structures, Cultivated assets, Research and development, and Other IPP assets. An estimate of the depreciation rate ("EU KLEMS Geometric depreciation rates") is available

[5]For more information on how those variables are constructed, see Erumban et al. (2012).

[6] The general expression proposed by Jorgenson also depends on taxes and on the inflation of investment goods, from which we have abstracted.

by sector-asset (common for all countries). We use the 2010 data. We compute the country-sector depreciation rates with the KLEMS industry classification and then convert them into the WIOD classification using real capital stock as weights whenever we need to aggregate the sectors up to the WIOD level. Both classifications are based on the ISIC 3 classification but regroup industries somewhat differently.

After converting the data into the WIOD classification, we obtain country-sector-specific depreciation rates for 346 country-sector pairs (out of 1,170). For the country-sector observations for which the KLEMS data set does not allow us to compute a depreciation rate, we use the US value, $\delta_i^{US}$, which we obtain directly from the Bureau of Economic Analysis data ("implied rates of depreciation of private nonresidential fixed assets"), available at `https://apps.bea.gov/national/FA2004/Details/Index.htm`. Here as well, we need to convert the data into the WIOD industry classification. To do so, we use the concordance from NAICS 2002 to ISIC 3.1 provided by the U.S. Census, available at `https://www.census.gov/eos/www/naics/concordances/concordances.html`, and we use the value of the net capital stock ("Current-Cost Net Capital Stock of Private Nonresidential Fixed Assets") as weights whenever we need to aggregate the sectors up to the WIOD level. We use the 2000 data for both the depreciation rate and the net capital stock data.

In the end, we have 6 countries for which the KLEMS data give us depreciation rate for all sectors (AUT, CZE, DNK, FIN, GRC, SVK), 9 countries with depreciation rates for some but not all sectors (BEL, ESP, EST, FRA, ITA, LTU, LUX, SVN, SWE), and 12 countries in KLEMS but for which we could not compute a depreciation rate (BGR, DEU, CYP, GBR, HUN, IRL, LVA, MLT, NLD, PRT, POL, ROU); the remaining countries are not in KLEMS.

To see the link between the price-cost margin and the markup, let us write the price-cost margin in terms of the model variables (country subscripts are omitted)

$$\mathsf{pcm}_i = \left(p_i Q_i - w L_i - \sum p_j X_{ij} - (r + \delta_i) p_{Ii} K_i\right) / (p_i Q_i),$$

where $r$ is the real interest rate, and $\delta_i$ and $p_{Ii}$ are the depreciation rate and the investment price index, respectively. Note that although the model described in Section 2 does not explicitly incorporate capital, we do take into account the capital cost when measuring the price-cost margin. Using the first-order conditions (equations [7] and [8] with $\tau_{Li} = \tau_{Xij} = \mu_i - 1$) and the corresponding equation for capital, Euler's theorem implies that $\eta L_i + \sum_j p_j X_{ij} + (r + \delta_i) p_{Ii} K_i = (p_i Q_i)/\mu_i$, and hence

$$\mu_i = \frac{1}{1 - \mathsf{pcm}_i}.$$

## C.3 Markups

Figure C.1 shows the distribution of $\mathsf{pcm}$ across sectors and countries, before setting the lower bound at zero. For each sector, we report the 25th, the median and 75th percentile of $\mathsf{pcm}$. The $\mathsf{pcm}$ is highly dispersed across both countries and sectors, with the lowest 25th percentile at -40% and the highest 75th percentile at 25%. In a few sectors—most notably in real estate, agriculture and transportation—the price-cost margin is negative for a large share of the observations. We attribute these negative values to measurement errors and set the lower bound of the price-cost margin to zero in our quantitative analysis. In a recent work, Hall (2018) also attributes the negative markups he finds in some US sectors (using a different estimation method) to measurement errors. He proposes a method to uncover the true distribution of the markups. However, estimating the parameters of the true distribution of markups would not be sufficient for our purpose since the productivity loss does not depend solely on the distribution of the markups but on the joint distribution between the markups and the sectors' production parameters.

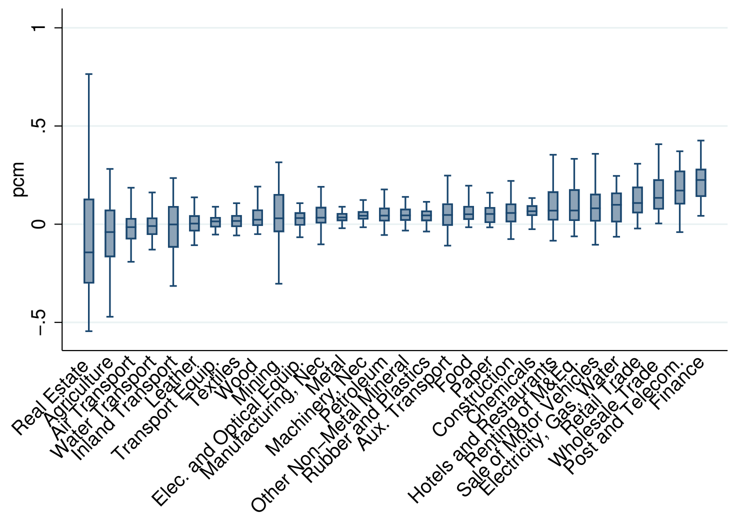
We report, in Figure C.2, each country's median pcm as a function of the country's income per capita, which we compute from the Penn World Table (Feenstra et al., 2015). As shown in the figure, price-cost margins tend to be higher in lower-income than in high-income countries and higher in North America than in Europe.

By construction, the price-cost margin is sensitive to the quality of the measures of sales and costs. One caveat with our capital cost measure is that it relies on the assumption of a common interest rate across countries. With this assumption, one may wonder whether the real interest rate is underestimated in the countries in which the pcms are higher. We find that, in high-pcm countries, the pcms are not particularly high in capital-intensive sectors which suggests that the high pcms are not due to the underestimation of the real interest rate in these countries. On the contrary, the pcm tends to be negative in capital-intensive sectors which suggest that we may have over-estimated the real interest rates (see Figure C.3). Another potential issue is that the quality of the measurement of sales, labor costs and intermediate-inputs costs may vary across countries. In particular, tax evasion and the size of the informal sector could bias the pcm because they lead to incorrect measures of both sales and costs. Tax evasion and informality constitute an issue inasmuch as the sales and costs are misreported to different extents. If firms underreport their sales but report their full costs to reduce their corporate tax or sales tax, the measured pcm will underestimate the actual price-cost margin. Our measure may therefore underestimate the price-cost margin in lower-income countries, where tax enforcement is often weaker. A related issue is the measure of the labor income of self-employed workers. For our purpose, we need to fully deduct all labor costs and therefore need a measure that adjusts for the labor compensation of self-employed workers. The WIOD labor-compensation variable includes such an adjustment.[7] The adjustment brings the labor share in developing countries closer to the levels observed for high-income countries.[8] Insofar as the adjustment does not fully account for the labor income of self-employed workers, the pcm will overestimate the price-cost margin in lower-income countries.

[7]In advanced economies, the adjustment for self-employed labor income is based on a compensation per hour equal to the compensation per hour of employees , and for emerging economies, the compensation of self-employed workers is imputed using additional information to infer the gap between the earnings of self-employed and employed workers.

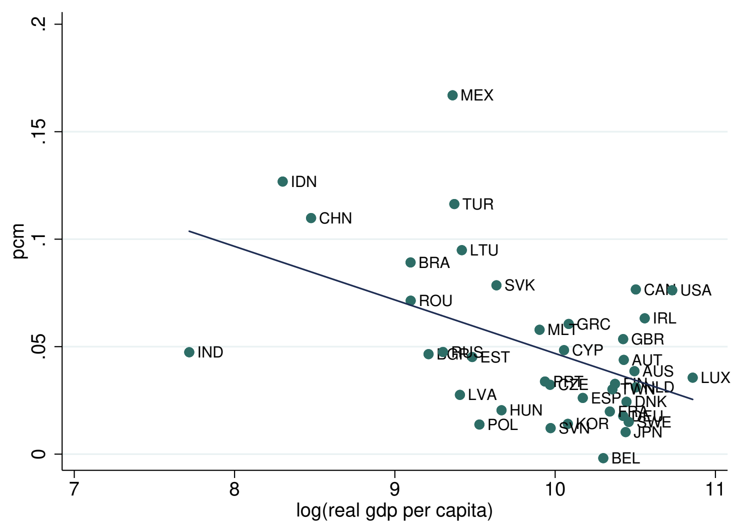
[8]On a related note, Gollin (2002) highlights that adjusting for the labor-income of self-employed workers is important when measuring the labor share (of GDP) in lower-income countries, since a large fraction of the workforce is self-employed in those countries. In Appendix C.7, we report each country's aggregate labor share as well as the aggregate capital stock over total value added.

Figure C.1: The price-cost margin across industries



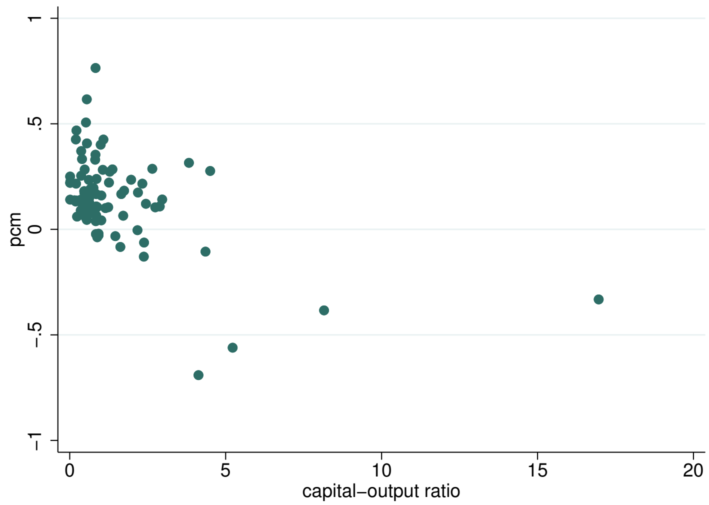

Note: The graph represents the distribution of pcm by industry. The upper and bottom sides of the box give the 25th and 75th percentiles, and the bar in the box represents the median. The industries are ranked from lowest to highest median pcm .

Figure C.2: The price-cost margin across countries




Note: The graph shows the median pcm (across industries) by country as a function of each country's real gdp per capita (1995–2007 mean).

Figure C.3: Capital-output ratio and pcm




Note: The capital-output ratio is computed as the sum the real capital stock times the investment deflator over the nominal value added. This is for year 2000. Each point is a country-industry pair for the following countries: Mexico, Indonesia and Turkey.

To check the validity of our measure, we plot each country's median pcm against the country's product-market-regulation indicator (OECD, 2019), which measures the intensity of barriers to competition, such as state intervention and impediments to trade, investment, and firm entry.[9] As shown in Figure C.4, and in line with intuition, the pcm tends to be higher in countries in which regulations are the least competition friendly (that is, with a high product-market-regulation indicator).

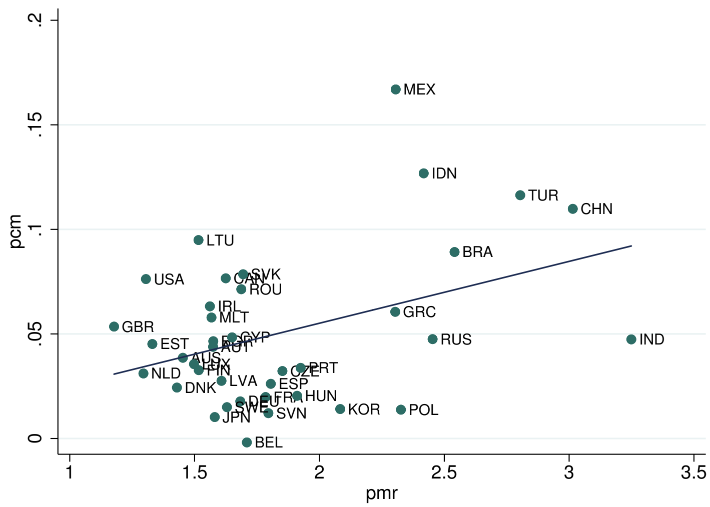
## C.4 Alternative calibration procedure

The inner nest of the production function is considered as a fictitious production sector. Similarly to the real sectors, we choose units in such a way that the price of the fictitious sector is equal to 1. So note that $\tilde{X}_i = \sum_j \tilde{X}_{ij}$, and $\tilde{X}_{ij}/\tilde{X}_i$ is the cost share of sector $j$ in the intermediate goods bundle. Using Klump et al. (2012) approach we have:

$$X_i = \tilde{X}_i \left( \sum_j \frac{\tilde{X}_{ij}}{\tilde{X}_i} \left( \frac{X_{ij}}{\tilde{X}_{ij}} \right)^{1-\frac{1}{\varepsilon_\rho}} \right)^{\frac{\varepsilon_\rho}{\varepsilon_\rho - 1}} = \left( \sum_j \left( \frac{\tilde{X}_{ij}}{\tilde{X}_i} \right)^{\frac{1}{\varepsilon_\rho}} X_{ij}^{1-\frac{1}{\varepsilon_\rho}} \right)^{\frac{\varepsilon_\rho}{\varepsilon_\rho - 1}},$$

[9]The data are publicly available at https://doi.org/10.1787/pmr-data-en. For each country, we use the mean of the overall product market regulation indicator over available years (1998, 2003, 2008 and/or 2013). Data are available for all the countries of the WIOD sample, except Taiwan.

Figure C.4: The price-cost margin and product market regulation across countries




Note: The graph shows the median `pcm` (across industries) by country as a function of the product market regulation indicator, `pmr`. A higher `pmr` captures a less competition-friendly environment.

which implies $v_{ij} = \tilde{X}_{ij}/\sum_j \tilde{X}_{ij}$.

In a similar fashion, looking at the outer loop, the total expenditure on inputs is $Q_i/\mu_i$ (since the price is 1 and there is a markup $\mu_i$). So the labor and intermediate bundle cost shares are $\mu_i\tilde{L}_i/\tilde{Q}_i$ and $\mu_i\tilde{X}_i/\tilde{Q}_i$ respectively. Then, we have

$$Q_i = \tilde{Q}_i\left(\frac{\mu_i\tilde{L}_i}{\tilde{Q}_i}\left(\frac{L_i}{\tilde{L}_i}\right)^{1-\frac{1}{\varepsilon_\sigma}} + \frac{\mu_i\tilde{X}_i}{\tilde{Q}_i}\left(\frac{X_i}{\tilde{X}_i}\right)^{1-\frac{1}{\varepsilon_\sigma}}\right)^{\frac{\varepsilon_\sigma}{\varepsilon_\sigma-1}} = \mu_i\left(\left(\frac{\mu_i\tilde{L}_i}{\tilde{Q}_i}\right)^{\frac{1}{\varepsilon_\sigma}} L_i^{1-\frac{1}{\varepsilon_\sigma}} + \left(\frac{\mu_i\tilde{X}_i}{\tilde{Q}_i}\right)^{\frac{1}{\varepsilon_\sigma}} X_i^{1-\frac{1}{\varepsilon_\sigma}}\right)^{\frac{\varepsilon_\sigma}{\varepsilon_\sigma-1}}.$$

This expression implies $\alpha_i = \mu_i\tilde{X}_i/Q_i = \mu_i\sum_j \tilde{X}_{ij}/\tilde{Q}_i$ and $A_i = \mu_i$. Also the choice to measure labor $L_i$ in terms of observed payroll, implicitly pins down $B_i = 1$ for all $i$. If labor is measured in terms of number of employees, the labor cost share is $\mu_i w\tilde{L}_i/\tilde{Q}_i$. Redoing the derivation will imply $B_i = w$. The parameterization given by this calibration is equivalent but harder to interpret than the one shown in the main text.

## C.5 Comparison with Jones (2011)

We give here additional details on how we replicate Jones's (2011) analysis. We specify and calibrate the model exactly as in Jones (2011). The productivity and distortions are jointly log-normally distributed: $a_i \equiv \ln A_i$ and $\omega_i \equiv \ln(1+\tau_i)$, with $a_i \backsim N(\bar{a}, \nu_a^2)$, $\omega_i \backsim N(\bar{\omega}, \nu_\omega^2)$, and $cov(\omega_i, a_i) = \nu_{a\omega}$. In the baseline scenario, the poor country has, relative to the rich country, a lower mean sectoral productivity, a higher variance of sectoral productivity, a higher mean of distortions, and a higher variance of distortions. In addition, high productivity firms face higher distortions in the poor country, whereas there is no correlation between distortion and productivity in the rich country.

The parameter values used by Jones (and in our replication) are reported in Table C.1. We simulate 1000 rich economies and 1000 poor economies, each with 100 sectors. To simulate our poor and rich economies, where productivities and distortions are potentially correlated, we first draw $\omega_i$, and then set $a_i = a_0 + \gamma\omega_i + \varepsilon_i^a$ with $\gamma = \nu_{a\omega}/\nu_\omega^2$, $\varepsilon_i^a \backsim N(0, \nu_\varepsilon^2)$, $\sigma_\varepsilon^2 = \nu_a^2 - \gamma^2\nu_\omega^2$, and $a_0 = \bar{a} - \gamma\bar{\omega}$.

We compute the TFP of each model economy using Proposition 1 and we take the mean of TFP across all the rich economies and the mean of TFP across all the poor economies, which we denote $TFP^{\text{rich}}$ and $TFP^{\text{poor}}$. We then compute the amplification factor, defined like Jones (2011) as the relative TFP ($TFP^{\text{rich}}/TFP^{\text{poor}}$) with sectoral linkages divided by the relative TFP without sectoral linkages.[10] We then consider how varying the intermediate-input elasticity between the (almost) Leontief ($\varepsilon_\rho = 1/(1+100) \approx 0.01$) and Cobb-Douglas ($\varepsilon_\rho = 1$) values affect the relative TFP and the amplification factor.

We report the results in Table C.2. In this baseline scenario in which the rich and the poor countries differ in both their productivity and their distortions, the TFP gap and the amplification factor increase when the intermediate-input elasticity is lower. This result, which is in line with Jones's (2011) finding, indicates that complementarity' amplification of low sectoral productivity more than offsets its mitigating impact on the effects of distortions in the baseline scenario considered in that paper.

Next, we consider a scenario that isolates the effects of distortions. In this scenario, we assume that the mean and the variance of sectoral productivities ($\bar{a}$, $\nu_a^2$) are identical in the two countries. In the identical TPFs scenario, complementarity mitigates the effects of distortions: the TFP ratio and the amplification factor are lower with Leontief than with Cobb-Douglas. This result is in line with our theoretical and quantitative analyses as well as with the results reported in Jones (2011) for that same scenario.[11] These two scenarios illustrate the fact that the mitigating effect of complementarity on distortions can fully be seen when distortions are studied in isolation.

Note that in Jones (2011) the two countries differ also in their physical and human capital, two dimensions that are absent from our framework. The TFP ratios of our replication exercise are therefore not directly comparable to the income ratios reported in Jones (2011). To complete the comparison with Jones (2011)'s approach, we computed the TFP ratio implied by Jones' solution. This amounts to shutting down the capital accumulation channel and the human capital disparities in Jones's Proposition 5 (that is, setting $\alpha = 0$ and $\bar{h}^{rich} = 1$). We report the TFP ratio we calculated as well as the income ratios in Table C.3. Note that the TFP ratios of Table C.2 are close but not identical to that of Table C.3. The discrepancy between the two tables most likely comes from

[10] In section4.3, we defined the amplification factor differently as $[TFP^{\text{no distortions}}/TFP^{\text{with distortions}} - 1]/[(TFP^{\text{no distortions}}/TFP^{\text{with distortions}})|_{\alpha=0} - 1)$. To facilitate comparison with Jones we adopt his definition in this section and define the amplification factor as $[TFP^{\text{rich}}/TFP^{\text{poor}}]/[(TFP^{\text{rich}}/TFP^{\text{poor}})|_{\alpha=0}]$.

[11] The "Identical TFPs" scenario is reported in Table 1 and 2 in Jones (2011). He finds an amplification factor (of the income ratio) lower with Leontief (1.1) than with Cobb-Douglas (1.4).

Table C.1: Jones (2011)'s baseline calibration

| Common parameters | | Specific parameters | | |
|---|---|---|---|---|
| | | | poor | rich |
| $v_{ij}$ | $1/n$ | $\nu_\omega$ | 0.68 | 0.45 |
| $\beta_i$ | $1/n$ | $\bar{\tau}$ | $1/0.8-1$ | 0 |
| $\varepsilon_\chi$ | 3 | $\bar{\omega}$ | $\ln(1+\bar{\tau})-0.5\nu_\omega^2$ | $\ln(1+\bar{\tau})-0.5\nu_\omega^2$ |
| $\varepsilon_\sigma$ | 1 | $\nu_a$ | 1.23 | 0.84 |
| $\alpha$ | 0.5 | $\bar{a}$ | $\ln(0.604)-0.5\nu_a^2$ | $-0.5\nu_a^2$ |
| | | $\nu_{a\omega}$ | $0.647\nu_a\nu_\omega$ | 0 |

Note: Parameters used in Jones's baseline scenario. See Jones (2011), section V.B and Table 1 Notes section.

Table C.2: Jones (2011)'s exercise in our framework

| | Leontief $\varepsilon_\rho = 0.01$ | Baseline $\varepsilon_\rho = 0.5$ | Cobb-Douglas $\varepsilon_\rho = 1$ |
|---|---|---|---|
| 1. Baseline | | | |
| TFP ratio - No sectoral linkages | 1.7 | 1.7 | 1.7 |
| TFP ratio - With sectoral linkages | 7.1 | 5.9 | 4.7 |
| Amplif. factor | 4.3 | 3.6 | 2.9 |
| 2. Identical TFPs | | | |
| TFP ratio - No sectoral linkages | 1.5 | 1.5 | 1.5 |
| TFP ratio - With sectoral linkages | 1.5 | 1.6 | 1.7 |
| Amplif. factor | 1.0 | 1.1 | 1.2 |

Notes: TFP ratio obtained from the numerical solution of Proposition 1 under Jones's parametrization. In the identical TFPs scenario, the mean and variance of sectoral productivities in the poor economy are equal to their values in the rich economy. Note that the amplification factor follows here Jones's (2011) definition and differs from that reported in Table 1 and 2. See explanations in the text.

differences in the solution methods: the results of Table C.2 are based on the numerical solution (averages of simulations of 1000 model economies of n=100 sectors) whereas Table C.3's results are obtained from Jones's analytical formula.

Table C.3: Income and TFP ratio in Jones (2011)

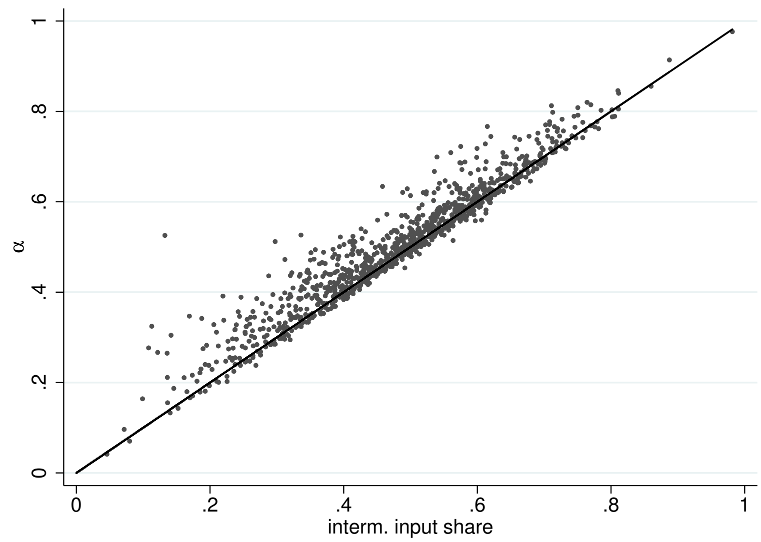
| | Leontief $\varepsilon_\rho = 0.01$ | Baseline $\varepsilon_\rho = 0.5$ | Cobb-Douglas $\varepsilon_\rho = 1$ | Leontief $\varepsilon_\rho = 0.0001$ | Baseline $\varepsilon_\rho = 0.5$ | Cobb-Douglas $\varepsilon_\rho = 1$ |
|---|---|---|---|---|---|---|
| | TFP ratios | | | Income ratios | | |
| 1. Baseline | | | | | | |
| No linkages | 1.6 | 1.6 | 1.6 | 4.7 | 4.7 | 4.7 |
| With linkages | 6.0 | 5.4 | 4.8 | 32.1 | 29.1 | 26.0 |
| Amplif. factor | 3.7 | 3.3 | 2.9 | 6.8 | 6.2 | 5.5 |
| 2. Identical TFPs | | | | | | |
| No linkages | 1.5 | 1.5 | 1.5 | 3.4 | 3.4 | 3.4 |
| With linkages | 1.5 | 1.7 | 1.8 | 3.8 | 4.3 | 4.8 |
| Amplif. factor | 1.0 | 1.1 | 1.2 | 1.1 | 1.3 | 1.4 |

Note: Income ratios are from Jones (2011, Table 1 and Table 2 [scenario 1 and 2]); TFP ratio are obtained from our own computation of Jones's Proposition 5 with $\alpha = 0$ and $\bar{h}^{rich} = 1$. The amplification factor follows Jones' definition (see text for more details).

## C.6 Preliminary tables and figures

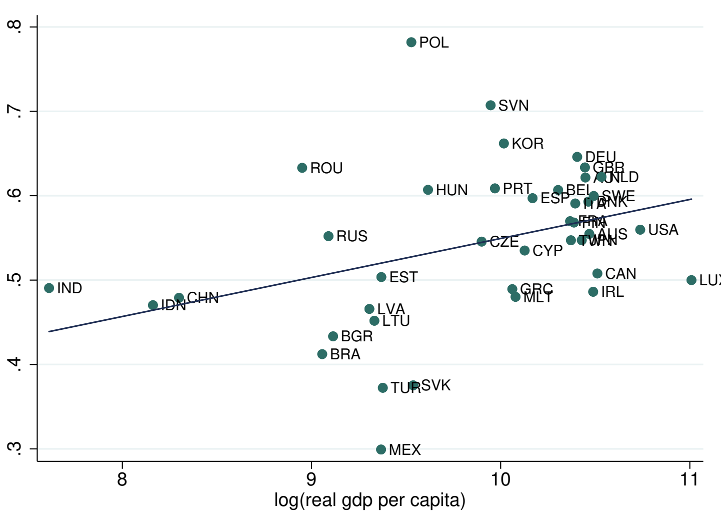
We provide here statistics on the production function parameters, as well as the aggregate labor and capital ratios. In Figure C.5, we plot the estimated $\alpha$ against the "naive" estimate $\sum_j p_j X_{ij}/p_i Q_i$. As we might expect, for most observations the estimated parameters are higher than the observed shares because of the presence of markups. However the markups cause the price of intermediates in some sectors to increase so much that the observed intermediate share is higher than the underlying technical parameter.

Figure C.5: Observed intermediate-input share vs $\alpha$




Note: Parameter $\alpha$ estimated for $(\varepsilon_\sigma, \varepsilon_\rho) = (0.70, 0.01)$.

Figure C.6: The aggregate labor share




Note: The labor share is computed as the sum of labor compensation over the nominal value added. The figure display data for year 2000.

Table C.4: Production function parameters, by country

| country | $\beta$ | | | $\alpha$ | | | $V$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | min | med | max | min | med | max | min | med | max |
| AUS | 0.00 | 0.02 | 0.17 | 0.30 | 0.56 | 0.91 | 0.00 | 0.01 | 0.59 |
| AUT | 0.00 | 0.03 | 0.10 | 0.25 | 0.45 | 0.63 | 0.00 | 0.01 | 0.76 |
| BEL | 0.00 | 0.03 | 0.09 | 0.23 | 0.50 | 0.78 | 0.00 | 0.01 | 0.64 |
| BGR | 0.01 | 0.02 | 0.12 | 0.18 | 0.50 | 0.72 | 0.00 | 0.01 | 0.72 |
| BRA | 0.00 | 0.03 | 0.10 | 0.04 | 0.54 | 0.80 | 0.00 | 0.01 | 0.59 |
| CAN | 0.00 | 0.02 | 0.11 | 0.17 | 0.48 | 0.86 | 0.00 | 0.01 | 0.59 |
| CHN | 0.00 | 0.02 | 0.20 | 0.20 | 0.70 | 0.81 | 0.00 | 0.01 | 0.66 |
| CZE | 0.00 | 0.02 | 0.10 | 0.33 | 0.55 | 0.76 | 0.00 | 0.01 | 0.60 |
| DEU | 0.00 | 0.02 | 0.11 | 0.19 | 0.53 | 0.76 | 0.00 | 0.01 | 0.67 |
| DNK | 0.00 | 0.02 | 0.11 | 0.25 | 0.45 | 0.98 | 0.00 | 0.01 | 0.83 |
| ESP | 0.00 | 0.02 | 0.15 | 0.22 | 0.59 | 0.77 | 0.00 | 0.01 | 0.53 |
| EST | 0.00 | 0.03 | 0.11 | 0.22 | 0.48 | 0.77 | 0.00 | 0.01 | 0.61 |
| FIN | 0.00 | 0.02 | 0.12 | 0.27 | 0.50 | 0.72 | 0.00 | 0.01 | 0.52 |
| FRA | 0.00 | 0.03 | 0.13 | 0.17 | 0.55 | 0.77 | 0.00 | 0.01 | 0.56 |
| GBR | 0.00 | 0.02 | 0.10 | 0.26 | 0.48 | 0.76 | 0.00 | 0.01 | 0.80 |
| GRC | 0.00 | 0.02 | 0.15 | 0.16 | 0.43 | 0.74 | 0.00 | 0.01 | 0.87 |
| HUN | 0.00 | 0.03 | 0.16 | 0.20 | 0.51 | 0.81 | 0.00 | 0.01 | 0.50 |
| IDN | 0.00 | 0.02 | 0.15 | 0.18 | 0.52 | 0.77 | 0.00 | 0.01 | 0.92 |
| IND | 0.00 | 0.02 | 0.18 | 0.10 | 0.60 | 0.82 | 0.00 | 0.01 | 0.66 |
| IRL | 0.00 | 0.01 | 0.17 | 0.27 | 0.44 | 0.69 | 0.00 | 0.01 | 0.64 |
| JPN | 0.00 | 0.02 | 0.16 | 0.13 | 0.55 | 0.73 | 0.00 | 0.01 | 0.60 |
| KOR | 0.00 | 0.02 | 0.15 | 0.24 | 0.55 | 0.85 | 0.00 | 0.01 | 0.70 |
| LTU | 0.00 | 0.02 | 0.12 | 0.14 | 0.39 | 0.65 | 0.00 | 0.01 | 0.60 |
| MEX | 0.00 | 0.02 | 0.11 | 0.07 | 0.51 | 0.84 | 0.00 | 0.01 | 0.85 |
| NLD | 0.00 | 0.03 | 0.10 | 0.27 | 0.46 | 0.73 | 0.00 | 0.01 | 0.59 |
| POL | 0.00 | 0.03 | 0.12 | 0.30 | 0.55 | 0.74 | 0.00 | 0.01 | 0.46 |
| PRT | 0.00 | 0.02 | 0.14 | 0.20 | 0.56 | 0.76 | 0.00 | 0.01 | 0.70 |
| ROU | 0.00 | 0.02 | 0.12 | 0.24 | 0.47 | 0.70 | 0.00 | 0.01 | 0.82 |
| RUS | 0.00 | 0.02 | 0.12 | 0.21 | 0.52 | 0.75 | 0.00 | 0.01 | 0.56 |
| SVK | 0.00 | 0.03 | 0.09 | 0.26 | 0.56 | 0.72 | 0.00 | 0.01 | 0.77 |
| SVN | 0.00 | 0.03 | 0.11 | 0.28 | 0.47 | 0.71 | 0.00 | 0.01 | 0.60 |
| SWE | 0.00 | 0.02 | 0.12 | 0.25 | 0.48 | 0.69 | 0.00 | 0.01 | 0.50 |
| TUR | 0.00 | 0.02 | 0.10 | 0.29 | 0.56 | 0.80 | 0.00 | 0.01 | 0.80 |
| TWN | 0.00 | 0.02 | 0.22 | 0.24 | 0.56 | 0.75 | 0.00 | 0.01 | 0.57 |
| USA | 0.00 | 0.02 | 0.14 | 0.28 | 0.57 | 0.77 | 0.00 | 0.01 | 0.73 |

Notes: Parameters calibrated under the assumption $\varepsilon_\sigma = 0.70$ and $\varepsilon_\rho = 0.01$.

## C.7 Additional tables of results

We report here additional tables of results.

Table C.6 and C.7 give the baseline results for the 35 countries and for additional combinations of ($\varepsilon_\sigma$, $\varepsilon_\rho$). The last column of the table gives the results when the elasticities of substitution are set in line with Peter and Ruane (2020)'s recent estimates. They find an elasticity of substitution (EOS) between value added and the intermediate bundle of 0.6, between materials, energy and services of

Table C.5: Production function parameters, by industry

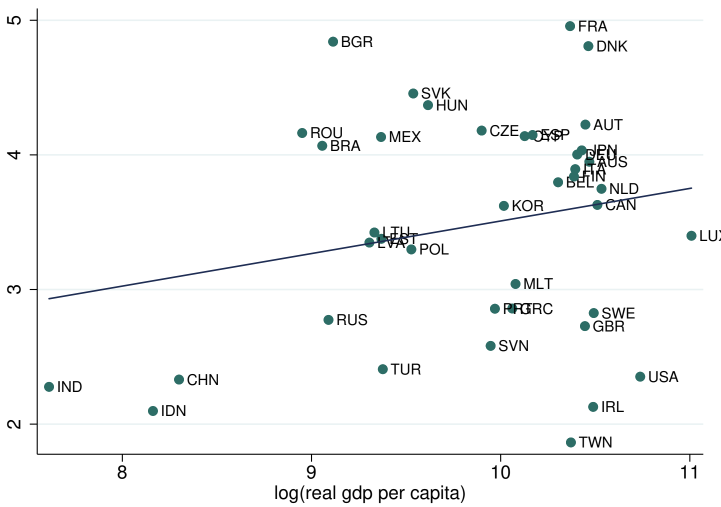
| industry | $\beta$ | | | $\alpha$ | | | $V$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | min | med | max | min | med | max | min | med | max |
| Agriculture | 0.01 | 0.03 | 0.18 | 0.22 | 0.42 | 0.59 | 0.00 | 0.01 | 0.73 |
| Mining | 0.00 | 0.00 | 0.10 | 0.19 | 0.39 | 0.56 | 0.00 | 0.01 | 0.51 |
| Food | 0.04 | 0.07 | 0.13 | 0.63 | 0.72 | 0.85 | 0.00 | 0.01 | 0.61 |
| Textiles | 0.00 | 0.02 | 0.10 | 0.32 | 0.56 | 0.78 | 0.00 | 0.01 | 0.57 |
| Leather | 0.00 | 0.00 | 0.02 | 0.45 | 0.55 | 0.81 | 0.00 | 0.01 | 0.41 |
| Wood | 0.00 | 0.00 | 0.05 | 0.44 | 0.60 | 0.77 | 0.00 | 0.01 | 0.49 |
| Paper | 0.01 | 0.02 | 0.10 | 0.42 | 0.56 | 0.72 | 0.00 | 0.01 | 0.62 |
| Petroleum | 0.00 | 0.02 | 0.08 | 0.25 | 0.62 | 0.98 | 0.00 | 0.01 | 0.92 |
| Chemicals | 0.01 | 0.03 | 0.15 | 0.31 | 0.62 | 0.78 | 0.00 | 0.01 | 0.52 |
| Rubber | 0.00 | 0.01 | 0.03 | 0.41 | 0.55 | 0.79 | 0.00 | 0.01 | 0.46 |
| Non-metal. mineral | 0.00 | 0.01 | 0.02 | 0.43 | 0.54 | 0.70 | 0.00 | 0.01 | 0.36 |
| Basic and fabr. metal | 0.01 | 0.03 | 0.08 | 0.39 | 0.61 | 0.79 | 0.00 | 0.01 | 0.70 |
| Machinery, nec | 0.01 | 0.03 | 0.07 | 0.38 | 0.53 | 0.75 | 0.00 | 0.01 | 0.41 |
| Elec. optical equip. | 0.01 | 0.04 | 0.22 | 0.38 | 0.57 | 0.79 | 0.00 | 0.01 | 0.52 |
| Transp. equip. | 0.01 | 0.05 | 0.11 | 0.27 | 0.59 | 0.80 | 0.00 | 0.01 | 0.60 |
| Manufacturing nec | 0.00 | 0.02 | 0.03 | 0.30 | 0.58 | 0.72 | 0.00 | 0.02 | 0.46 |
| Elec., gas, water | 0.01 | 0.02 | 0.05 | 0.23 | 0.51 | 0.67 | 0.00 | 0.01 | 0.80 |
| Construction | 0.06 | 0.11 | 0.20 | 0.41 | 0.55 | 0.72 | 0.00 | 0.01 | 0.50 |
| Car sales | 0.00 | 0.01 | 0.03 | 0.21 | 0.39 | 0.54 | 0.00 | 0.01 | 0.57 |
| Wholesale trade | 0.02 | 0.05 | 0.12 | 0.25 | 0.41 | 0.55 | 0.00 | 0.01 | 0.41 |
| Retail trade | 0.01 | 0.04 | 0.10 | 0.16 | 0.33 | 0.52 | 0.00 | 0.01 | 0.55 |
| Hotels and rest. | 0.01 | 0.03 | 0.13 | 0.24 | 0.48 | 0.73 | 0.00 | 0.01 | 0.59 |
| Inland transportation | 0.01 | 0.02 | 0.09 | 0.27 | 0.41 | 0.60 | 0.00 | 0.01 | 0.52 |
| Water transportation | 0.00 | 0.00 | 0.07 | 0.33 | 0.53 | 0.78 | 0.00 | 0.01 | 0.83 |
| Air transportation | 0.00 | 0.01 | 0.02 | 0.36 | 0.54 | 0.81 | 0.00 | 0.01 | 0.61 |
| Other transp. support | 0.00 | 0.01 | 0.06 | 0.29 | 0.48 | 0.70 | 0.00 | 0.01 | 0.82 |
| Post, telecomm. | 0.01 | 0.02 | 0.04 | 0.18 | 0.44 | 0.63 | 0.00 | 0.01 | 0.64 |
| Finance | 0.01 | 0.04 | 0.09 | 0.18 | 0.43 | 0.54 | 0.00 | 0.00 | 0.63 |
| Real estate | 0.01 | 0.09 | 0.14 | 0.04 | 0.23 | 0.53 | 0.00 | 0.00 | 0.87 |
| Renting of m&eq | 0.01 | 0.05 | 0.10 | 0.25 | 0.38 | 0.56 | 0.00 | 0.01 | 0.64 |

Notes: Parameters calibrated under the assumption $\varepsilon_\sigma = 0.70$ and $\varepsilon_\rho = 0.01$. In the last panel, the statistics are computed by purchasing industry.

0.4, and between various types of materials of 4.7. In our framework, the elasticity of substitution is the same between all intermediate inputs. We therefore computed $\varepsilon_\rho$ as the weighted average of the two elasticities estimated by Peter and Ruane (2020), using the 2007 US spending on materials and other intermediates (over total intermediate input spending) as weights, that is $\varepsilon_\rho = 0.36 \times 4.7 + (1 - 0.36) \times 0.4$.

Table C.8 reports the TFP gain when the markups distort only the firms' labor. We consider two economies without sectoral linkages: in the first one, the $\beta_i$ are set to the same value as in the economy with linkages and in the second one, the $\beta_i$ are recalibrated so as to yield sector of identical sizes (undistorted value added shares) in the economy with and without linkages. Table C.11 performs a sensitivity analysis of these results with respect to the elasticity of substitution in consumption.

Figure C.7: The aggregate capital-output ratio




Note: The capital-output ratio is computed as the sum the real capital stock times the investment deflator over the nominal value added. This is for year 2000.

We report the results for alternative values of the consumption elasticity in Table C.10 and C.11.

We then report the results of our robustness checks. In the first robustness check, we recompute the TFP gain after removing the pcm outliers. We winsorize the price-cost margin measure by capping the gap between the pcm and the industry median at the 1st and 99th percentile (computed on the pooled data). The gap is pcm - median(pcm). The results are presented in Table C.9.

Table C.6: TFP gain and the elasticity of substitution

| | $(\varepsilon_\sigma, \varepsilon_\rho)$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| country | $(0.01, 0.01)$ | $(0.7, 0.7)$ | $(1, 1)$ | $(1, 0.01)$ | $(1, 0.7)$ | $(0.01, 1)$ | $(0.7, 1)$ | $(0.6, 2)$ |
| BEL | 0.001 | 0.002 | 0.003 | 0.002 | 0.003 | 0.002 | 0.003 | 0.004 |
| FRA | 0.001 | 0.003 | 0.004 | 0.003 | 0.004 | 0.002 | 0.004 | 0.005 |
| AUT | 0.002 | 0.005 | 0.006 | 0.005 | 0.006 | 0.003 | 0.005 | 0.006 |
| CAN | 0.002 | 0.005 | 0.007 | 0.005 | 0.006 | 0.004 | 0.006 | 0.007 |
| HUN | 0.001 | 0.005 | 0.007 | 0.005 | 0.007 | 0.004 | 0.006 | 0.008 |
| DEU | 0.002 | 0.005 | 0.007 | 0.005 | 0.007 | 0.004 | 0.006 | 0.008 |
| CZE | 0.001 | 0.005 | 0.007 | 0.005 | 0.007 | 0.003 | 0.006 | 0.008 |
| KOR | 0.002 | 0.006 | 0.007 | 0.005 | 0.007 | 0.004 | 0.006 | 0.009 |
| FIN | 0.004 | 0.008 | 0.010 | 0.007 | 0.009 | 0.006 | 0.008 | 0.011 |
| ESP | 0.003 | 0.009 | 0.012 | 0.008 | 0.011 | 0.006 | 0.010 | 0.013 |
| SWE | 0.005 | 0.010 | 0.012 | 0.008 | 0.011 | 0.009 | 0.011 | 0.015 |
| AUS | 0.004 | 0.011 | 0.014 | 0.009 | 0.013 | 0.008 | 0.012 | 0.017 |
| NLD | 0.006 | 0.013 | 0.017 | 0.011 | 0.015 | 0.011 | 0.015 | 0.022 |
| EST | 0.004 | 0.013 | 0.018 | 0.012 | 0.016 | 0.008 | 0.014 | 0.018 |
| JPN | 0.005 | 0.014 | 0.018 | 0.011 | 0.016 | 0.011 | 0.016 | 0.022 |
| ROU | 0.006 | 0.014 | 0.019 | 0.013 | 0.017 | 0.010 | 0.016 | 0.020 |
| DNK | 0.011 | 0.017 | 0.020 | 0.014 | 0.018 | 0.017 | 0.019 | 0.028 |
| PRT | 0.005 | 0.016 | 0.021 | 0.015 | 0.019 | 0.010 | 0.017 | 0.022 |
| RUS | 0.006 | 0.016 | 0.022 | 0.015 | 0.020 | 0.011 | 0.018 | 0.023 |
| USA | 0.004 | 0.016 | 0.022 | 0.019 | 0.021 | 0.007 | 0.017 | 0.019 |
| LTU | 0.007 | 0.019 | 0.025 | 0.019 | 0.023 | 0.011 | 0.020 | 0.024 |
| SVK | 0.004 | 0.018 | 0.025 | 0.021 | 0.024 | 0.007 | 0.019 | 0.022 |
| BGR | 0.006 | 0.020 | 0.028 | 0.017 | 0.024 | 0.014 | 0.023 | 0.035 |
| POL | 0.006 | 0.021 | 0.029 | 0.020 | 0.026 | 0.013 | 0.024 | 0.031 |
| GBR | 0.007 | 0.022 | 0.031 | 0.022 | 0.028 | 0.013 | 0.025 | 0.032 |
| SVN | 0.022 | 0.028 | 0.032 | 0.024 | 0.029 | 0.029 | 0.031 | 0.042 |
| GRC | 0.014 | 0.028 | 0.036 | 0.027 | 0.033 | 0.021 | 0.031 | 0.039 |
| IRL | 0.018 | 0.030 | 0.036 | 0.029 | 0.034 | 0.023 | 0.032 | 0.037 |
| IND | 0.007 | 0.026 | 0.036 | 0.025 | 0.033 | 0.016 | 0.030 | 0.040 |
| BRA | 0.010 | 0.029 | 0.039 | 0.031 | 0.037 | 0.016 | 0.031 | 0.036 |
| CHN | 0.008 | 0.042 | 0.061 | 0.048 | 0.057 | 0.016 | 0.045 | 0.052 |
| MEX | 0.014 | 0.049 | 0.068 | 0.058 | 0.065 | 0.020 | 0.051 | 0.053 |
| TWN | 0.014 | 0.051 | 0.074 | 0.050 | 0.067 | 0.028 | 0.057 | 0.073 |
| IDN | 0.028 | 0.076 | 0.106 | 0.070 | 0.095 | 0.048 | 0.085 | 0.109 |
| TUR | 0.066 | 0.143 | 0.198 | 0.142 | 0.179 | 0.094 | 0.158 | 0.204 |
| median | 0.006 | 0.016 | 0.021 | 0.015 | 0.019 | 0.011 | 0.017 | 0.022 |

Notes: The model is re-calibrated for each value of the elasticities of substitution. The case ($\varepsilon_\sigma = \varepsilon_\rho = 1$) is repeated for convenience. Because we set only the positive markups to zero, the TFP gain can be negative; this occurs for some parameter combinations when the distribution of markups is skewed below zero as observed in Korea. The last column of the table uses elasticities values in line with the Indian estimates of Peter and Ruane (2020).

Table C.7: TFP gain and IO amplification

| | TFP gain | | | IO amplification factor | | |
|---|---|---|---|---|---|---|
| $(\varepsilon_\sigma, \varepsilon_\rho)$ | (0.01,0.01) | (0.70,0.01) | (1.00,1.00) | (0.01,0.01) | (0.70,0.01) | (1.00,1.00) |
| BEL | 0.001 | 0.002 | 0.003 | 1.1 | 1.7 | 2.9 |
| FRA | 0.001 | 0.002 | 0.004 | 1.4 | 2.4 | 4.0 |
| AUT | 0.002 | 0.004 | 0.006 | 1.2 | 2.5 | 4.1 |
| CAN | 0.002 | 0.004 | 0.007 | 1.8 | 3.4 | 5.4 |
| HUN | 0.001 | 0.004 | 0.007 | 1.3 | 3.4 | 6.4 |
| DEU | 0.002 | 0.004 | 0.007 | 1.4 | 3.4 | 6.3 |
| CZE | 0.001 | 0.004 | 0.007 | 1.4 | 4.3 | 8.1 |
| KOR | 0.002 | 0.004 | 0.007 | 1.2 | 2.9 | 5.7 |
| FIN | 0.004 | 0.006 | 0.010 | 1.4 | 2.2 | 3.7 |
| ESP | 0.003 | 0.007 | 0.012 | 1.2 | 2.6 | 4.6 |
| SWE | 0.005 | 0.007 | 0.012 | 1.2 | 1.7 | 2.8 |
| AUS | 0.004 | 0.007 | 0.014 | 1.5 | 2.7 | 5.4 |
| NLD | 0.006 | 0.009 | 0.017 | 1.5 | 2.1 | 3.8 |
| EST | 0.004 | 0.010 | 0.018 | 1.2 | 2.9 | 5.3 |
| JPN | 0.005 | 0.009 | 0.018 | 1.3 | 2.3 | 4.4 |
| ROU | 0.006 | 0.011 | 0.019 | 1.5 | 2.8 | 4.8 |
| DNK | 0.011 | 0.013 | 0.020 | 1.4 | 1.6 | 2.5 |
| PRT | 0.005 | 0.012 | 0.021 | 1.4 | 3.3 | 5.8 |
| RUS | 0.006 | 0.012 | 0.022 | 1.1 | 2.4 | 4.2 |
| USA | 0.004 | 0.014 | 0.022 | 1.0 | 3.4 | 5.6 |
| LTU | 0.007 | 0.015 | 0.025 | 1.1 | 2.5 | 4.2 |
| SVK | 0.004 | 0.015 | 0.025 | 1.8 | 6.7 | 11.2 |
| BGR | 0.006 | 0.013 | 0.028 | 1.3 | 3.1 | 6.3 |
| POL | 0.006 | 0.015 | 0.029 | 1.1 | 3.0 | 5.7 |
| GBR | 0.007 | 0.017 | 0.031 | 1.4 | 3.4 | 6.4 |
| SVN | 0.022 | 0.023 | 0.032 | 1.0 | 1.1 | 1.5 |
| GRC | 0.014 | 0.022 | 0.036 | 0.9 | 1.5 | 2.4 |
| IRL | 0.018 | 0.025 | 0.036 | 1.3 | 1.8 | 2.6 |
| IND | 0.007 | 0.019 | 0.036 | 0.7 | 1.7 | 3.3 |
| BRA | 0.010 | 0.024 | 0.039 | 1.4 | 3.4 | 5.5 |
| CHN | 0.008 | 0.034 | 0.061 | 2.0 | 8.9 | 16.1 |
| MEX | 0.014 | 0.043 | 0.068 | 1.1 | 3.5 | 5.5 |
| TWN | 0.014 | 0.037 | 0.074 | 1.1 | 3.0 | 6.1 |
| IDN | 0.028 | 0.056 | 0.106 | 1.5 | 2.9 | 5.6 |
| TUR | 0.066 | 0.114 | 0.198 | 1.1 | 1.9 | 3.3 |
| median | 0.006 | 0.012 | 0.021 | 1.3 | 2.8 | 5.3 |

Table C.8: IO amplification factor - markups on labor only

| | no recalibration of $\beta_i$ | | | recalibration $\beta_i = (1-\alpha_i)s_i$ | | |
|---|---|---|---|---|---|---|
| $(\varepsilon_\sigma, \varepsilon_\rho)$ | (0.01,0.01) | (0.70,0.01) | (1.00,1.00) | (0.01,0.01) | (0.70,0.01) | (1.00,1.00) |
| FRA | 0.4 | 0.6 | 1.0 | 0.4 | 0.6 | 1.0 |
| BEL | 0.5 | 0.7 | 1.2 | 0.5 | 0.6 | 1.0 |
| CZE | 0.4 | 0.7 | 1.5 | 0.3 | 0.5 | 1.0 |
| CAN | 0.5 | 0.7 | 1.3 | 0.4 | 0.6 | 1.0 |
| AUT | 0.5 | 0.7 | 1.1 | 0.4 | 0.6 | 1.0 |
| HUN | 0.5 | 0.8 | 1.6 | 0.3 | 0.5 | 1.0 |
| KOR | 0.4 | 0.7 | 1.4 | 0.3 | 0.5 | 1.0 |
| DEU | 0.6 | 0.9 | 1.9 | 0.3 | 0.5 | 1.0 |
| SVK | 0.4 | 0.6 | 1.2 | 0.4 | 0.5 | 1.0 |
| FIN | 0.4 | 0.6 | 1.0 | 0.4 | 0.6 | 1.0 |
| ESP | 0.4 | 0.7 | 1.2 | 0.4 | 0.6 | 1.0 |
| AUS | 0.5 | 0.7 | 1.3 | 0.4 | 0.5 | 1.0 |
| USA | 0.4 | 0.6 | 1.0 | 0.4 | 0.6 | 1.0 |
| SWE | 0.4 | 0.6 | 1.0 | 0.5 | 0.6 | 1.0 |
| PRT | 0.5 | 0.7 | 1.2 | 0.4 | 0.6 | 1.0 |
| EST | 0.6 | 0.7 | 1.4 | 0.5 | 0.6 | 1.0 |
| ROU | 0.5 | 0.7 | 1.3 | 0.4 | 0.5 | 1.0 |
| JPN | 0.4 | 0.7 | 1.4 | 0.3 | 0.5 | 1.0 |
| RUS | 0.5 | 0.7 | 1.1 | 0.4 | 0.6 | 1.0 |
| CHN | 0.4 | 0.8 | 1.7 | 0.3 | 0.5 | 1.0 |
| POL | 0.4 | 0.7 | 1.3 | 0.3 | 0.5 | 1.0 |
| LTU | 0.6 | 0.7 | 1.2 | 0.5 | 0.6 | 1.0 |
| BRA | 0.4 | 0.6 | 1.0 | 0.4 | 0.6 | 1.0 |
| GBR | 0.6 | 0.8 | 1.6 | 0.4 | 0.5 | 1.0 |
| NLD | 0.8 | 1.1 | 2.0 | 0.5 | 0.6 | 1.0 |
| BGR | 0.6 | 0.8 | 2.0 | 0.4 | 0.5 | 1.0 |
| DNK | 0.8 | 0.9 | 1.4 | 0.8 | 0.7 | 1.0 |
| IRL | 0.6 | 0.6 | 0.8 | 0.7 | 0.8 | 1.0 |
| MEX | 0.6 | 0.8 | 1.2 | 0.5 | 0.7 | 1.0 |
| IND | 0.5 | 0.7 | 1.4 | 0.4 | 0.5 | 1.0 |
| GRC | 0.6 | 0.8 | 1.2 | 0.5 | 0.7 | 1.0 |
| SVN | 0.7 | 0.7 | 0.9 | 0.8 | 0.8 | 1.0 |
| TWN | 0.7 | 1.0 | 2.1 | 0.4 | 0.5 | 1.0 |
| IDN | 0.8 | 1.0 | 1.9 | 0.6 | 0.6 | 1.0 |
| TUR | 0.5 | 0.5 | 0.8 | 0.7 | 0.7 | 1.0 |
| median | 0.5 | 0.7 | 1.3 | 0.4 | 0.6 | 1.0 |

Notes: The Input-Output amplification factor is computed under two cases. The first panel shows the results when $\beta_i$ are identical in the no-linkages economy and in the economy with linkages. The second panel shows the results when the no-linkages economy's $\beta_i$ are recalibrated to maintain the size of each sector identical to their size in the economy with sectoral linkages.

Table C.9: TFP gain and IO amplification - Winsorized

| | TFP gain | | | IO amplification factor | | |
|---|---|---|---|---|---|---|
| $(\varepsilon_\sigma, \varepsilon_\rho)$ | (0.01,0.01) | (0.70,0.01) | (1.00,1.00) | (0.01,0.01) | (0.70,0.01) | (1.00,1.00) |
| BEL | 0.001 | 0.002 | 0.003 | 1.1 | 1.7 | 2.9 |
| FRA | 0.001 | 0.002 | 0.004 | 1.4 | 2.4 | 4.0 |
| AUT | 0.002 | 0.004 | 0.006 | 1.2 | 2.5 | 4.1 |
| CAN | 0.002 | 0.004 | 0.007 | 1.8 | 3.4 | 5.4 |
| SVN | 0.003 | 0.004 | 0.007 | 1.0 | 1.4 | 2.4 |
| HUN | 0.001 | 0.004 | 0.007 | 1.3 | 3.4 | 6.4 |
| DEU | 0.002 | 0.004 | 0.007 | 1.4 | 3.4 | 6.3 |
| CZE | 0.001 | 0.004 | 0.007 | 1.4 | 4.3 | 8.1 |
| KOR | 0.002 | 0.004 | 0.007 | 1.2 | 2.9 | 5.7 |
| DNK | 0.004 | 0.006 | 0.009 | 1.3 | 1.7 | 2.8 |
| FIN | 0.004 | 0.006 | 0.010 | 1.4 | 2.2 | 3.7 |
| NLD | 0.004 | 0.006 | 0.011 | 1.4 | 2.3 | 3.9 |
| ESP | 0.003 | 0.007 | 0.012 | 1.2 | 2.6 | 4.6 |
| SWE | 0.005 | 0.007 | 0.012 | 1.2 | 1.7 | 2.8 |
| AUS | 0.004 | 0.007 | 0.014 | 1.5 | 2.7 | 5.4 |
| EST | 0.004 | 0.010 | 0.018 | 1.2 | 2.9 | 5.3 |
| JPN | 0.005 | 0.009 | 0.018 | 1.3 | 2.3 | 4.4 |
| ROU | 0.006 | 0.011 | 0.019 | 1.5 | 2.8 | 4.8 |
| PRT | 0.005 | 0.012 | 0.021 | 1.4 | 3.3 | 5.8 |
| RUS | 0.006 | 0.012 | 0.022 | 1.1 | 2.4 | 4.2 |
| USA | 0.004 | 0.014 | 0.022 | 1.0 | 3.4 | 5.6 |
| BGR | 0.005 | 0.012 | 0.023 | 1.3 | 3.3 | 6.5 |
| GRC | 0.008 | 0.015 | 0.024 | 1.0 | 1.8 | 2.9 |
| LTU | 0.007 | 0.015 | 0.025 | 1.1 | 2.5 | 4.2 |
| SVK | 0.004 | 0.015 | 0.025 | 1.8 | 6.7 | 11.2 |
| POL | 0.006 | 0.015 | 0.029 | 1.1 | 3.0 | 5.7 |
| GBR | 0.007 | 0.017 | 0.031 | 1.4 | 3.4 | 6.4 |
| TWN | 0.007 | 0.019 | 0.035 | 1.0 | 2.6 | 4.7 |
| IND | 0.007 | 0.019 | 0.036 | 0.7 | 1.8 | 3.4 |
| IRL | 0.018 | 0.025 | 0.036 | 1.3 | 1.8 | 2.6 |
| BRA | 0.010 | 0.024 | 0.039 | 1.4 | 3.4 | 5.5 |
| IDN | 0.015 | 0.031 | 0.052 | 1.5 | 3.2 | 5.5 |
| CHN | 0.008 | 0.034 | 0.061 | 2.0 | 8.9 | 16.1 |
| MEX | 0.014 | 0.043 | 0.068 | 1.1 | 3.5 | 5.5 |
| TUR | 0.020 | 0.057 | 0.111 | 1.3 | 3.7 | 7.1 |
| median | 0.005 | 0.012 | 0.021 | 1.3 | 2.7 | 4.8 |

Table C.10: Sensitivity analysis - Alternative consumption elasticities

| | TFP gain | | | | |
|---|---|---|---|---|---|
| $(\varepsilon_\sigma, \varepsilon_\rho)$ | (0.70,0.01) | (0.01,0.01) | (0.70,0.70) | (1.00,1.00) | (3.00,3.00) |
| $\varepsilon_\chi = 0.01$ | 0.006 | 0.000 | 0.009 | 0.013 | 0.074 |
| $\varepsilon_\chi = 0.70$ | 0.010 | 0.004 | 0.014 | 0.019 | 0.083 |
| $\varepsilon_\chi = 1.00$ | 0.012 | 0.006 | 0.016 | 0.021 | 0.086 |
| $\varepsilon_\chi = 3.00$ | 0.024 | 0.018 | 0.028 | 0.034 | 0.103 |
| | IO amplification factor | | | | |
| $(\varepsilon_\sigma, \varepsilon_\rho)$ | (0.70,0.01) | (0.01,0.01) | (0.70,0.70) | (1.00,1.00) | (3.00,3.00) |
| $\varepsilon_\chi = 0.01$ | 146.5 | 4.4 | 247.7 | 361.2 | 1,827.9 |
| $\varepsilon_\chi = 0.70$ | 3.3 | 1.3 | 5.1 | 7.0 | 26.9 |
| $\varepsilon_\chi = 1.00$ | 2.8 | 1.3 | 3.9 | 5.3 | 19.0 |
| $\varepsilon_\chi = 3.00$ | 1.7 | 1.3 | 2.0 | 2.4 | 6.8 |

Note: This table presents the median TFP gain and the median Input-Output amplification factor from removing markups. The TFP gain is computed as (TFP without markups)/(TFP with markups) -1. The IO amplification factor is the ratio of the TFP gain in the baseline economy over the TFP gain in the economy with $\alpha_i = 0$, $\forall i = 1, ..., n$. . The median is taken across all countries in our sample. The first column corresponds to our baseline production elasticities.

Table C.11: Sensitivity analysis - Markups on labor only

| | TFP gain | | | | |
|---|---|---|---|---|---|
| $(\varepsilon_\sigma, \varepsilon_\rho)$ | (0.70,0.01) | (0.01,0.01) | (0.70,0.70) | (1.00,1.00) | (3.00,3.00) |
| $\varepsilon_\chi = 0.01$ | 0.001 | 0.000 | 0.003 | 0.004 | 0.015 |
| $\varepsilon_\chi = 0.70$ | 0.002 | 0.001 | 0.004 | 0.005 | 0.016 |
| $\varepsilon_\chi = 1.00$ | 0.003 | 0.002 | 0.004 | 0.006 | 0.017 |
| $\varepsilon_\chi = 3.00$ | 0.007 | 0.007 | 0.008 | 0.009 | 0.020 |
| | IO amplification factor | | | | |
| $(\varepsilon_\sigma, \varepsilon_\rho)$ | (0.70,0.01) | (0.01,0.01) | (0.70,0.70) | (1.00,1.00) | (3.00,3.00) |
| $\varepsilon_\chi = 0.01$ | 26.9 | 1.2 | 56.0 | 82.7 | 352.8 |
| $\varepsilon_\chi = 0.70$ | 0.8 | 0.5 | 1.3 | 1.6 | 5.0 |
| $\varepsilon_\chi = 1.00$ | 0.7 | 0.5 | 1.0 | 1.3 | 3.5 |
| $\varepsilon_\chi = 3.00$ | 0.5 | 0.5 | 0.6 | 0.7 | 1.3 |

Note: This table presents the median TFP gain and the median Input-Output amplification factor when markups are applied only to labor costs. The TFP gain is computed as (TFP without markups)/(TFP with markups) -1. The IO amplification factor is the ratio of the TFP gain in the baseline economy over the TFP gain in the economy with $\alpha_i = 0$, $\forall i = 1, ..., n$. . The median is taken across all countries in our sample. The first column corresponds to our baseline production elasticities.

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