# On-line Appendix
# "Non-rational Beliefs in an Open Economy"*

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## Abstract

This file provides additional material for the paper.

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# 1 A Small Open Economy Model

This section extends the model of Justiniano and Preston (2010) to have non-rational beliefs. Following Eusepi, Giannoni, and Preston (2020), we allow agents to have arbitrary beliefs about expected future returns of available assets. A contribution of this paper is to provide a first step in the empirical evaluation of the resulting open-economy asset pricing implications.

**Households.** A continuum of large households $i \in [0,1]$ maximize intertemporal utility

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[ \xi_T \frac{C_{b,T}(i)^{1-\sigma}}{1-\sigma} - \varphi_T \int_0^1 \frac{N_T(i,j)^{1+\phi^{-1}}}{1+\phi^{-1}} dj \right]$$

where

$$C_{b,t} = \frac{C_t(i)}{Z_t} - b \frac{C_{t-1}(i)}{Z_{t-1}}$$

with $\sigma, \phi > 0$, $0 < b < 1$ and $Z_t = \gamma Z_{t-1}$ an exogenous technology trend. Each household comprises a large family, whose members $j \in [0,1]$ supply specialized labor, $N(i,j)$, to the production of each differentiated good $j$. The large family assumption insures each household against labor market risk from nominal wage contracting. The consumption index is defined as

$$C_t = \left[ \alpha^{\frac{1}{\eta}} (C_{H,t})^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} (C_{F,t})^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

where

$$\begin{aligned} C_{H,t} &= \left[ \int_0^1 C_{H,t}(f)^{\frac{\theta_{H,t}-1}{\theta_{H,t}}} df \right]^{\frac{\theta_{H,t}}{\theta_{H,t}-1}} \\ C_{F,t} &= \left[ \int_0^1 C_{F,t}(k)^{\frac{\theta_{F,t}-1}{\theta_{F,t}}} dk \right]^{\frac{\theta_{F,t}}{\theta_{F,t}-1}} \end{aligned}$$

denote domestic consumption of the bundles of differentiated home and foreign produced goods, with associated elasticities of substitution $\eta > 0$ and $\theta_{H,t}, \theta_{F,t} > 1$, the latter exogenous processes

$$\begin{aligned} \log \theta_{H,t} &= (1-\rho_{\theta_H}) \log \theta_H + \rho_{\theta_H} \log \theta_{H,t-1} + \sigma_{\theta_H} \varepsilon_t^{\theta_H} \\ \log \theta_{F,t} &= (1-\rho_{\theta_F}) \log \theta_F + \rho_{\theta_F} \log \theta_{F,t-1} + \sigma_{\theta_F} \varepsilon_t^{\theta_F} \end{aligned}$$

where $\varepsilon_t^{\theta_H}$ and $\varepsilon_t^{\theta_F}$ are i.i.d. $N(0,1)$, $\sigma_{\theta_H}, \sigma_{\theta_F} > 0$, $0 < \rho_{\theta_H}, \rho_{\theta_F} < 1$, $E[\theta_{H,t}] = \theta_H$ and $E[\theta_{F,t}] = \theta_F$. The parameter $0 < \alpha < 1$ determines the degree of home bias in total consumption. The preference and dis-utilitity of labor supply shocks are stationary exogenous processes

$$\begin{aligned} \log \xi_t &= \rho_\xi \log \xi_{t-1} + \sigma_\xi \varepsilon_t^\xi \\ \log \varphi_t &= \rho_\varphi \log \varphi_{t-1} + \sigma_\varphi \varepsilon_t^\varphi \end{aligned}$$

where $\varepsilon_t^\xi$ and $\varepsilon_t^\varphi$ are i.i.d. $N(0,1)$, $\sigma_\xi, \sigma_\varphi > 0$, $0 < \rho_\xi, \rho_\varphi < 1$, $E[\xi_t] = 1$ and $E[\varphi_t] = 1$. Conditional expectations of households, $\hat{E}_t^i$, is discussed below.

The household's flow budget constraint is

$$C_t(i) + \frac{P_t^B B_t(i)}{P_t} + S_t\left(\frac{P_t^{B^*} B_t^*(i)}{P_t}\right) + Z_t\Phi\left(\frac{B_t(i)}{Z_t P_t} - \frac{B_{t-1}(i)}{Z_{t-1}P_{t-1}}\right) + Z_t\Phi^*\left(\frac{S_t B_t^*(i)}{Z_t P_t} - \frac{S_{t-1}B_{t-1}^*(i)}{Z_{t-1}P_{t-1}}\right)$$
$$\leq \frac{B_{t-1}(i)}{P_t} + S_t\frac{B_{t-1}^*(i)}{P_t} + (1-\bar{\tau}_w)\int_0^1 \frac{W_t(i)}{P_t}N_t(i,j)\,dj + \Gamma_{H,t} + \Gamma_{F,t} - T_t + T_{w,t} + T_{H,t} + T_{F,t}$$

where $P_t^B$ and $P_t^{B^*}$ are the prices of domestic and foreign one-period nominal government debt, $B_t(i)$ and $B_t^*(i)$; $\Gamma_{H,t}$ and $\Gamma_{F,t}$ are the dividend payments, net of sales taxes, from an equal share of all equity claims on domestic and imported goods firms; $\bar{\tau}_\omega$ the labor income tax rate whose proceeds are rebated lump-sum to households as $T_{w,t}$; $T_{H,t}$ and $T_{F,t}$ the lump-sum rebate of sales revenue taxes from domestic and import goods firms; and $T_t$ lump-sum taxes and transfers.

The functions $\Phi(\cdot)$ and $\Phi^*(\cdot)$ denote transaction costs with the property

$$\Phi(0) = \Phi^*(0) = \Phi'(0) = \Phi^{*\prime}(0) = 0$$

and

$$\Phi''(0) \geq 0 \text{ and } \Phi^{*\prime\prime}(0) \geq 0.$$

Section **??** discusses these costs in detail as they are central to the methodological insights of the paper, but for now note they are required for a bounded solution to a first-order approximation of the household's problem under arbitrary beliefs.[1] Household's optimal consumption and portfolio choice must satisfy the no-Ponzi condition

$$\lim_{T\to\infty} \hat{E}_t^i\left(\prod_{s=0}^{\infty} R_{t+s}\pi_{t+s}^{-1}\right)^{-1}\left(\frac{B_T}{P_T} + S_T\frac{B_T^*}{P_T}\right) \geq 0.$$

Because households must hold domestic and foreign debt which is in non-negative supply in equilibrium, and because all households have the same beliefs, this constraint will be satisfied under arbitrary beliefs.

Households have market power in the supply of differentiated labor inputs. The demand for labor type $j$ by firm $f$ is

$$N_t(j,f) = \left(\frac{W_t(j)}{W_t}\right)^{-\theta_{w,t}} N_t(f) \tag{1}$$

where

$$N_t(f) = \left[\int_0^1 N_t(j,f)^{\frac{\theta_{w,t}-1}{\theta_{w,t}}}\,dj\right]^{\frac{\theta_{w,t}}{\theta_{w,t}-1}} \text{ and } W_t = \left[\int_0^1 W_t(j)^{1-\theta_{w,t}}\,dj\right]^{\frac{1}{1-\theta_{w,t}}}$$

define the composite labor input used in production and the associated wage rate. The elasticity of demand across differentiated inputs satisfies the exogenous process

$$\log\theta_{w,t} = (1-\rho_{\theta_w})\log\theta_w + \rho_{\theta_w}\log\theta_{w,t-1} + \sigma_{\theta_w}\varepsilon_t^{\theta_w}$$

[1] An alternative to transactions costs would be to solve the model non-linearly and use risk aversion to bound portfolio decisions. While this would be interesting in its own right, we employ transactions costs to permit use of linear econometric techniques.

and $\theta_{w,t} > 1$ and $\varepsilon_t^{\theta_w}$ is i.i.d. $N(0,1)$, $\sigma_{\theta_w} > 0$, $0 < \rho_{\theta_w} < 1$ and $E[\theta_{w,t}] = \theta_w$. Following Erceg, Henderson, and Levin (2000) a fraction of household members $0 < \xi_\omega < 1$ cannot optimally reset their wage, but follow the indexation rules

$$W_t(j) = W_{t-1}(j)\left(P_{t-1}/P_{t-2}\right)^{\iota_\omega}\gamma$$

for $0 < \iota_\omega < 1$. For the remaining fraction $\xi_\omega$, each member $j$ of household $i$, choose optimally their nominal wage, $W_t(j)$, to maximize

$$\hat{E}_t^i \sum_{T=t}^{\infty} (\xi_\omega\beta)^{T-t}\left[\Lambda_T(i)\, N_t(j)\,(1-\bar{\tau}_w)\frac{W_t(j)}{P_T}\left(\frac{P_{T-1}}{P_{t-1}}\right)^{\iota_\omega}\frac{Z_{T-1}}{Z_{t-1}} - \varphi_T\frac{N_T(j)^{1+\phi^{-1}}}{1+\phi^{-1}}\right]$$

subject to (1). Note the value of future wage income for member $j$ is valued by marginal utility of the larger household, $\Lambda_T(i)$. The subsidy $\bar{\tau}_w$ is set to eliminate the distortion from monopolistic competition in steady state.

**Domestic Firms**. A continuum of monopolistically competitive firms $f \in [0,1]$ produce differentiated goods, $Y_{H,t}$, facing a linear production technology in composite labor services, $N(f)$, and demand curve

$$\begin{aligned} Y_{H,t} &= A_t Z_t N_t(f) \\ Y_{H,T}(f) &= \left(\frac{p_H\left(P_{H,T-1}/P_{H,t-1}\right)^{\iota_H}}{P_{H,T}}\right)^{-\theta_{H,t}} Y_{H,T} \end{aligned}$$

where

$$Y_{H,t} = C_{H,t} + C_{H,t}^* + G_t$$

defines total demand for domestic goods, the sum domestic demand for home goods plus foreign demand, $C_{H,t}^*$, and government demand, $G_t$. Stationary technological progress, $A_t$, follows an exogenous processes

$$\log A_t = \rho_a \log A_{t-1} + \sigma_a \varepsilon_t^a$$

where $\varepsilon_t^a$ is i.i.d. $N(0,1)$, $\sigma_a > 0$, $0 < \rho_a < 1$ and $E[A_t] = 1$.

Following Calvo (1983) and Yun (1996) a fraction of firms $0 < \xi_H < 1$ cannot optimally choose their price, but reset it according to the indexation rule

$$P_{H,t}(f) = P_{H,t-1}(f)\left(P_{H,t-1}/P_{H,t-2}\right)^{\iota_H}$$

where $0 < \iota_H < 1$. The remaining fraction of firms choose a price $p_H$ to maximize the expected discounted value of profits

$$\hat{E}_t^f \sum_{T=t}^{\infty} \xi_H^{T-t} Q_{t,T}\Gamma_{H,T}(f)$$

where the stochastic discount factor, $Q_{t,T} = \beta^{T-t}\Lambda_T/\Lambda_t$, values future profits

$$\Gamma_{H,T}(f) = Y_{H,T}(f)\left((1-\bar{\tau}_H)\frac{p_H}{P_T}\left(\frac{P_{H,T-1}}{P_{H,t-1}}\right)^{\iota_H} - \frac{W_T}{A_T Z_T P_T}\right)$$

with $\bar{\tau}_H$ a subsidy to eliminate the steady-state distortion from monopolistic competition.

**Imported Goods Firms.** Following Monacelli (2005), a continuum of imported goods firms $k \in [0,1]$ distribute foreign produced goods. While the law of one price holds at the docks for these goods, imported goods firms have market power in their distribution. Each firm faces demand curve

$$C_{F,T}(k) = \left( \frac{p_F}{P_{F,T}} \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} \right)^{-\theta_{F,t}} C_{F,T} \tag{2}$$

for all $T > t$ and take aggregate prices and consumption bundles as given.

A fraction of firms $0 < \xi_F < 1$ cannot optimally reset their price and apply the indexation rule

$$P_{F,t}(k) = P_{F,t-1}(k) \left( P_{F,t-1}/P_{F,t-2} \right)^{\iota_F}$$

where $0 < \iota_F < 1$. The firm's price-setting problem in period $t$ is to choose a price $p_F$ to maximize the expected present discounted value of profits

$$\hat{E}_t^k \sum_{T=t}^{\infty} \xi_F^{T-t} Q_{t,T} C_{F,T}(k) \left[ (1 - \bar{\tau}_F) \frac{p_F}{P_T} \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} - \frac{S_T P_{F,T}^*}{P_T} \right]$$

subject to the demand curve (3). The government again sets a subsidy $\bar{\tau}_F$ to eliminate the steady-state distortion from monopolistic competition.

**Foreign Economy.** For brevity we do not present details of the foreign economy in the main text. Following Gali and Monacelli (2005) we assume foreign economy evolves independently of the home country. This block of the model is identical to the Eusepi, Giannoni, and Preston (2019) closed-economy model for the United States. Detailed derivations are available in the on-line appendix. It determines the dynamics of a standard set of macroeconomic variables which together with the following two assumptions provides a complete description of private sector behavior.

First, following Kollmann (2002) foreign demand for the domestically produced good is given by the demand function

$$C_{H,t}^* = \alpha^* \left( \frac{P_{H,t}^*}{P_t^*} \right)^{-\eta^*} Y_t^*$$

where $\eta^* > 0$ and $0 < \alpha^* < 1$ and $Y_t^*$ foreign output; $P_t^*$ the foreign price level; and $P_{H,t}^*$ the foreign currency price of the domestic good. Second, following Schmitt-Grohe and Uribe (2003) we assume domestic residents pay a premium to hold foreign debt. The foreign debt price is

$$\frac{1}{P_t^{B^*}} = R_t^* \exp\left\{ -\phi^{R^*} \left( \frac{S_t B_t^*}{c_F P_t Z_t} + rp_t \right) \right\}$$

where $R_t^*$ is the foreign one-period risk free rate; $c_F$ the state-state level of imports; $\phi^{R^*} > 0$; and the risk premium $rp_t$ follows an exogenous process

$$rp_t = \rho_{rp} rp_{t-1} + \sigma_{rp} \varepsilon_t^{rp}$$

where $\varepsilon_t^{rp}$ is i.i.d. $N(0,1)$, $\sigma_{rp} > 0$, and $0 < \rho_{rp} < 1$. This assumption ensures stationarity of domestic holdings of foreign debt in equilibrium.[2]

[2]Because we will study the model in a zero-transactions-cost limit, this assumption is not redundant.

**Government Policy.** The central bank implements monetary policy using the interest rate rule

$$R_t = (R_{t-1})^{\rho_R} \left[ R \left( P_t / P_{t-1} \right)^{\phi_\pi} \left( Y_{H,t} / Y_{H,t-1} \right)^{\phi_y} S_t^{\phi_s} \right]^{1-\rho_R} m_t$$

where $\phi_\pi$, $\phi_y$ and $\phi_S \geq 0$ and $R$ the steady-state gross interest rate. $Y_{H,t}$ is the domestic output. The steady-state inflation rate is zero; $\log m_t = \sigma_m \varepsilon_t^m$ denotes a mean-zero i.i.d. monetary shock.

Following Eusepi, Giannoni, and Preston (2019) we assume fiscal policy is Ricardian, and that this is understood by agents.[3] To focus on foreign-debt holdings we consider an economy with zero government debt and balanced-budget policy

$$T_t = \frac{P_{H,t}}{P_t} G_t.$$

We assume the government purchases only the domestically produced good.[4] To further simplify, we assume agents know the tax rebates and dividend policy rules are

$$\begin{aligned}
T_{H,t} &= \bar{\tau}_H \int_0^1 Y_{H,t}(f) \frac{P_{H,t}(f)}{P_t} df. \\
T_{F,t} &= \bar{\tau}_F \int_0^1 C_{F,t}(k) \frac{P_{F,t}(k)}{P_t} dk \\
T_{w,t} &= \bar{\tau}_w \int_0^1 \frac{W_t(i)}{P_t} N_t(i)\, di
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_{H,t} &= \int_0^1 Y_{H,t}(f) \left( (1-\bar{\tau}_H) \frac{P_{H,t}(f)}{P_t} - \frac{W_t}{A_t Z_t P_t} \right) df \\
\Gamma_{F,t} &= \int_0^1 C_{F,t}(k) \left( (1-\bar{\tau}_F) \frac{P_{F,t}(k)}{P_t} - \frac{S_t P_{F,t}^*}{P_t} \right) dk
\end{aligned}$$

in every period $t$. This knowledge implies households don't need to forecast future taxes and dividend payments to make optimal decisions. See the on-line appendix for details.

**Market Clearing and Equilibrium.** Within the domestic and foreign economies, we consider a symmetric equilibrium in which all households are identical, even though they do not know this to be true. Given that households have the same initial asset holdings, preferences, and beliefs, and face common constraints, they make identical state-contingent decisions. Equilibrium requires all goods, labor and asset markets to clear with initial condition $B_{-1}(i) = B_{-1}^*(i) = 0$. Equilibrium then is a sequence of prices and allocations satisfying individual optimality and market clearing conditions. A statement of the complete system is found in the on-line appendix.

[3] Eusepi and Preston (2018) show that in general learning will imply departures from Ricardian equivalence, so holdings of the public debt are perceived as net wealth. The associated wealth effects on aggregate demand can be sizable, and impair the standard intertemporal substitution channel of monetary policy. We choose to focus on how belief distortions affect long-term interest rates and monetary policy design, understanding that imperfect knowledge about fiscal and monetary policy both serve to complicate inflation policy.

[4] This follows Froot and Rogoff (1991) and Rogoff (1992) which argue government spending falls more heavily on non-traded goods in many advanced economies.

**Optimality.** The first-order conditions for the household are

$$\begin{aligned}
\partial C_t(i) &: \xi_t \left(\frac{C_t(i)}{Z_t} - b\frac{C_{t-1}(i)}{Z}\right)^{-\sigma} \frac{1}{Z_t} - \beta b E_t^i \xi_{t+1} \left(\frac{C_{t+1}(i)}{Z_{t+1}} - b\frac{C_t(i)}{Z_t}\right)^{-\sigma} \frac{1}{Z_t} = \Lambda_t(i) \\
\partial B_t(i) &: \Lambda_t(i) P_t^B = E_t^i \left[\Lambda_{t+1}(i) \beta \frac{P_t}{P_{t+1}}\right] - \Lambda_t(i) \Phi' \left(\frac{B_t}{Z_t P_t} - \frac{B_{t-1}}{Z_t P_{t-1}}\right) \\
&\quad + \beta E_t^i \left[\Lambda_{t+1}(i) \frac{Z_{t+1}}{Z_t} \Phi' \left(\frac{B_{t+1}}{Z_{t+1}P_{t+1}} - \frac{B_t}{Z_t P_t}\right)\right] \\
\partial B_t^*(i) &: \Lambda_t(i) P_t^{B^*} = E_t^i \left[\Lambda_{t+1}(i) \beta \frac{P_t}{P_{t+1}} \frac{S_{t+1}}{S_t}\right] - \Lambda_t(i) \Phi^{*\prime} \left(\frac{S_t B_t^*}{Z_t P_t} - \frac{S_{t-1} B_{t-1}^*}{Z_{t-1} P_{t-1}}\right) \\
&\quad + \beta E_t^i \left[\Lambda_{t+1}(i) \frac{Z_{t+1}}{Z_t} \Phi^{*\prime} \left(\frac{S_{t+1} B_{t+1}^*}{Z_{t+1} P_{t+1}} - \frac{S_t B_t^*}{Z_{t+1} P_t}\right)\right]
\end{aligned}$$

The demand for each category of consumption good is

$$C_{H,t}(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\theta_{w,t}} C_{H,t}; \quad C_{F,t}(i) = \left(\frac{P_{F,t}}{P_{F,t}}\right)^{-\theta_{w,t}} C_{F,t}$$

and

$$C_{H,t} = \alpha \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t; \quad C_{F,t} = (1-\alpha) \left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t$$

with the home CPI defined as

$$P_t = \left[\alpha P_{H,t}^{1-\eta} + (1-\alpha) P_{F,t}^{1-\eta}\right]^{\frac{1}{1-\eta}}.$$

Let

$$\Pi_{w,T} \equiv \left(\frac{Z_{T-1}}{Z_{t-1}}\right) \left(\frac{P_{T-1}}{P_{t-1}}\right)^{\iota_\omega}$$

define the cumulative index for wages in period $T$, for wages last reset in period $t$. Then the first-order condition for wages is

$$0 = E_t^i \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} (1-\bar{\tau}_w) \Lambda_T(i) P_T^{-1} \Pi_{w,T} \left(N_T(j) + W_t(j) \frac{\partial N_T(j)}{\partial W_t(j)}\right) \\ -\varphi_T N_T(j)^{\phi^{-1}} \frac{\partial N_T(j)}{\partial W_t(j)} \end{array} \right]$$

Now

$$N_T(j,f) = \left(\frac{W_t(j)}{W_T} \Pi_{w,T}\right)^{-\theta_{\omega,t}} N_T(f)$$

so

$$\frac{\partial N_T(j)}{\partial W_t(j)} = -\theta_{\omega,t} \left(\frac{W_t(j)}{W_T} \Pi_{w,T}\right)^{-\theta_{\omega,t}-1} \frac{1}{W_T} \Pi_{w,T} N_T(f)$$

Therefore we have

$$
\begin{aligned}
0 &= E_t^i \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} (1-\bar{\tau}_w) \Lambda_T(i) P_T^{-1} \Pi_{w,T} \times \\ \left( \left( \frac{W_t(j)}{W_T} \Pi_{w,T} \right)^{-\theta_{\omega,t}} N_T(f) - W_t(j) \theta_{\omega,T} \left( \frac{W_t(j)}{W_T} \Pi_{w,T} \right)^{-\theta_{\omega,t}-1} \frac{\Pi_{w,T}}{W_T} N_T(f) \right) \\ +\varphi_T N_T(j)^{\phi^{-1}} \theta_{\omega,T} \left( \frac{W_t(j)}{W_T} \Pi_{w,T} \right)^{-\theta_{\omega,t}-1} \frac{1}{W_T} \Pi_{w,T} N_T(f) \end{array} \right] \\
&= E_t^i \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} (1-\bar{\tau}_w) \Lambda_T(i) P_T^{-1} \Pi_{w,T} \times \\ \left( \left( \frac{W_t(j)}{W_T} \Pi_{w,T} \right)^{-\theta_{\omega,t}} N_T(f) - \theta_{\omega,T} \left( \frac{W_t(j)}{W_T} \Pi_{w,T} \right)^{-\theta_{\omega,t}} N_T(f) \right) \\ +\varphi_T N_T(j)^{\phi^{-1}} \theta_{\omega,T} \left( \frac{W_t(j)}{W_T} \Pi_{w,T} \right)^{-\theta_{\omega,T}} W_t(j)^{-1} N_T(f) \end{array} \right] \\
&= E_t^i \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} \left( \frac{W_t(j)}{W_T} \Pi_{w,T} \right)^{-\theta_{\omega,t}} N_T(f) \times \\ \left\{ (1-\bar{\tau}_w) \Lambda_T(i) P_T^{-1} \Pi_{w,T} (1-\theta_{\omega,T}) + \varphi_T N_T(j)^{\phi^{-1}} W_t(j)^{-1} \theta_{\omega,T} \right\} \end{array} \right] \\
&= E_t^i \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} N_T(j) W_t(j)^{-1} (1-\theta_{\omega,T}) \times \\ \left\{ (1-\bar{\tau}_w) \Lambda_T(i) P_T^{-1} \Pi_{w,T} W_t(j) - \varphi_T N_T(j)^{\phi^{-1}} \frac{\theta_{\omega,T}}{\theta_{\omega,T}-1} \right\} \end{array} \right]
\end{aligned}
$$

Now we can write in normalized variables

$$
\begin{aligned}
0 &= E_t^i \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} (1-\theta_{\omega,T}) N_T(j) P_T^{-1} W_T Z_T^{-1} \times \\ \left\{ (1-\bar{\tau}_w) Z_T \Lambda_T(i) \Pi_{w,T} \frac{W_t(j)}{W_T} - \varphi_T N_T(j)^{\phi^{-1}} \frac{\theta_{\omega,T}}{\theta_{\omega,T}-1} \frac{Z_T P_T}{W_T} \right\} \end{array} \right] \\
&= E_t^i \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} (1-\theta_{\omega,T}) w_T^r \times \\ \left\{ (1-\bar{\tau}_w) \tilde{\Lambda}_T(i) \Pi_{w,T} \frac{W_t(j)}{W_T} - \varphi_T N_T(j)^{\phi^{-1}} \frac{\theta_{\omega,T}}{\theta_{\omega,T}-1} w_T^{r-1} \right\} \end{array} \right]
\end{aligned}
$$

where

$$w^r = \frac{W_t}{P_t Z_t}; \; \tilde{\Lambda}_T(i) = Z_T \Lambda_T(i)$$

is the real wage.

**Domestic Firms**. Domestic goods producing firms, $f \in [0,1]$, maximize

$$E_t \sum_{T=t}^{\infty} \xi_H^{T-t} Q_{t,T} \Gamma_{H,T}(f)$$

where

$$
\begin{aligned}
\Gamma_{H,T}(f) &= Y_{H,T}(f) \left( (1-\bar{\tau}_H) \frac{p}{P_T} \left( \frac{P_{H,T-1}}{P_{H,t-1}} \right)^{\iota_H} - \frac{W_T}{A_T Z_T P_T} \right) \\
Y_{H,T}(f) &= \left( \frac{p \left( P_{H,T-1}/P_{H,t-1} \right)^{\iota_H}}{P_{H,T}} \right)^{-\theta_{H,t}} Y_{H,T} \\
Y_{H,T}(f) &= A_T Z_T H_T(f)
\end{aligned}
$$

where

$$Y_{H,t} = C_{H,t} + C_{H,t}^* + G_t$$

defines total demand for domestic goods and $G_t$ government purchases and $Z_t$ a stochastic technology trend, and $A_t$ a stationary technology process

$$\log A_t = \rho_a \log A_{t-1} + \sigma_a \varepsilon_t^a$$

It follows that total real tax collections from domestic goods firms is

$$T_{H,T} = \bar{\tau}_H \int_0^1 Y_{H,t}(f) \frac{P_{H,t}(f)}{P_t} df.$$

and total dividends paid by all firms

$$\Gamma_{H,t} = \int_0^1 Y_{H,t}(f) \left( (1-\bar{\tau}_H) \frac{P_{H,t}(f)}{P_t} - \frac{W_t}{A_t Z_t P_t} \right) df$$

in period $t$. As in wage setting, firms not having the opportunity to adjust their price apply the indexation rule

$$P_{H,t}(f) = P_{H,t-1}(f) \pi_{H,t}^{i_H}.$$

These relations imply

$$\frac{\partial Y_{H,T}(f)}{\partial p} = -\theta_{H,t} p^{-\theta_H - 1} \left( \frac{(P_{H,T-1}/P_{H,t-1})^{\iota_H}}{P_{H,T}} \right)^{-\theta_{H,t}} Y_{H,T}.$$

The first-order condition for price setting is then

$$\begin{aligned}
0 &= E_t \sum_{T=t}^{\infty} \xi_H^{T-t} Q_{t,T} \left[ \begin{array}{c} -\theta_{H,t} p^{-\theta_H - 1} \left( \frac{(P_{H,T-1}/P_{H,t-1})^{\iota_H}}{P_{H,T}} \right)^{-\theta_{H,t}} Y_{H,T} \left( (1-\bar{\tau}_H) \frac{p}{P_T} \left( \frac{P_{H,T-1}}{P_{H,t-1}} \right)^{\iota_H} - \frac{W_T}{A_T Z_T P_T} \right) \\ +Y_{H,T}(f) \left( (1-\bar{\tau}_H) \frac{1}{P_T} \left( \frac{P_{H,T-1}}{P_{H,t-1}} \right)^{\iota_H} \right) \end{array} \right] \\
&= E_t \sum_{T=t}^{\infty} \xi_H^{T-t} Q_{t,T} \left[ \begin{array}{c} -\theta_{H,t} p^{-1} Y_{H,T}(f) \left( (1-\bar{\tau}_H) \frac{p}{P_T} \left( \frac{P_{H,T-1}}{P_{H,t-1}} \right)^{\iota_H} - \frac{W_T}{A_T Z_T P_T} \right) \\ +Y_{H,T}(f) \left( (1-\bar{\tau}_H) \frac{1}{P_T} \left( \frac{P_{H,T-1}}{P_{H,t-1}} \right)^{\iota_H} \right) \end{array} \right] \\
&= E_t \sum_{T=t}^{\infty} \xi_H^{T-t} Q_{t,T} Y_{H,T}(f) (1-\theta_{H,t}) \left[ (1-\bar{\tau}_H) \frac{p}{P_T} \left( \frac{P_{H,T-1}}{P_{H,t-1}} \right)^{\iota_H} - \frac{\theta_{H,t}}{\theta_{H,t}-1} \frac{W_T}{A_T Z_T P_T} \right] \\
&= E_t \sum_{T=t}^{\infty} \xi_H^{T-t} Q_{t,T} Y_{H,T}(f) (1-\theta_{H,t}) \frac{P_{H,T}}{P_T} \left[ (1-\bar{\tau}_H) \frac{p}{P_{H,T}} \left( \frac{P_{H,T-1}}{P_{H,t-1}} \right)^{\iota_H} - \frac{\theta_{H,t}}{\theta_{H,t}-1} \frac{W_T}{A_T Z_T P_T} \frac{P_T}{P_{H,T}} \right]
\end{aligned}$$

**Imported Goods Firms.** Imported goods firms $k \in [0,1]$ face demand curve

$$C_{F,T}(k) = \left( \frac{P_{F,t}(k)}{P_{F,T}} \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} \right)^{-\theta_{F,t}} C_{F,T} \tag{3}$$

for all $t$ and take aggregate prices and consumption bundles as given. Firms not having the opportunity to re-optimize apply the indexation rule

$$P_{F,t}(k) = P_{H,t-1}(f) \pi_{F,t}^{\iota_F}.$$

The firm's price setting problem in period $t$ is to maximize the expected present discounted value of profits

$$E_t \sum_{T=t}^{\infty} \xi_F^{T-t} Q_{t,T} C_{F,T}(k) \left[ (1-\bar{\tau}_F) \frac{P_{F,t}(k)}{P_T} \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} - \frac{S_T P_{F,T}^*}{P_T} \right]$$

subject to the demand curve (3). Total taxes from imported goods are therefore

$$T_{F,t} = \bar{\tau}_F \int_0^1 C_{F,t}(k) \frac{P_{F,t}(k)}{P_t} dk$$

The first-order condition is then

$$\begin{aligned}
0 &= E_t \sum_{T=t}^{\infty} \xi_F^{T-t} Q_{t,T} \left[ \begin{array}{c} \frac{\partial C_{F,T}(k)}{\partial P_{F,t}(k)} \left( (1-\bar{\tau}_F) \frac{P_{F,t}(k)}{P_T} \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} - \frac{S_T P_{F,T}^*}{P_T} \right) \\ + C_{F,T}(k) \frac{1}{P_T} (1-\bar{\tau}_F) \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} \end{array} \right] \\
&= E_t \sum_{T=t}^{\infty} \xi_F^{T-t} Q_{t,T} \left[ \begin{array}{c} -\theta_{F,t} P_{F,t}(k)^{-\theta_{F,t}-1} \left( \frac{\left( P_{F,T-1}/P_{F,t-1} \right)^{\iota_F}}{P_{F,T}} \right)^{-\theta_{F,t}} \left( (1-\bar{\tau}_F) \frac{P_{F,t}(k)}{P_T} \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} - \frac{S_T P_{F,T}^*}{P_T} \right) \\ + C_{F,T}(k) \frac{1}{P_T} (1-\bar{\tau}_F) \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} \end{array} \right] \\
&= E_t \sum_{T=t}^{\infty} \xi_F^{T-t} Q_{t,T} \left[ \theta_{F,t} C_{F,T}(k) P_{F,t}(k)^{-1} \frac{S_T P_{F,T}^*}{P_T} + (1-\theta_{F,t})(1-\bar{\tau}_F) \frac{1}{P_T} C_{F,T}(k) \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} \right] \\
&= E_t \sum_{T=t}^{\infty} \xi_F^{T-t} Q_{t,T} (1-\theta_F) C_{F,T}(k) P_{F,t}(k)^{-1} \left[ \frac{\theta_{F,t}}{1-\theta_{F,t}} \frac{S_T P_{F,T}^*}{P_T} + (1-\bar{\tau}_F) \frac{P_{F,t}(k)}{P_T} \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} \right] \\
&= E_t \sum_{T=t}^{\infty} \xi_F^{T-t} Q_{t,T} (1-\theta_F) C_{F,T}(k) P_{F,t}(k)^{-1} \left[ (1-\bar{\tau}_F) \frac{P_{F,t}(k)}{P_T} \left( \frac{P_{F,T-1}}{P_{F,t-1}} \right)^{\iota_F} - \frac{\theta_{F,t}}{\theta_{F,t}-1} \frac{S_T P_{F,T}^*}{P_T} \right].
\end{aligned}$$

## 2 Steady State

Look for a steady state in which taxes eliminate distortions from monopolistic competition with the following properties:

$$\begin{aligned}
i_t &= i_t^* = i; \quad \frac{P_t}{P_{t-1}} = \frac{S_t}{S_{t-1}} = \frac{S_t P_t^*}{P_t} = 1; \; B_t = B_t^* = 0; \; \Lambda_t Z_t = \Lambda \\
\frac{C_t}{Z_t} &= c; \quad \frac{C_{H,t}}{Z_t} = c_H; \quad \frac{C_{F,t}}{Z_t} = c_F; \; \frac{P_{H,t}}{P_t} = 1; \; \frac{P_{F,t}}{P_t} = 1; \quad \frac{Y_t}{Z_t} = y \\
\gamma_H &= \frac{\Gamma_{H,t}}{Z_t}; \; \tau_H = \frac{T_{H,t}}{Z_t}; \; \gamma_F = \frac{\Gamma_{F,t}}{Z_t}; \; \tau_F = \frac{T_{F,t}}{Z_t}; \quad \tau_t = \frac{T_t}{Z_t}; \; g_t = \frac{G_t}{Z_t}.
\end{aligned}$$

From the household first-order conditions we have

$$\begin{aligned}
(1-\beta b)(c-bc)^{-\sigma} &= \Lambda \\
P^B &= \gamma^{-1} \pi^{-1} \beta = (1+i)^{-1} \\
P^{B^*} &= \gamma^{-1} \pi^{-1} \beta = (1+i)^{-1}
\end{aligned}$$

From the domestic producing firms first order condition and definition of profits we have

$$\begin{aligned}(1-\bar{\tau}_H)\frac{P_H}{P} &= \frac{\theta}{\theta-1}w \\ \gamma_H &= y_H\left[(1-\bar{\tau}_H)-w\right]\end{aligned}$$

We choose the tax rate to eliminate the distortions from sticky prices in steady state. Hence

$$1-\bar{\tau}_H = \frac{\theta_p}{\theta_p-1}$$

or

$$\bar{\tau}_H = 1-\frac{\theta_p}{\theta_p-1}.$$

Using results from below we have

$$w = 1.$$

It follows that profits and taxes on domestic goods production are

$$\begin{aligned}\gamma_H &= -y_H\bar{\tau}_H \\ \tau_H &= y_H\bar{\tau}_H\end{aligned}$$

From foreign import goods producing firms we have

$$\gamma_F = c_F\left[(1-\bar{\tau}_F)\frac{P_F}{P}-\frac{SP_F^*}{P}\right]$$

and

$$(1-\bar{\tau}_F)\frac{P_F}{P} = \frac{\theta_F}{\theta_F-1}\frac{SP_F^*}{P}$$

Let's set the steady state subsidy to eliminate the markup so that

$$1-\bar{\tau}_F = \frac{\theta_F}{\theta_F-1}$$

or

$$\bar{\tau}_F = 1-\frac{\theta_F}{\theta_F-1} \tag{4}$$

Hence the law of one price will hold in steady state

$$\frac{SP_F^*}{P_F} = \frac{SP^*}{P}\frac{P}{P_F} = \frac{P}{P_F} = 1$$

and

$$\begin{aligned}\gamma_F &= c_F\left[(1-\bar{\tau}_F)\frac{P_F}{P}-\frac{SP_F^*}{P}\right] \\ &= c_F\left[(1-\bar{\tau}_F)-1\right] \\ &= -c_F\bar{\tau}_F\end{aligned}$$

and associate sales taxes

$$\tau_F = c_F \bar{\tau}_F$$

From the households' wage problem we have

$$(1 - \bar{\tau}_w)\,\tilde{\Lambda} = N^{\phi^{-1}} \frac{\theta_w}{\theta_w - 1}$$

where

$$1 - \bar{\tau}_w = \frac{\theta_w}{\theta_w - 1}$$

so that

$$\tilde{\Lambda} = N^{\phi^{-1}}$$

From the domestic price index

$$P_t = \left[\alpha P_{H,t}^{1-\eta} + (1-\alpha)\, P_{F,t}^{1-\eta}\right]^{\frac{1}{1-\eta}}.$$

we have

$$\begin{aligned} 1 &= \left[\alpha \left(\frac{P_H}{P_F}\right)^{1-\eta} + (1-\alpha)\right]^{\frac{1}{1-\eta}} \\ &= \left(\frac{P_H}{P_F}\right)^{1-\eta} \end{aligned}$$

so that

$$\frac{P_H}{P_F} = \frac{P_H}{P}\frac{P}{P_F} = \frac{P_H}{P} = 1$$

Given these steady-state taxes all relative prices are equal to unity in steady state and we have

$$y = \frac{P_H y_H}{P} = y_{H.}$$

From goods market clear for the domestic good

$$Y_{H,t} = C_{H,t} + C_{H,t}^{*} + G_t$$

implies

$$\frac{Y_{H,t}}{Z_t} = \frac{C_{H,t}}{Z_t} + \frac{C_{H,t}^{*}}{Z_t^{*}}\frac{Z_t^{*}}{Z_t} + \frac{G_t}{Z_t}$$

or

$$y_H = c_H + c_H^{*} + g$$

And

$$C_{H,t}^{*} = \alpha^{*} \left(\frac{P_{H,t}^{*}}{P_t^{*}}\right)^{-\eta^{*}} Y_t^{*}$$

which implies

$$c_H^{*} = \alpha^{*} y^{*}$$

so that

$$
\begin{aligned}
1 &= \frac{c_H}{y} + \frac{a^* y^*}{y} + \frac{g}{y} \\
&= \frac{c_H}{y} + \frac{c_F}{y} + \frac{g}{y}
\end{aligned}
$$

where

$$
\frac{a^* y^*}{y} = \frac{c_F}{y}
$$

is steady state exports and the equality following with balanced trade.

## 3 Log-linear Approximation

Take a log-linear approximation in a neighborhood in which $W_t(j)/W_t$, $P_t/P_{t-1}$ and $w_t^r$ remain close to their steady state values 1, 1, and $\bar{w}$. Define

$$
\begin{aligned}
\hat{w}_t^r &= \log\left(\frac{w_t^r}{\bar{w}}\right); \; \hat{w}_t^* = \log\left(\frac{W_t(j)}{W_t}\right); \; \pi_t = \log\left(\frac{P_t}{P_{t-1}}\right); \\
\pi_t^w &= \log\left(\frac{W_t}{W_{t-1}}\bar{\gamma}^{-1}\right); \; \hat{\gamma}_{z,t} = \log\left(\frac{Z_t}{Z_{t-1}\bar{\gamma}}\right); \; \hat{\theta}_{w,t} = \log\left(\frac{\theta_{w,t}}{\theta_w}\right); \; a_t = \log A_t
\end{aligned}
$$

and

$$
\gamma_{H,t} = \frac{\Gamma_{H,t}}{Z_t}; \; \tau_{H,t} = \frac{T_{H,t}}{Z_t}; \; \gamma_{F,t} = \frac{\Gamma_{F,t}}{Z_t}; \; \tau_{F,t} = \frac{T_{F,t}}{Z_t}; \;\; \tau_t = \frac{T_t}{Z_t}; \; g_t = \frac{G_t}{Z_t}
$$

and

$$
\tilde{\gamma}_{H,t} = \frac{\gamma_{H,t} - \gamma_H}{y}; \tilde{\tau}_{H,t} = \frac{\tau_{H,t} - \tau_H}{y} \text{ and so on.}
$$

**Households.** The first-order condition can be written

$$
\begin{aligned}
0 &= E_t^j \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} \left(1 - \theta_w e^{\hat{\theta}_{w,t}}\right) \bar{w} e^{\hat{w}_T^r} \times \\ \left\{ (1-\tau_w) \Lambda e^{\hat{\lambda}_T(j)} e^{\hat{w}_t^*} e^{\sum_{\tau=t}^{T-1}\left(\hat{\gamma}_\tau + \iota_w \pi_\tau - \pi_{\tau+1}^w\right)} - \varphi e^{\hat{\varphi}_T} N^{\phi^{-1}} e^{\phi^{-1}\hat{N}_T(j)} \frac{\theta_w e^{\hat{\theta}_{w,T}}}{\theta_w e^{\hat{\theta}_{w,T}} - 1} \bar{w}^{-1} e^{-\hat{w}_T^r} \right\} \end{array} \right] \\
&= E_t^j \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} (1-\tau_w)\Lambda\left(\hat{\lambda}_T(j) + \hat{w}_t^* + \sum_{\tau=t}^{T-1}\left(\hat{\gamma}_\tau^z + \iota_w \pi_\tau - \pi_{\tau+1}^w\right)\right) \\ -\varphi N^{\phi^{-1}} \bar{w}^{-1} \frac{\theta_w}{\theta_w - 1}\left(\hat{\varphi}_T + \phi^{-1}\hat{N}_T(j) - \hat{w}_T^r\right) - \varphi N^{\phi^{-1}} \bar{w}^{-1}\left(\frac{\theta_w \hat{\theta}_{w,T}}{\theta_w - 1} - \frac{\theta_w}{(\theta_w-1)^2}\theta_w \hat{\theta}_{w,T}\right) \end{array} \right] \\
&= E_t^j \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} \left(\hat{\lambda}_T(j) + \hat{w}_t^* + \sum_{\tau=t}^{T-1}\left(\hat{\gamma}_\tau^z + \iota_w \pi_\tau - \pi_{\tau+1}^w\right)\right) \\ -\frac{\varphi N^{\phi^{-1}} \bar{w}^{-1}}{(1-\tau_w)\Lambda} \frac{\theta_w}{\theta_w - 1}\left(\hat{\varphi}_T + \phi^{-1}\hat{N}_T(j) - \hat{w}_T^r\right) - \frac{\varphi N^{\phi^{-1}} \bar{w}^{-1}}{(1-\tau_w)\Lambda}\left(-\frac{\theta_w \hat{\theta}_{w,T}}{(\theta_w-1)^2}\right) \end{array} \right] \\
&= E_t^j \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \left(\hat{\lambda}_T(j) + w_t^* + \sum_{\tau=t}^{T-1}\left(\hat{\gamma}_\tau^z + \iota_w \pi_\tau - \pi_{\tau+1}^w\right)\right) - \left(\hat{\varphi}_T + \phi^{-1}\hat{N}_T(j) - \hat{w}_T^r\right) + \frac{\hat{\theta}_{w,T}}{\theta_w - 1} \right].
\end{aligned}
$$

From the labor demand equation we have

$$
\hat{N}_T(j) = N_T - \theta_w \left(\hat{w}_t^* + \sum_{\tau=t}^{T-1}\left(\hat{\gamma}_{z,\tau} + \gamma_w \pi_\tau - \pi_{\tau+1}^w\right)\right)
$$

which permits writing the first-order condition as

$$\begin{aligned} 0 &= E_t^j \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} \left(\hat{\lambda}_T(j) + \hat{w}_t^* + \sum_{\tau=t}^{T-1} \left(\hat{\gamma}_{z,\tau} + \iota_w \pi_\tau - \pi_{\tau+1}^w\right)\right) \\ -\left(\hat{\varphi}_T + \phi^{-1}\left(N_T - \theta_w\left(\hat{w}_t^* + \sum_{\tau=t}^{T-1}\left(\hat{\gamma}_{z,\tau} + \iota_w \pi_\tau - \pi_{\tau+1}^w\right)\right)\right) - \hat{w}_T^r\right) + \frac{\hat{\theta}_{w,T}}{\theta_w - 1} \end{array} \right] \\ &= E_t^j \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} \left(\hat{\lambda}_T(j) + (1+\theta_w\phi^{-1})\left(\hat{w}_t^* + \sum_{\tau=t}^{T-1}\left(\hat{\gamma}_{z,\tau} + \iota_w \pi_\tau - \pi_{\tau+1}^w\right)\right)\right) \\ -\left(\hat{\varphi}_T + \phi^{-1} N_T - \hat{w}_T^r\right) + \frac{\hat{\theta}_{w,T}}{\theta_w - 1} \end{array} \right] \end{aligned}$$

Rearranging provides

$$\begin{aligned} \frac{(1+\theta_w\phi^{-1})}{1-\xi_w\beta}\hat{w}_t^* &= E_t^j \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} -(1+\theta_w\phi^{-1}) \sum_{\tau=t}^{T-1}\left(\hat{\gamma}_{z,\tau} + \iota_w \pi_\tau - \pi_{\tau+1}^w\right) \\ +\left(\hat{\varphi}_T + \phi^{-1} N_T - \hat{w}_T^r - \hat{\lambda}_T(j)\right) - \frac{\hat{\theta}_{w,T}}{\theta_w - 1} \end{array} \right] \\ &= E_t^j \sum_{T=t}^{\infty} (\xi_\omega \beta)^{T-t} \left[ \begin{array}{c} \frac{(1+\theta_w\phi^{-1})\xi_w\beta}{1-\xi_w\beta}\left(\pi_{T+1}^w - \hat{\gamma}_{z,T} - \iota_w \pi_T\right) \\ +\left(\hat{\varphi}_T + \phi^{-1} N_T - \hat{w}_T^r - \hat{\lambda}_T(j)\right) - \frac{\hat{\theta}_{w,T}}{\theta_w - 1} \end{array} \right] \\ &= \frac{(1+\theta_w\phi^{-1})\xi_w\beta}{1-\xi_w\beta} E_t^j\left(\pi_{t+1}^w - \hat{\gamma}_{z,t} - \iota_w\pi_t\right) + \left(\hat{\varphi}_t + \phi^{-1}N_t - \hat{w}_t^r - \hat{\lambda}_t(j)\right) - \frac{\hat{\theta}_{w,t}}{\theta_w - 1} \\ &\quad + \xi_w\beta\frac{(1+\theta_w\phi^{-1})}{1-\xi_w\beta} E_t^j \hat{w}_{t+1}^* \end{aligned}$$

or

$$\hat{w}^*(j) = \xi_w\beta E_t^j\left(\pi_{t+1}^w - \hat{\gamma}_{z,t} - \iota_w\pi_t\right) + \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_t + \phi^{-1}N_t - \hat{w}_t^r - \hat{\lambda}_t(j) - \frac{\hat{\theta}_{w,t}}{\theta_w - 1}\right] + \xi_w\beta E_t^j \hat{w}_{t+1}^*(j)$$

Note that the wage index satisfies the approximation

$$\hat{w}_t^* = \frac{\xi_w}{1-\xi_w}\left(\pi_t^w - \iota_w\pi_{t-1} - \hat{\gamma}_{z,t-1}\right)$$

Log-linearizing the remaining three first-order conditions gives

$$(1-\beta b)\hat{\lambda}_t(i) = \hat{\xi}_t + \frac{b\sigma}{1-b}\hat{c}_{t-1}(i) - \frac{\sigma}{1-b}\left(1+\beta b^2\right)\hat{c}_t(i) - \beta b E_t \hat{\xi}_{t+1} + \beta b \frac{\sigma}{1-b} E_t \hat{c}_{t+1}(i).$$

The Euler equation for domestic bond holdings provides

$$\tilde{\Lambda}_t(i) P_t^B = E_t^i\left[\tilde{\Lambda}_{t+1}(i)\gamma_{z,t+1}^{-1}\beta\frac{P_t}{P_{t+1}}\right] - \tilde{\Lambda}_t(i)\Phi'\left(\frac{B_t}{Z_tP_t} - \frac{B_{t-1}}{Z_{t-1}P_{t-1}}\right) - \beta E_t^i\left[\tilde{\Lambda}_{t+1}(i)\Phi'\left(\frac{B_{t+1}}{Z_{t+1}P_{t+1}} - \frac{B_t}{Z_tP_t}\right)\right]$$

in normalized terms. Letting $b_t = B_t/(Z_tP_t)$ and

$$\tilde{b}_t = \frac{b_t - b}{y}$$

we have

$$\Lambda e^{\hat{\lambda}_t(i)} P^B e^{\hat{P}_t^B} = \beta E_t \left[ \Lambda e^{\hat{\lambda}_{t+1}(i)} \bar{\gamma} e^{-\hat{\gamma}_{z,t}} e^{-\pi_{t+1}} \right] - \Lambda e^{\hat{\lambda}_t(i)} \Phi' \left( b_t - b_{t-1} \right) - \beta E_t^i \left[ \Lambda e^{\hat{\lambda}_{t+1}(i)} \Phi' \left( b_{t+1} - b_t \right) \right]$$

Then

$$\begin{aligned} LHS &= \Lambda P^B \left( \hat{\lambda}_t(i) + \hat{P}_t^B \right) \\ RHS &= \beta \bar{\Lambda} \bar{\gamma} E_t^i \left( \hat{\lambda}_{t+1}(i) - \hat{\gamma}_{z,t} - \pi_{t+1} \right) - \Lambda \left( \hat{\lambda}_t(i) \Phi'(0) + \Phi''(0) Y \left( \tilde{b}_t - \tilde{b}_{t-1} \right) \right) \\ &\quad - \beta \Lambda E_t^i \left[ \Phi'(0) \hat{\lambda}_{t+1}(i) + \Phi''(0) Y \left( \tilde{b}_{t+1} - \tilde{b}_t \right) \right] \\ &= \beta \bar{\Lambda} \bar{\gamma} E_t^i \left( \hat{\lambda}_{t+1}(i) - \hat{\gamma}_{z,t} - \pi_{t+1} \right) - \Lambda \Phi''(0) Y \left( \tilde{b}_t - \tilde{b}_{t-1} \right) - \beta \Lambda \Phi''(0) Y E_t^i \left( \tilde{b}_{t+1} - \tilde{b}_t \right) \end{aligned}$$

or rearranging

$$\begin{aligned} \hat{\lambda}_t(i) + \hat{P}_t^B &= E_t^i \left( \hat{\lambda}_{t+1}(i) - \hat{\gamma}_{z,t} - \pi_{t+1} \right) - \frac{\Lambda \Phi''(0) Y}{P^B} \left( \tilde{b}_t - \tilde{b}_{t-1} \right) - \beta \frac{\Lambda \Phi''(0) Y}{P^B} E_t^i \left( \tilde{b}_{t+1} - \tilde{b}_t \right) \\ &= E_t^i \left( \hat{\lambda}_{t+1}(i) - \hat{\gamma}_{z,t} - \pi_{t+1} \right) - \chi \left[ (1+\beta) \tilde{b}_t - \tilde{b}_{t-1} - \beta E_t^i \tilde{b}_{t+1} \right] \end{aligned}$$

where

$$\chi = \frac{\Lambda \Phi''(0) Y}{P^B}$$

The Euler equation for foreign bond holding is

$$\begin{aligned} \Lambda_t(i) P_t^B &= E_t^i \left[ \Lambda_{t+1}(i) \beta \frac{P_t}{P_{t+1}} \frac{S_{t+1}}{S_t} \right] - \Lambda_t(i) \Phi^{*\prime} \left( \frac{S_t B_t^*}{Z_t P_t} - \frac{S_{t-1} B_{t-1}^*}{Z_{t-1} P_{t-1}} \right) \\ &\quad - \beta E_t^i \left[ \Lambda_{t+1}(i) \frac{Z_{t+1}}{Z_t} \Phi^{*\prime} \left( \frac{S_{t+1} B_{t+1}^*}{Z_{t+1} P_{t+1}} - \frac{S_t B_t^*}{Z_{t+1} P_t} \right) \right] \end{aligned}$$

which in normalized variables provides

$$\begin{aligned} \tilde{\Lambda}_t(i) P_t^{B^*} &= E_t^i \left[ \tilde{\Lambda}_{t+1}(i) \gamma_{z,t}^{-1} \beta \frac{P_t}{P_{t+1}} \frac{S_{t+1}}{S_t} \right] - \tilde{\Lambda}_t(i) \Phi^{*\prime} \left( \frac{S_t B_t^*}{Z_t P_t} - \frac{S_{t-1} B_{t-1}^*}{Z_{t-1} P_{t-1}} \right) \\ &\quad - \beta E_t^i \left[ \tilde{\Lambda}_{t+1}(i) \Phi^{*\prime} \left( \frac{S_{t+1} B_{t+1}^*}{Z_{t+1} P_{t+1}} - \frac{S_t B_t^*}{Z_t P_t} \right) \right] \end{aligned}$$

Let

$$b_t^* = \frac{S_t B_t^*}{Z_t P_t}$$

be the real value of normalized domestic currency face value of foreign debt holdings. Hence we have

$$\begin{aligned} \hat{\lambda}_t(i) + \hat{P}_t^{B^*} &= E_t^i \left( \hat{\lambda}_{t+1}(i) - \hat{\gamma}_{z,t} - \pi_{t+1} + \log S_{t+1} - \log S_t \right) \\ &\quad - \frac{\Lambda {\Phi^*}''(0) Y}{P^{B^8}} \left( \tilde{b}_t^*(i) - \tilde{b}_{t-1}^*(i) \right) - \beta \frac{\Lambda \Phi^{*\prime\prime}(0) Y}{P^{B^*}} E_t^i \left( \tilde{b}_{t+1}^*(i) - \tilde{b}_t^*(i) \right) \\ &= E_t^i \left( \hat{\lambda}_{t+1}(i) - \hat{\gamma}_{z,t} - \pi_{t+1} + \log S_{t+1} - \log S_t \right) - \chi^* \left[ (1+\beta) \tilde{b}_t^*(i) - \tilde{b}_{t-1}^*(i) - \beta E_t^i \tilde{b}_{t+1}^*(i) \right]. \end{aligned}$$

Approximating the overall demand for domestic and imported goods

$$C_{H,t} = \alpha \left( \frac{P_{H,t}}{P_t} \right)^{-\eta} C_t \text{ and } C_{F,t} = (1-\alpha) \left( \frac{P_{F,t}}{P_t} \right)^{-\eta} C_t$$

gives

$$\hat{c}_{H,t} = \hat{c}_t - \eta \hat{p}_{H,t} \text{ and } \hat{c}_{F,t} = \hat{c}_t - \eta \hat{p}_{F,t}.$$

And the price index

$$P_t = \left[ (1-\alpha) P_{H,t}^{1-\eta} + \alpha P_{F,t}^{1-\eta} \right]^{\frac{1}{1-\eta}}$$

provides

$$1 = \left[ \sigma \left( \frac{P_{H,t}}{P_t} \right)^{1-\eta} + (1-\alpha) \left( \frac{P_{F,t}}{P_t} \right)^{1-\eta} \right]$$

or

$$0 = \alpha (1-\eta) \hat{p}_{H,t} + (1-\alpha)(1-\eta) \hat{p}_{F,t}$$

or

$$\hat{p}_{H,t} = -\frac{(1-\alpha)}{\alpha} \hat{p}_{F,t}.$$

A question will arise later about what variables households are forecasting. To keep things simple, assume households forecast domestic value added, $\hat{y}_{H,t}$ and international prices given by the real and nominal exchange rate, $q_t$ and $\log S_t$, as well as the terms of trade, $\log(P_{F,t}/P_{H,t})$. Note that movements in $\hat{p}_{F,t}$ and $\hat{p}_{H,t}$ are proportional to the terms of trade since

$$\log \left( \frac{P_{F,t}}{P_{H,t}} \right) = \hat{p}_{F,t} - \hat{p}_{H,t}$$

and

$$\hat{p}_{H,t} = -\frac{1-\alpha}{\alpha} \hat{p}_{F,t}$$

so that

$$\begin{aligned} \log \left( \frac{P_{F,t}}{P_{H,t}} \right) &= \hat{p}_{F,t} + \frac{1-\alpha}{\alpha} \hat{p}_{F,t} \\ &= \frac{1}{\alpha} \hat{p}_{F,t} \\ &= -\frac{1}{1-\alpha} \hat{p}_{H,t} \end{aligned}$$

or

$$\begin{aligned} \hat{p}_{F,t} &= \alpha \log \left( \frac{P_{F,t}}{P_{H,t}} \right) \\ \hat{p}_{H,t} &= -(1-\alpha) \log \left( \frac{P_{F,t}}{P_{H,t}} \right). \end{aligned}$$

Goods market clearing for any good $j$ requires

$$Y_{H,t}(j) = C_{H,t}(j) + C^*_{H,t}(j) + G(j)$$

which satisfies the approximation

$$\hat{y}_{H,t}(j) = \frac{c_H}{y}\hat{c}_{H,t}(j) + \frac{c_H^*}{y}\hat{c}_{H,t}^*(j) + \frac{c_H^*}{y}\log\left(\frac{Z_t^*}{Z_t}\right) + \tilde{g}_t(j)$$

for each good $j$ assuming different trend technology, and

$$\begin{aligned}\hat{y}_{H,t} &= \frac{c_H}{y}\hat{c}_{H,t} + \frac{c_H^*}{y}\hat{c}_{H,t}^* + \frac{c_H^*}{y}\log\left(\frac{Z_t^*}{Z_t}\right) + \tilde{g}_t \\ &= \frac{c_H}{y}\hat{c}_{H,t} + \frac{c_F}{y}\hat{c}_{H,t}^* + \frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right) + \tilde{g}_t\end{aligned}$$

in the aggregate, where the second equality holds if trade is balanced in steady state. Using the demand functions

$$C_{H,t} = \alpha\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t \text{ and } C_{H,t}^* = \alpha^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta^*} Y_t^*$$

we have

$$\begin{aligned}\hat{y}_{H,t} &= \frac{c_H}{y}\left(\hat{c}_t - \eta\hat{p}_{H,t}\right) + \frac{c_F}{y}\left(\hat{c}_t^* - \eta^*\hat{p}_{H,t}^*\right) + \frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right) + \tilde{g}_t \\ &= \frac{c_H}{y}\left(\hat{c}_t - \eta\hat{p}_{H,t}\right) + \frac{c_F}{y}\left(\hat{c}_t^* - \eta^*\hat{p}_{H,t}^*\right) + \frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right) + \tilde{g}_t \\ &= \frac{c_H}{y}\left(\hat{c}_t - \eta\alpha\log\left(\frac{P_{H,t}}{P_{F,t}}\right)\right) + \frac{c_F}{y}\left(\left(y_t^* - \eta^*\left(\log\left(\frac{P_{H,t}}{P_{F,t}}\right) - \psi_{F,t}\right)\right) + \log\left(\frac{Z_t^*}{Z_t}\right)\right) + \tilde{g}_t\end{aligned}$$

Finally, we need to approximate the flow budget constraint

$$\begin{aligned}& C_t(i) + \frac{P_t^B B_t(i)}{P_t} + S_t\left(\frac{P_t^{B^*} B_t^*(i)}{P_t}\right) + Z_t\Phi\left(\frac{B_t}{Z_tP_t} - \frac{B_{t-1}}{Z_{t-1}P_{t-1}}\right) + Z_t\Phi^*\left(\frac{S_tB_t^*}{Z_tP_t} - \frac{S_{t-1}B_{t-1}^*}{Z_{t-1}P_{t-1}}\right) \\ \leq\ & \frac{B_{t-1}}{P_t} + S_t\frac{B_{t-1}^*}{P_t} + (1-\tau_w)\int_0^1 \frac{W_t(j)}{P_t}N_t(i,j)\,dj + \Gamma_{H,t} + \Gamma_{F,t} - T_{H,t} + T_{w,t} + T_{H,t} + T_{F,t}.\end{aligned}$$

Writing in normalized terms

$$\begin{aligned}& \frac{C_t(i)}{Z_t} + \frac{P_t^B B_t(i)}{Z_tP_t} + P_t^{B^*}\left(\frac{S_tB_t^*(i)}{Z_tP_t}\right) + \Phi\left(\frac{B_t}{Z_tP_t} - \frac{B_{t-1}}{Z_{t-1}P_{t-1}}\right) + \Phi^*\left(\frac{S_tB_t^*}{Z_tP_t} - \frac{S_{t-1}B_{t-1}^*}{Z_{t-1}P_{t-1}}\right) \\ \leq\ & \frac{B_{t-1}}{Z_tP_t} + \frac{S_tB_{t-1}^*}{Z_tP_t} + (1-\tau_w)\int_0^1 \frac{W_t(j)}{P_tZ_t}N_t(i,j)\,dj + \frac{\Gamma_{H,t}}{Z_t} + \frac{\Gamma_{F,t}}{Z_t} - \frac{T_{H,t}}{Z_t} + \frac{T_{w,t}}{Z_t} + \frac{T_{H,t}}{Z_t} + \frac{T_{F,t}}{Z_t}.\end{aligned}$$

or

$$\begin{aligned}& c_t(i) + P_t^B b_t + P_t^{B^*} b_t^* + \Phi\left(b_t - b_{t-1}\right) + \Phi^*\left(b_t^* - b_{t-1}^*\right) \\ \leq\ & \frac{Z_{t-1}}{Z_t}\frac{P_{t-1}}{P_t}b_{t-1} + \frac{S_t}{S_{t-1}}\frac{P_{t-1}}{P_t}\frac{Z_{t-1}}{Z_t}b_{t-1}^* + (1-\tau_w)\int_0^1 \frac{W_t(i,j)}{P_tZ_t}N_t(i,j)\,dj \\ +\ & \frac{\Gamma_{H,t}}{Z_t} + \frac{\Gamma_{F,t}}{Z_t} - \frac{T_t}{Z_t} + \frac{T_{w,t}}{Z_t} + \frac{T_{H,t}}{Z_t} + \frac{T_{F,t}}{Z_t}.\end{aligned}$$

Taking a log-linear approximation gives

$$LHS = c\hat{c}_t(i) + P^B b\hat{P}_t^B + P^B(b_t - b) + P^{B^*}b^*\hat{P}_t^{B^*} + P^{B^*}(b_t^* - b)$$

and

$$\begin{aligned}
RHS &= \gamma_z^{-1}(b_{t-1} - b) - \gamma_z^{-1}b(\gamma_{z,t} + \pi_t) + \gamma_z^{-1}\left(b_{t-1}^* - b^*\right) - \gamma_z^{-1}b^*(\gamma_{z,t} + \pi_t) + \gamma_z^{-1}b^*(\log(S_t) - \log S_{t-1}) \\
&\quad + (1 - \tau_w)w^r N\int_0^1 \left(\hat{w}_t^r(j) + \hat{N}_t(j)\right)dj \\
&\quad + (\gamma_{H,t} - \gamma_H) + (\gamma_{F,t} - \gamma_F) - (\tau_t - \tau) + (\tau_{w.t} - \tau_w) + (\tau_{H,t} - \tau_H) + (\tau_{F,t} - \tau_F) \\
&= \gamma_z^{-1}(b_{t-1} - b) - \gamma_z^{-1}b(\gamma_{z,t} + \pi_t) + \gamma_z^{-1}\left(b_{t-1}^* - b^*\right) - \gamma_z^{-1}b^*(\gamma_{z,t} + \pi_t) + \gamma_z^{-1}b^*(\log(S_t) - \log S_{t-1}) \\
&\quad + (1 - \tau_w)w^r N\left((1 - \theta_w)\int_0^1 \hat{w}_t^r(i)\,di + \theta_w\hat{w}_t^r + \hat{N}_t\right) \\
&\quad + (\gamma_{H,t} - \gamma_H) + (\gamma_{F,t} - \gamma_F) - (\tau_t - \tau) + (\tau_{w.t} - \tau_w) + (\tau_{H,t} - \tau_H) + (\tau_{F,t} - \tau_F) \\
&= \gamma_z^{-1}(b_{t-1} - b) - \gamma_z^{-1}b(\gamma_{z,t} + \pi_t) + \gamma_z^{-1}\left(b_{t-1}^* - b^*\right) - \gamma_z^{-1}b^*(\gamma_{z,t} + \pi_t) + \gamma_z^{-1}b^*(\log(S_t) - \log S_{t-1}) \\
&\quad + (1 - \tau_w)w^r N\left(\hat{w}_t^r + \hat{N}_t\right) \\
&\quad + (\gamma_{H,t} - \gamma_H) + (\gamma_{F,t} - \gamma_F) - (\tau_t - \tau) + (\tau_{w.t} - \tau_w) + (\tau_{H,t} - \tau_H) + (\tau_{F,t} - \tau_F)
\end{aligned}$$

This permits

$$\begin{aligned}
&\frac{c}{y}\hat{c}_t(i) + \frac{P^B b}{y}\hat{P}_t^B + P^B\left(\frac{b_t - b}{y}\right) + \frac{P^{B^*}b^*}{y}\hat{P}_t^{B^*} + P^{B^*}\left(\frac{b_t^* - b}{y}\right) \\
= \; &\gamma_z^{-1}\left(\frac{b_{t-1} - b}{y}\right) - \gamma_z^{-1}\frac{b(\gamma_{z,t} + \pi_t)}{y} + \gamma_z^{-1}\left(\frac{b_{t-1}^* - b^*}{y}\right) - \gamma_z^{-1}\frac{b^*}{y}(\gamma_{z,t} + \pi_t) + \gamma_z^{-1}\frac{b^*}{y}(\log(S_t) - \log S_{t-1}) \\
&+ (1 - \tau_w)\frac{w^r N}{y}\left(\hat{w}_t^r + \hat{N}_t\right) \\
&+ \frac{(\gamma_{H,t} - \gamma_H)}{y} + (\frac{\gamma_{F,t} - \gamma_F}{y}) - \left(\frac{\tau_t - \tau}{y}\right) + \left(\frac{\tau_{w.t} - \tau_w}{y}\right) + \left(\frac{\tau_{H,t} - \tau_H}{y}\right) + \left(\frac{\tau_{F,t} - \tau_F}{y}\right)
\end{aligned}$$

or

$$\begin{aligned}
&\frac{c}{y}\hat{c}_t(i) + \frac{P^B b}{y}\hat{P}_t^B + P^B\tilde{b}_t(i) + \frac{P^{B^*}b^*}{y}\hat{P}_t^{B^*} + P^{B^*}\tilde{b}_t^*(i) \\
= \; &\gamma_z^{-1}\tilde{b}_{t-1}(i) - \gamma_z^{-1}\frac{b}{c}(\gamma_{z,t} + \pi_t) + \gamma_z^{-1}\tilde{b}_{t-1}^*(i) - \gamma_z^{-1}\frac{b^*}{c}(\gamma_{z,t} + \pi_t) + \gamma_z^{-1}\frac{b^*}{y}(\log(S_t) - \log S_{t-1}) \\
&+ (1 - \tau_w)\frac{w^r N}{y}\left(\hat{w}_t^r + \hat{N}_t\right) + \tilde{\gamma}_{H,t} + \tilde{\gamma}_{F,t} - \tilde{\tau}_t + \tilde{\tau}_{w,t} + \tilde{\tau}_{H,t} + \tilde{\tau}_{F,t}
\end{aligned}$$

We can therefore write, because

$$P^B = \gamma_z^{-1}\beta,$$

as

$$\begin{aligned}
P^{B}\tilde{b}_{t}(i)+P^{B^{*}}\tilde{b}_{t}^{*}(i) &= \beta^{-1}\left(P^{B}\tilde{b}_{t-1}(i)+P^{B}\tilde{b}_{t-1}^{*}(i)\right)-\frac{c}{y}\hat{c}_{t}(i)\\
&-\frac{P^{B^{*}}b^{*}}{y}\hat{P}_{t}^{B^{*}}-\frac{P^{B}b}{y}\hat{P}_{t}^{B}+\gamma_{z}^{-1}\frac{b^{*}}{y}\left(\log\left(S_{t}\right)-\log S_{t-1}\right)\\
&+\left(1-\tau_{w}\right)\frac{w^{r}N}{y}\left(\hat{w}_{t}^{r}+\hat{N}_{t}\right)-\gamma_{z}^{-1}\frac{b}{y}\left(\gamma_{z,t}+\pi_{t}\right)-\gamma_{z}^{-1}\frac{b^{*}}{y}\left(\gamma_{z,t}+\pi_{t}\right)\\
&+\tilde{\gamma}_{H,t}+\tilde{\gamma}_{F,t}-\tilde{\tau}_{t}+\tilde{\tau}_{t}^{w}+\tilde{\tau}_{H,t}+\tilde{\tau}_{F,t}
\end{aligned}$$

Taxes and dividends are

$$T_{T}^{p}=\bar{\tau}_{H}\int_{0}^{1}Y_{H,t}(f)\,\frac{P_{H,t}(f)}{P_{t}}df.$$

$$\Gamma_{H,t}=\int_{0}^{1}Y_{H,t}(f)\left(\left(1-\bar{\tau}_{H}\right)\frac{P_{H,t}(f)}{P_{t}}-\frac{W_{t}}{A_{t}Z_{t}P_{t}}\right)df$$

Total disposable wage income and income tax are

$$\left(1-\tau_{w}\right)w_{t}^{r}N_{t}\text{ and }T_{w,t}$$

Hence

$$\frac{\Gamma_{H,t}+T_{H,T}}{Z_{t}}+\left(1-\tau_{w}\right)w_{t}^{r}N_{t}+T_{w,t}=\int_{0}^{1}\frac{Y_{H,t}(f)}{Z_{t}}\frac{P_{H,t}(f)}{P_{t}}df$$

so that

$$\begin{aligned}
&\tilde{\gamma}_{H,t}+\tilde{\tau}_{H,t}+\left(1-\tau_{w}\right)\frac{w^{r}N}{y}\left(\hat{w}_{t}^{r}+\hat{N}_{t}\right)+\tilde{\tau}_{w,t}\\
=\;&\frac{y_{H}}{y}\left[\hat{p}_{H,t}+\hat{y}_{Ht}\right]=\frac{y_{H}}{y}\left[\hat{p}_{H,t}+\frac{c_{H}}{y_{H}}\hat{c}_{H,t}+\frac{c_{F}}{y_{H}}\hat{c}_{H,t}^{*}+\tilde{g}_{t}\right]
\end{aligned}\tag{5}$$

And

$$\begin{aligned}
T_{F,t} &= \bar{\tau}_{F}\int_{0}^{1}C_{F,t}(k)\,\frac{P_{F,t}(k)}{P_{t}}dk\\
\Gamma_{F,t} &= C_{F,T}(k)\left[\left(1-\bar{\tau}_{F}\right)\frac{P_{F,t}(k)}{P_{t}}-\frac{S_{t}P_{F,t}^{*}}{P_{t}}\right]
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\Gamma_{F,t}+T_{F,T}}{Z_{t}} &= \int_{0}^{1}\frac{C_{F,T}(k)}{Z_{t}}\left[\frac{P_{F,t}(k)}{P_{t}}-\frac{S_{t}P_{F,t}^{*}}{P_{t}}\right]dk\\
&= \int_{0}^{1}\frac{C_{F,T}(k)}{Z_{t}}\left[\frac{P_{F,t}(k)}{P_{F,t}}-\frac{S_{t}P_{F,t}^{*}}{P_{F,t}}\right]\frac{P_{F,t}}{P_{t}}dk
\end{aligned}$$

where

$$\Psi_{F,t}\equiv\frac{S_{t}P_{F,t}^{*}}{P_{F,t}}$$

is the law of one price gap. Therefore we have

$$\begin{aligned}\tilde{\gamma}_{F,t}+\tilde{\tau}_{F,t} &= -\frac{c_F}{y}\psi_{F,t} \\ &= -\frac{c_F}{y}\left[q_t-\hat{p}_{F,t}\right]\end{aligned}$$

$$\tilde{\gamma}_{H,t}+\tilde{\tau}_{H,t}=\frac{y_H}{y}\left[\hat{p}_{H,t}-\hat{w}_t^r-a_t\right] \tag{6}$$

Finally

$$\tilde{T}_t \equiv \frac{T_t}{Z_t}=\frac{P_{H,t}}{P_t}\frac{G_t}{Z_t}$$

so that

$$\tilde{\tau}_t=\frac{\tilde{T}_t-\tilde{T}}{y}=\frac{g}{y}\hat{p}_{H,t}+\tilde{g}_t$$

Substituting into the FBC gives

$$\begin{aligned}P^B\tilde{b}_t(i)+P^{B^*}\tilde{b}_t^*(i) &= \beta^{-1}\left(P^B\tilde{b}_{t-1}(i)+P^B\tilde{b}_{t-1}^*(i)\right)-\frac{c}{y}\hat{c}_t(i) \\ &\quad -\frac{P^{B^*}b^*}{y}\hat{P}_t^{B^*}-\frac{P^Bb}{y}\hat{P}_t^B+\gamma_z^{-1}\frac{b^*}{y}\left(\log\left(S_t\right)-\log S_{t-1}\right) \\ &\quad -\gamma_z^{-1}\frac{b}{y}\left(\gamma_{z,t}+\pi_t\right)-\gamma_z^{-1}\frac{b^*}{y}\left(\gamma_{z,t}+\pi_t\right) \\ &\quad +\left[\hat{p}_{H,t}+\frac{c_H}{y_H}\hat{c}_{H,t}+\frac{c_F}{y_H}\hat{c}_{H,t}^*+\tilde{g}_t\right]-\tilde{g}_t-\frac{g}{y}\hat{p}_{Ht}-\frac{c_F}{y}\left[q_t-\hat{p}_{F.t}\right] \\ &= \beta^{-1}\left(P^B\tilde{b}_{t-1}(i)+\tilde{b}_{t-1}^*(i)\right)-\frac{c}{y}\hat{c}_t(i)-\gamma_z^{-1}\frac{b}{y}\left(\gamma_{z,t}+\pi_t\right)-\gamma_z^{-1}\frac{b^*}{y}\left(\gamma_{z,t}+\pi_t\right) \\ &\quad -\frac{P^{B^*}b^*}{y}\hat{P}_t^{B^*}-\frac{P^Bb}{y}\hat{P}_t^B+\gamma_z^{-1}\frac{b^*}{y}\left(\log\left(S_t\right)-\log S_{t-1}\right) \\ &\quad +\left[\left(1-\frac{g}{y}\right)\hat{p}_{H,t}+\frac{c_H}{y}\hat{c}_{H,t}+\frac{c_F}{y}\hat{c}_{H,t}^*\right]-\frac{c_F}{y}\left[q_t-\hat{p}_{F.t}\right]\end{aligned}$$

In the case of zero debt in steady state we have

$$\begin{aligned}P^B\tilde{b}_t(i)+P^{B^*}\tilde{b}_t^*(i) &= \beta^{-1}\left(P^B\tilde{b}_{t-1}(i)+P^B\tilde{b}_{t-1}^*(i)\right)-\frac{c}{y}\hat{c}_t(i) \\ &\quad +\left[\left(1-\frac{g}{y}\right)\hat{p}_{H,t}+\frac{c_H}{y}\hat{c}_{H,t}+\frac{c_F}{y}\hat{c}_{H,t}^*\right]-\frac{c_F}{y}\left[q_t-\hat{p}_{F.t}\right].\end{aligned}$$

Because we assume households forecast domestic value added and international prices, let's rewrite the flow budget constraint as in terms of $\hat{y}_{H,t}$, $q_t$ and $\log\left(P_{F,t}/P_{H,t}\right)$. From the definition of these variables

$$\hat{y}_{H,t}=\frac{c_H}{y}\hat{c}_{H,t}+\frac{c_F}{y}\hat{c}_{H,t}^*+\frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right)+\tilde{g}_t$$

so that

$$\frac{c_H}{y}\hat{c}_{H,t}+\frac{c_F}{y}\hat{c}_{H,t}^*=\hat{y}_{H,t}-\frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right)-\tilde{g}_t$$

and

$$\begin{aligned}\hat{p}_{F,t} &= \alpha \log\left(\frac{P_{F,t}}{P_{H,t}}\right)\\ \hat{p}_{H,t} &= -(1-\alpha)\log\left(\frac{P_{F,t}}{P_{H,t}}\right).\end{aligned}$$

Substituting gives

$$\begin{aligned}P^B\tilde{b}_t(i)+P^{B^*}\tilde{b}_t^*(i) &= \beta^{-1}\left(P^B\tilde{b}_{t-1}(i)+P^B\tilde{b}_{t-1}^*(i)\right)-\frac{c}{y}\hat{c}_t(i)\\ &\quad+\left[\left(1-\frac{g}{y}\right)\hat{p}_{H,t}+\hat{y}_{H,t}-\frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right)-\tilde{g}_t\right]-\frac{c_F}{y}\left[q_t-\hat{p}_{F.t}\right]\\ &= \beta^{-1}\left(P^B\tilde{b}_{t-1}(i)+P^B\tilde{b}_{t-1}^*(i)\right)-\frac{c}{y}\hat{c}_t(i)\\ &\quad+\left[-\left(1-\frac{g}{y}\right)(1-\alpha)\log\left(\frac{P_{F,t}}{P_{H,t}}\right)+\hat{y}_{H,t}-\frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right)-\tilde{g}_t\right]\\ &\quad-\frac{c_F}{y}\left[q_t-\alpha\log\left(\frac{P_{F,t}}{P_{H,t}}\right)\right]\end{aligned}$$

or

$$\begin{aligned}P^B\tilde{b}_t(i)+P^{B^*}\tilde{b}_t^*(i) &= \beta^{-1}\left(P^B\tilde{b}_{t-1}(i)+P^B\tilde{b}_{t-1}^*(i)\right)-\frac{c}{y}\hat{c}_t(i)\\ &\quad+\left(\frac{c_F}{y}\alpha-\left(1-\frac{g}{y}\right)(1-\alpha)\right)\log\left(\frac{P_{F,t}}{P_{H,t}}\right)+\hat{y}_{H,t}-\frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right)-\tilde{g}_t-\frac{c_F}{y}q_t\end{aligned}$$

As a check, let's make sure that net exports drive the net foreign asset position. Recall

$$P_tC_t = P_{Ht}C_{Ht}+P_{Ft}C_{Ft}$$

so that

$$\hat{c}_t = \frac{c_H}{c}\left(\hat{c}_{H,t}+\hat{p}_{Ht}\right)+\frac{c_F}{c}\left(\hat{c}_{F,t}+\hat{p}_{Ft}\right)$$

Using this in the aggregate FBC

$$\begin{aligned}P^B\tilde{b}_t+P^{B^*}\tilde{b}_t^* &= \beta^{-1}\left(P^B\tilde{b}_{t-1}+P^B\tilde{b}_{t-1}^*\right)-\gamma_z^{-1}\frac{b}{y}\left(\gamma_{z,t}+\pi_t\right)-\gamma_z^{-1}\frac{b^*}{y}\left(\gamma_{z,t}+\pi_t\right)\\ &\quad-\frac{P^{B^*}b^*}{y}\hat{P}_t^{B^*}-\frac{P^Bb}{y}\hat{P}_t^B+\gamma_z^{-1}\frac{b^*}{y}\left(\log(S_t)-\log S_{t-1}\right)\\ &\quad+\left[\left(1-\frac{g}{y}\right)\hat{p}_{H,t}+\frac{c_H}{y}\hat{c}_{H,t}+\frac{c_F}{y}\hat{c}_{H,t}^*\right]-\frac{c_F}{y}\left[q_t-\hat{p}_{F.t}\right]-\frac{c_H}{y}\left(\hat{c}_{H,t}+\hat{p}_{Ht}\right)-\frac{c_F}{y}\left(\hat{c}_{F,t}+\hat{p}_{Ft}\right)\\ &= \beta^{-1}\left(P^B\tilde{b}_{t-1}+P^B\tilde{b}_{t-1}^*\right)-\gamma_z^{-1}\frac{b}{y}\left(\gamma_{z,t}+\pi_t\right)-\gamma_z^{-1}\frac{b^*}{y}\left(\gamma_{z,t}+\pi_t\right)\\ &\quad-\frac{P^{B^*}b^*}{y}\hat{P}_t^{B^*}-\frac{P^Bb}{y}\hat{P}_t^B+\gamma_z^{-1}\frac{b^*}{y}\left(\log(S_t)-\log S_{t-1}\right)\\ &\quad+\left[\left(1-\frac{g}{y}-\frac{c_H}{y}\right)\hat{p}_{H,t}+\frac{c_F}{y}\hat{c}_{H,t}^*\right]-\frac{c_F}{y}\left[q_t+\hat{c}_{Ft}\right]\end{aligned}$$

Note that

$$\frac{c_H^*}{y} = \frac{c_H^*}{y_H} = 1 - \frac{c_H}{y_H} - \frac{g}{y_H} = 1 - \frac{g}{y} - \frac{c_H}{y} \text{ where } y_H = y$$

Hence,

$$\begin{aligned} P^B \tilde{b}_t + P^{B^*} \tilde{b}_t^* &= \beta^{-1}\left(P^B \tilde{b}_{t-1} + P^B \tilde{b}_{t-1}^*\right) - \gamma_z^{-1}\frac{b}{y}\left(\gamma_{z,t}+\pi_t\right) - \gamma_z^{-1}\frac{b^*}{y}\left(\gamma_{z,t}+\pi_t\right) \\ & -\frac{P^{B^*}b^*}{y}\hat{P}_t^{B^*} - \frac{P^B b}{y}\hat{P}_t^B + \gamma_z^{-1}\frac{b^*}{y}\left(\log\left(S_t\right) - \log S_{t-1}\right) \\ & + \left[\frac{c_H^*}{y}\hat{p}_{H,t} + \frac{c_F}{y}\hat{c}_{H,t}^*\right] - \frac{c_F}{y}\left[q_t + \hat{c}_{Ft}\right] \end{aligned}$$

In our benchmark model we will have zero domestic debt at all times, and zero steady state foreign assets holdings which gives the simple expression

$$P^{B^*}\tilde{b}_t^* = \beta^{-1}P^B\tilde{b}_{t-1}^* + \left[\frac{c_H^*}{y}\hat{p}_{H,t} + \frac{c_F}{y}\hat{c}_{H,t}^*\right] - \frac{c_F}{y}\left[q_t + \hat{c}_{Ft}\right]$$

Now we can see that term

$$\left[\frac{c_H^*}{y}\hat{p}_{H,t} + \frac{c_F}{y}\hat{c}_{H,t}^*\right] - \frac{c_F}{y}\left[q_t + \hat{c}_{Ft}\right]$$

matches with $\widetilde{nx}_t$ from an approximation of

$$NX_t = \frac{P_{H,t}}{P_t}C_{H,t}^* - \frac{e_t P_t^*}{P_t}C_{F,t}$$

**Domestic Firms.** The first order condition is

$$\begin{aligned} 0 &= E_t \sum_{T=t}^{\infty} \xi_H^{T-t} Q_{t,T} Y_{H,T}(f)\left(1-\theta_{H,t}\right)\frac{P_{H,T}}{P_T}\left[\left(1-\tau^f\right)\frac{p}{P_{H,T}}\left(\frac{P_{H,T-1}}{P_{H,t-1}}\right)^{\iota_H} - \frac{\theta_{H,T}}{\theta_{H,T}-1}\frac{W_T}{A_T Z_T P_T}\frac{P_T}{P_{H,T}}\right] \\ &= E_t \sum_{T=t}^{\infty} \left(\xi_H \beta\right)^{T-t} \Lambda_T Y_{H,T}(f)\left(1-\theta_{H,t}\right)\frac{P_{H,T}}{P_T}\left[\left(1-\tau^f\right)\frac{p}{P_{H,T}}\left(\frac{P_{H,T-1}}{P_{H,t-1}}\right)^{\iota_H} - \frac{\theta_{H,T}}{\theta_{H,T}-1}\frac{W_T}{A_T Z_T P_T}\frac{P_T}{P_{H,T}}\right] \\ &= E_t \sum_{T=t}^{\infty} \left(\xi_H \beta\right)^{T-t} \Lambda_T Y_{H,T}(f)\left(1-\theta_{H,t}\right)\frac{P_{H,T}}{P_T}\left[\left(1-\tau^f\right)e^{\hat{p}_t^*}e^{\sum_{\tau=t}^{T-1}\left(\iota_H \pi_{H,\tau} - \pi_{H,\tau+1}\right)} - \frac{\theta_H e^{\hat{\theta}_{H,T}}}{\theta_H e^{\hat{\theta}_{H,T}}-1} w e^{\hat{w}_T} e^{-\hat{p}_{H,T}} e^{-a_T}\right] \\ &= E_t \sum_{T=t}^{\infty} \left(\xi_H \beta\right)^{T-t}\left[\left(1-\tau^f\right)\left(\hat{p}_t^* + \sum_{\tau=t}^{T-1}\left(\iota_H \pi_{H,\tau} - \pi_{H,\tau+1}\right)\right) - \frac{\theta_H}{\theta_H - 1}\left(\hat{w}_T - \hat{p}_{H,T} - a_T\right) + \frac{\hat{\theta}_{H,T}}{\theta_H - 1}\right] \\ &= E_t \sum_{T=t}^{\infty} \left(\xi_H \beta\right)^{T-t}\left[\left(\hat{p}_t^* + \sum_{\tau=t}^{T-1}\iota_H \pi_{H,\tau} - \pi_{H,\tau+1}\right) - \left(\hat{w}_T - \hat{p}_{H,T} - a_T\right) + \frac{\hat{\theta}_{H,T}}{\theta_H - 1}\right] \end{aligned}$$

or

$$\begin{aligned}
\frac{\hat{p}^*_{H,t}}{1-\xi_H\beta} &= E_t\sum_{T=t}^{\infty}(\xi_H\beta)^{T-t}\left[-\left(\sum_{\tau=t}^{T-1}\iota_H\pi_{H,\tau}-\pi_{H,\tau+1}\right)+(\hat{w}_T-\hat{p}_{H,T}-a_T)-\frac{\hat{\theta}_{H,T}}{\theta_H-1}\right] \\
&= E_t\sum_{T=t}^{\infty}(\xi_H\beta)^{T-t}\left[-\left(\sum_{\tau=t}^{T-1}\iota_H\pi_{H,\tau}-\pi_{H,\tau+1}\right)+(\hat{w}_T-\hat{p}_{H,T}-a_T)-\frac{\hat{\theta}_{H,T}}{\theta_H-1}\right] \\
&= E_t\sum_{T=t}^{\infty}(\xi_H\beta)^{T-t}\left[\frac{\xi_p\beta}{1-\xi_p\beta}(\pi_{H,T+1}-\iota_H\pi_{H,T})+(\hat{w}_T-\hat{p}_{H,T}-a_T)-\frac{\hat{\theta}_{H,T}}{\theta_H-1}\right]
\end{aligned}$$

and from the aggregate price index

$$\hat{p}_{H,t}=\frac{\xi_H}{1-\xi_H}(\pi_{H,t}-\iota_H\pi_{H,t-1})$$

so that

$$\pi_{H,t}-\iota_H\pi_{H,t-1}=\frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H}E_t\sum_{T=t}^{\infty}(\xi_p\beta)^{T-t}\left[\frac{\xi_H\beta}{1-\xi_H\beta}(\pi_{H,T+1}-\iota_H\pi_{H,T})+(\hat{w}_T-\hat{p}_{H,T}-a_T)-\frac{\hat{\theta}_{H,T}}{(\theta_H-1)}\right.$$

Note that the optimal re-set price satisfies

$$\begin{aligned}
\hat{p}^{opt}_{H,t} &= (1-\xi_H\beta)E_t\sum_{T=t}^{\infty}(\xi_H\beta)^{T-t}\left[\frac{\xi_p\beta}{1-\xi_p\beta}(\pi_{H,T+1}-\iota_H\pi_{H,T})+(\hat{w}_T-\hat{p}_{H,T}-a_T)-\frac{\hat{\theta}_{H,T}}{\theta_H-1}\right] \\
&= (1-\xi_H\beta)\left(E_t\frac{\xi_p\beta}{1-\xi_p\beta}(\pi_{H,T+1}-\iota_H\pi_{H,T})+(\hat{w}_T-\hat{p}_{H,T}-a_T)-\frac{\hat{\theta}_{H,T}}{\theta_H-1}\right)+\xi_H\beta E_t\hat{p}^{opt}_{H,t+1} \\
&= \xi_H\beta E_t\hat{p}^{opt}_{H,t+1}+\delta^H_t
\end{aligned}$$

where

$$\delta^H_t=(1-\xi_H\beta)\left(E_t\frac{\xi_p\beta}{1-\xi_p\beta}(\pi_{H,t+1}-\iota_H\pi_{H,t})+(\hat{w}_t-\hat{p}_{H,t}-a_t)-\frac{\hat{\theta}_{H,t}}{\theta_H-1}\right)$$

**Foreign firms.** The first order condition is

$$\begin{aligned}
0 &= E_t\sum_{T=t}^{\infty}\xi_F^{T-t}Q_{t,T}(1-\theta_{F,t})C_{F,T}(k)P_{F,t}(k)^{-1}\frac{P_{F,T}}{P_T}\left[(1-\tau_F)\frac{P_{F,t}(k)}{P_{F,T}}\left(\frac{P_{F,T-1}}{P_{F,t-1}}\right)^{\iota_F}-\frac{\theta_{f,T}}{(\theta_{f,T}-1)}\frac{S_TP^*_{F,T}}{P_{F,T}}\right] \\
&= E_t\sum_{T=t}^{\infty}\xi_F^{T-t}Q_{t,T}(1-\theta_{F,t})C_{F,T}(k)P_{F,t}(k)^{-1}\frac{P_{F,T}}{P_T}\left[(1-\tau_F)e^{\hat{p}^*_{F,t}}e^{\sum_{\tau=t}^{T-1}\iota_F\pi_{F,\tau}-\pi_{F,\tau+1}}-\frac{\theta_Fe^{\hat{\theta}_{F,T}}}{\left(\theta_Fe^{\hat{\theta}_{F,T}}-1\right)}e^{\psi_T}\right] \\
&= E_t\sum_{T=t}^{\infty}(\xi_F\beta)^{T-t}\left[(1-\tau_F)\left(\hat{p}^*_{F,t}+\sum_{\tau=t}^{T-1}(\iota_F\pi_{F,\tau}-\pi_{F,\tau+1})\right)-\frac{\theta_F}{(\theta_F-1)}\psi_T+\frac{\theta_F\hat{\theta}_{F,T}}{(\theta_F-1)^2}\right] \\
&= E_t\sum_{T=t}^{\infty}(\xi_F\beta)^{T-t}\left[\left(\hat{p}^*_{F,t}+\sum_{\tau=t}^{T-1}(\iota_F\pi_{F,\tau}-\pi_{F,\tau+1})\right)-\psi_T+\frac{\hat{\theta}_{F,T}}{(\theta_F-1)}\right]
\end{aligned}$$

or

$$\begin{aligned}\frac{1}{1-\xi_F\beta}\hat{p}^*_{F,t} &= E_t\sum_{T=t}^{\infty}(\xi_F\beta)^{T-t}\left[\sum_{\tau=t}^{T-1}(\pi_{F,\tau+1}-\iota_F\pi_{F,\tau})+\psi_T-\frac{\hat{\theta}_{F,T}}{(\theta_F-1)}\right]\\ &= E_t\sum_{T=t}^{\infty}(\xi_F\beta)^{T-t}\left[\frac{\xi_f\beta}{1-\xi_f\beta}(\pi_{F,T+1}-\iota_F\pi_{F,T})+\psi_T-\frac{\hat{\theta}_{F,T}}{(\theta_F-1)}\right].\end{aligned}$$

From the price index

$$\hat{p}^*_{F,t}=\frac{\xi_F}{1-\xi_F}(\pi_{F,t}-\iota_F\pi_{F,t-1})$$

so that

$$\pi_{F,t}-\iota_F\pi_{F,t-1}=\frac{(1-\xi_F)(1-\xi_F\beta)}{\xi_F}E_t\sum_{T=t}^{\infty}(\xi_f\beta)^{T-t}\left[\frac{\xi_F\beta}{1-\xi_F\beta}(\pi_{F,T+1}-\iota_F\pi_{F,T})+\psi_T-\frac{\hat{\theta}_{F,T}}{(\theta_F-1)}\right].$$

The optimal re-set price satisfies

$$\begin{aligned}\hat{p}^{opt}_{F,t} &= (1-\xi_F\beta)E_t\sum_{T=t}^{\infty}(\xi_F\beta)^{T-t}\left[\frac{\xi_f\beta}{1-\xi_f\beta}(\pi_{F,T+1}-\iota_F\pi_{F,T})+\psi_T-\frac{\hat{\theta}_{F,T}}{(\theta_F-1)}\right]\\ &= (1-\xi_F\beta)E_t\left(\frac{\xi_f\beta}{1-\xi_f\beta}(\pi_{F,T+1}-\iota_F\pi_{F,T})+\psi_T-\frac{\hat{\theta}_{F,T}}{(\theta_F-1)}\right)+\xi_F\beta E_t\hat{p}^{opt}_{F,t+1}\\ &= \xi_F\beta E_t\hat{p}^{opt}_{F,t+1}+\delta^F_t\end{aligned}$$

where

$$\delta^F_t=(1-\xi_F\beta)E_t\left(\frac{\xi_f\beta}{1-\xi_f\beta}(\pi_{F,t+1}-\iota_F\pi_{F,t})+\psi_t-\frac{\hat{\theta}_{F,t}}{(\theta_F-1)}\right)$$

# 4 Summary of First-order Conditions

## 4.1 Domestic Economy

List of variables: $w^{opt}_t, w_t, \pi_t, \pi^w, N_t, \lambda_t, c_t(i), b_t(i), S_t, c_{F,t}, c_{H,t}, q_t, \psi_{F,t}, y_{H,t}, \pi_{H,t}, \pi_{F,t}, R_t, c^*_{H,t}, b^*_t(i)$, $\log(P_{F,t}/P_{H,t})$ — 22 variables with $R^*_t, P^{B^*}_t, \pi^*_t$ determined in the foreign block.

1. Optimal Reset wage: $w^{opt}_t$

$$\begin{aligned}\hat{w}^{opt}(j) &= \xi_w\beta E^j_t(\pi^w_{t+1}-\hat{\gamma}_{z,t}-\iota_w\pi_t)\\ &+\frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_t+\phi^{-1}(\hat{y}_{H,t}-a_t)-\hat{w}^r_t-\hat{\lambda}_t(j)-\frac{\hat{\theta}_{w,t}}{(\theta_w-1)}\right]+\xi_w\beta E^j_t\hat{w}^{opt}_{t+1}(j)\end{aligned}$$

2. Wage inflation: $\pi^w_t$

$$w^{opt}_t=\frac{\xi_w}{1-\xi_w}(\pi^w_t-\iota_w\pi_{t-1}-\hat{\gamma}_{z,t-1})$$

3. The real wage inflation: $w_t$

$$w_t = \pi_t^w - \pi_t - \hat{\gamma}_{z,t} + w_{t-1}$$

4. Consumption: $c_t(i)$

$$(1-\beta b)\hat{\lambda}_t(i) = \hat{\xi}_t + \frac{b\sigma}{1-b}\hat{c}_{t-1}(i) - \frac{\sigma}{1-b}\left(1+\beta b^2\right)\hat{c}_t(i) - \beta b E_t \hat{\xi}_{t+1} + \beta b \frac{\sigma}{1-b} E_t \hat{c}_{t+1}(i).$$

5. Marginal Utility: $\lambda_t$

$$\hat{\lambda}_t(i) - R_t = E_t^i\left(\hat{\lambda}_{t+1}(i) - \hat{\gamma}_{z,t} - \pi_{t+1}\right) - \chi\left[(1+\beta)\tilde{b}_t(i) - \tilde{b}_{t-1}(i) - \beta E_t^i \tilde{b}_{t+1}(i)\right]$$

6. Nominal exchange rate: $S_t$

$$\begin{aligned}\hat{\lambda}_t(i) - R_t^* &= E_t^i\left(\hat{\lambda}_{t+1}(i) - \hat{\gamma}_{z,t} - \pi_{t+1} + \log S_{t+1} - \log S_t\right)\\ &\quad -\chi^*\left[\left(1+\beta+\frac{\phi^{R^*}\left(\frac{c_F}{y}\right)^{-1}}{\chi^*}\right)\tilde{b}_t^*(i) - \tilde{b}_{t-1}^*(i) - \beta E_t^i \tilde{b}_{t+1}^*(i)\right] + \phi^{R^*}\varepsilon_t^{R^*}\end{aligned}$$

7. Foreign bond holdings: $b_t^*(i)$

$$\begin{aligned}P^B\tilde{b}_t(i) + P^{B^*}\tilde{b}_t^*(i) &= \beta^{-1}\left(P^B\tilde{b}_{t-1}(i) + P^B\tilde{b}_{t-1}^*(i)\right) - \frac{c}{y}\hat{c}_t(i)\\ &\quad + \left(\frac{c_F}{y}\alpha - \left(1-\frac{g}{y}\right)(1-\alpha)\right)\log\left(\frac{P_{F,t}}{P_{H,t}}\right) + \hat{y}_{H,t} - \frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right) - \tilde{g}_t - \frac{c_F}{y}q_t\end{aligned}$$

8. Domestic value demand: $\hat{y}_{H,t}$

$$\hat{y}_{H,t} = \frac{c_H}{y}\hat{c}_{H,t} + \frac{c_F}{y}\hat{c}_{H,t}^* + \frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right) + \tilde{g}_t$$

9. Domestic home produced goods demand: $c_{H,t}$

$$\hat{c}_{H,t} = \hat{c}_t + \eta(1-\alpha)\log\left(\frac{P_{Ft}}{P_{Ht}}\right)$$

10. Imported good demand: $c_{F,t}$

$$\hat{c}_{F,t} = \hat{c}_t - \eta\alpha\log\left(\frac{P_{Ft}}{P_{Ht}}\right)$$

11. Domestic goods inflation: $\pi_{H,t}$

$$\pi_{H,t} - \iota_H\pi_{H,t-1} = \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H}E_t\sum_{T=t}^{\infty}(\xi_H\beta)^{T-t}\left[\begin{array}{c}\frac{\xi_H\beta}{1-\xi_H\beta}\left(\pi_{H,T+1} - \iota_H\pi_{H,T}\right)\\ +\left(\hat{w}_T + (1-\alpha)\log\left(\frac{P_{F,t}}{P_{H,t}}\right) - a_T\right) - \frac{\hat{\theta}_{H,T}}{(\theta_H-1)}\end{array}\right]$$

12. Imported goods inflation: $\pi_{F,t}$

$$\pi_{F,t}-\iota_F\pi_{F,t-1}=\frac{(1-\xi_F)(1-\xi_F\beta)}{\xi_F}E_t\sum_{T=t}^{\infty}(\xi_F\beta)^{T-t}\left[\frac{\xi_F\beta}{1-\xi_F\beta}(\pi_{F,T+1}-\iota_F\pi_{F,T})+\psi_T-\frac{\hat{\theta}_{F,T}}{(\theta_F-1)}\right]$$

13: Monetary policy: $\hat{P}_t^B=-R_t$

$$-\hat{P}_t^B=R_t=\rho_R R_{t-1}+(1-\rho_R)(\phi_\pi\pi_t+\phi_y y_{Ht}+\phi_s\log S_t)+m_t$$

In our model $y_{Ht}$ is domestic total output.

14. The real exchange rate: $q_t$

$$q_t=q_{t-1}+\log S_t-\log S_{t-1}+\pi_t^*-\pi_t$$

15: Labor input: $N_t$

$$\hat{y}_{H,t}=a_t+\hat{N}_t$$

16: Law of one price gap: $\psi_{F,t}$

$$\psi_{F,t}=\hat{q}_t-\alpha\log\left(\frac{P_{Ft}}{P_{Ht}}\right)$$

17. CPI inflation: $\pi_t$

$$\pi_t=\alpha\pi_{H,t}+(1-\alpha)\pi_{F,t}$$

18. Foreign demand of domestic goods: $\hat{c}_{H,t}^*$

$$\hat{c}_{H,t}^*=\left(y_t^*-\eta^*\left(-\log\left(\frac{P_{F,t}}{P_{H,t}}\right)-\psi_{F,t}\right)\right)+\log\left(\frac{Z_t^*}{Z_t}\right)$$

19. Terms of trade: $\log(P_{F,t}/P_{H,t})$

$$\log\left(\frac{P_{F,t}}{P_{H,t}}\right)-\log\left(\frac{P_{F,t-1}}{P_{H,t-1}}\right)=\pi_{Ft}-\pi_{Ht}$$

20. Domestic bond holdings. Market clearing:

$$b_t=\int_0^1 b_t(i)\,di=0$$

The above calculations assume a debt elastic interest rate: $\hat{P}_t^{B^*}$

$$\begin{aligned}-\hat{P}_t^{B^*} &= R_t^*-\phi^{R^*}\left(\frac{S_tB_t^*}{P_t(c_FZ_t)}+\varepsilon_t^{R^*}\right)\\ &= R_t^*-\phi^{R^*}\left(\frac{b_t^*}{y}\left(\frac{c_F}{y}\right)^{-1}+\varepsilon_t^{R^*}\right)\\ &= R_t^*-\phi^{R^*}\left(\left(\frac{c_F}{y}\right)^{-1}\tilde{b}_t^*+\varepsilon_t^{R^*}\right)\end{aligned}$$

Expression follows from the assumption that

$$\frac{1}{P_t^{B^*}} = R_t^{B^*} = R_t^* \exp\left\{-\phi^{R^*}\left(\frac{S_t B_t^*}{P_t\left(c_F Z_t\right)} + \varepsilon_t^{R^*}\right)\right\}$$

and have used the fact that the steady state of $b_t^* = 0$. $R_t^*$ is the interest in Foreign. A debt-elastic interest-rate premium is the real quantity of outstanding foreign debt expressed in terms of domestic currency as a fraction of steady-state consumption of the imported good multiplied by the trend, $c_F Z_t$, (if we let $Z_t = 1$, it converges to the standard assumption) and $\varepsilon_t^{R^*}$ a risk premium shock. $\phi^{R^*}$ governs the interest rate elasticity of debt. Plugging $\hat{P}_t^{B^*}$ into foreign bond holding condition, the no arbitrage condition (point 5) changes.

## 4.2 The Foreign Economy

List of variables: $w_t^{opt^*}, \pi_t^*, \pi^{w^*}, N_t^*, \lambda_t^*, c_t^*(i), R_t^*, y_t^*$ , $b_t^*(i), b_t^*$— 10 variables

1. Optimal reset wage: $w_t^{opt^*}$

$$\begin{aligned}\hat{w}^{opt^*}(j) &= \xi_w \beta E_t^j\left(\pi_{t+1}^{w^*} - \hat{\gamma}_t^{z^*} - \iota_w \pi_t^*\right) \\ &+ \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_t^* + \phi^{-1}\left(\frac{c}{y}c_t^* + \tilde{g}_t^* - a_t^*\right) - \hat{w}_t^{r^*} - \hat{\lambda}_t^*(j) - \frac{\hat{\theta}_{w,t}^*}{(\theta_w - 1)}\right] + \xi_w\beta E_t^j \hat{w}_{t+1}^{opt^*}(j)\end{aligned}$$

2. Wage inflation: $\pi_t^{w^*}$

$$w_t^{opt^*} = \frac{\xi_w}{1-\xi_w}\left(\pi_t^{w^*} - \iota_w \pi_{t-1}^* - \hat{\gamma}_{t-1}^*\right)$$

3. Real Wage: $w_t^*$

$$w_t^* = \pi_t^{w^*} - \pi_t^* - \gamma_t^* + w_{t-1}^*$$

4. Consumption: $c_t^*$

$$(1-\beta b)\hat{\lambda}_t^*(i) = \hat{\xi}_t^* + \frac{b\sigma}{1-b}\hat{c}_{t-1}^*(i) - \frac{\sigma}{1-b}\left(1+\beta b^2\right)\hat{c}_t^*(i) - \beta b E_t \hat{\xi}_{t+1}^* + \beta b\frac{\sigma}{1-b}E_t \hat{c}_{t+1}^*(i).$$

5. Marginal utility: $\lambda_t$

$$\hat{\lambda}_t^*(i) - R_t^* = E_t^i\left(\hat{\lambda}_{t+1}^*(i) - \hat{\gamma}_{z,t}^* - \pi_{t+1}^*\right)$$

6. Bond holdings: $b_t^*$ (will assume in zero supply)

$$P^{B^*}\tilde{b}_t^*(i) = \beta^{-1}\left(P^B \tilde{b}_{t-1}^*(i)\right) - \frac{c}{y}\hat{c}_t^*(i) + \frac{c}{y}\hat{c}_t^*$$

7. Aggregate Production: $N_t^*$

$$\hat{y}_t^* = a_t^* + N_t^*$$

8. Inflation: $\pi_t^*$

$$\pi^*_{H,t}-\iota_H\pi^*_{H,t-1}=\frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H}E_t\sum_{T=t}^{\infty}(\xi_p\beta)^{T-t}\left[\frac{\xi_H\beta}{1-\xi_H\beta}\left(\pi^*_{H,T+1}-\iota_H\pi^*_{H,T}\right)+\hat{w}^*_T-a^*_T-\frac{\hat{\theta}^*_{H,T}}{(\theta_H-1)}\right]$$

9. Market Clearing Goods: $y_t^*$

$$y_t^*=\frac{c}{y}c_t^*+\tilde{g}_t^*$$

10. Monetary Policy: $R_t^*$

$$R_t^*=\rho_{R^*}R^*_{t-1}+(1-\rho_{R^*})\left(\phi_{\pi^*}\pi_t^*+\phi_{y^*}y_t^*\right)+m_t^*$$

11. Market clearing bonds:

$$b_t^*=\int b_t^*(i)\,di=0$$

Note, as in the domestic economy, we assume agents know the dividend and tax policies which permits simplification of the flow budget constraint as follows. Then

$$\frac{\Gamma^*_{H,t}+T^*_{H,T}}{Z_t^*}+(1-\tau_w)\,w_t^{r^*}N_t^*+T^*_{w,t}=\int_0^1\frac{Y_t^*(f)}{Z_t^*}df$$

satisfies

$$\tilde{\gamma}^*_{H,t}+\tilde{\tau}^*_{H,t}+(1-\tau_w)\frac{w^rN}{y}\left(\hat{w}_t^{r^*}+\hat{N}_t^*\right)+\tilde{\tau}^*_{w,t}=\hat{y}_t^*$$

so that

$$\begin{aligned}P^B\tilde{b}_t^*(i) &= \beta^{-1}\left(P^B\tilde{b}^*_{t-1}(i)\right)-\frac{c}{y}\hat{c}_t^*(i)-\tilde{\tau}_t^*+\hat{y}_t^*\\ &= \beta^{-1}\left(P^B\tilde{b}^*_{t-1}(i)\right)-\frac{c}{y}\hat{c}_t^*(i)+\frac{c}{y}\hat{c}_t^*\end{aligned}$$

12. Natural rate of consumption: $c_t^{nr^*}$

$$(1-\beta b)\,\hat{\lambda}_t^*=\hat{\xi}_t^*+\frac{b\sigma}{1-b}c_{t-1}^{nr^*}-\frac{\sigma}{1-b}\left(1+\beta b^2\right)c_t^{nr^*}-\beta bE_t\hat{\xi}^*_{t+1}+\beta b\frac{\sigma}{1-b}E_tc_{t+1}^{nr^*}$$

13. Natural rate of marginal utility: $\lambda_t^{nr^*}$

$$\hat{\xi}_t^*+\phi_n^{-1}\left(\frac{c}{y}\hat{c}_t^{nr^*}+\hat{G}_t^*\right)=\lambda_t^{nr^*}+\left(1+\phi_n^{-1}\right)a_t^*$$

14. Natural rate of real interest: $r_t^{nr^*}$

$$\lambda_t^{nr^*}=r_t^{nr^*}+E_t\lambda_{t+1}^{nr^*}$$

## 5 Solving under Rational expectations

When solving under rational expectations the model simplifies slightly. Because the domestic economy domestic debt is in zero supply we can set

$$b_t(i) = b_t = 0.$$

Similarly, in the foreign economy its debt is in zero net supply so that

$$b_t^* = 0$$

and we drop equations 6 and 11. Because of the small open economy assumption this doesn't mean that domestic holdings of the foreign debt are zero — they are small relative to the large economy. To make this clear the code and notes below employ the notation $b_{H,t}$ and $b_{F,t}$ to denote the home demand for domestic and foreign debt respectively. Under rational expectations $b_{H,t} = b_t^* = 0$ while $b_{F,t}$ is to be determined. There are three other simplifications. Under the rational expectations assumption domestic goods price inflation satisfies

$$\begin{aligned}
\pi_{H,t} - \iota_H \pi_{H,t-1} &= \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H} E_t \sum_{T=t}^{\infty} (\xi_H\beta)^{T-t} \left[ \begin{array}{c} \frac{\xi_H\beta}{1-\xi_H\beta}(\pi_{H,T+1} - \iota_H\pi_{H,T}) \\ + \left(\hat{w}_T + (1-\alpha)\log\left(\frac{P_{F,t}}{P_{H,t}}\right) - a_T\right) - \frac{\hat{\theta}_{H,T}}{(\theta_H-1)} \end{array} \right] \\
&= \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H} E_t \left[ \begin{array}{c} \frac{\xi_H\beta}{1-\xi_H\beta}(\pi_{H,t+1} - \iota_H\pi_{H,t}) \\ + \left(\hat{w}_t + (1-\alpha)\log\left(\frac{P_{F,t}}{P_{H,t}}\right) - a_t\right) - \frac{\hat{\theta}_{H,t}}{(\theta_H-1)} \end{array} \right] \\
&\quad + (\xi_H\beta) E_t (\pi_{H,t+1} - \iota_H\pi_{H,t}) \\
&= \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H} E_t \left[ \left(\hat{w}_t + (1-\alpha)\log\left(\frac{P_{F,t}}{P_{H,t}}\right) - a_t\right) - \frac{\hat{\theta}_{H,t}}{(\theta_H-1)} \right] \\
&\quad + (1-\xi_H)\beta E_t (\pi_{H,t+1} - \iota_H\pi_{H,t}) + (\xi_H\beta) E_t (\pi_{H,t+1} - \iota_H\pi_{H,t}) \\
&= \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H} E_t \left[ \left(\hat{w}_t + (1-\alpha)\log\left(\frac{P_{F,t}}{P_{H,t}}\right) - a_t\right) - \frac{\hat{\theta}_{H,t}}{(\theta_H-1)} \right] \\
&\quad + \beta E_t (\pi_{H,t+1} - \iota_H\pi_{H,t})
\end{aligned}$$

Replace equation 10 with this. Replace equation 11 with domestic imported goods inflation

$$\begin{aligned}
\pi_{F,t} - \iota_F \pi_{F,t-1} &= \frac{(1-\xi_F)(1-\xi_F\beta)}{\xi_F} E_t \sum_{T=t}^{\infty} (\xi_F\beta)^{T-t} \left[ \frac{\xi_F\beta}{1-\xi_F\beta}(\pi_{F,T+1} - \iota_F\pi_{F,T}) + \psi_T - \frac{\hat{\theta}_{F,T}}{(\theta_F-1)} \right] \\
&= \frac{(1-\xi_F)(1-\xi_F\beta)}{\xi_F} \left[ \frac{\xi_F\beta}{1-\xi_F\beta} E_t (\pi_{F,t+1} - \iota_F\pi_{F,t}) + \psi_t - \frac{\hat{\theta}_{F,t}}{(\theta_F-1)} \right] \\
&\quad + (\xi_F\beta) E_t (\pi_{F,t+1} - \iota_F\pi_{F,t}) \\
&= \frac{(1-\xi_F)(1-\xi_F\beta)}{\xi_F} \left[ \psi_t - \frac{\hat{\theta}_{F,t}}{(\theta_F-1)} \right] + \beta E_t (\pi_{F,t+1} - \iota_F\pi_{F,t})
\end{aligned}$$

Replace equation 7 with the foreign goods price inflation

$$\begin{aligned}
\pi^*_{H,t} - \iota_H \pi^*_{H,t-1} &= \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H} E_t \sum_{T=t}^{\infty} (\xi_p \beta)^{T-t} \left[ \frac{\xi_H \beta}{1-\xi_H\beta} \left(\pi^*_{H,T+1} - \iota_H \pi^*_{H,T}\right) + \hat{w}^*_T - a^*_T - \frac{\hat{\theta}^*_{H,t}}{(\theta_H - 1)} \right] \\
&= \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H} E_t \left[ \frac{\xi_H \beta}{1-\xi_H\beta} \left(\pi^*_{H,t+1} - \iota_H \pi^*_{H,t}\right) + \hat{w}^*_t - a^*_t - \frac{\hat{\theta}^*_{H,t}}{(\theta_H - 1)} \right] \\
&\quad + \xi_p \beta E_t \left(\pi^*_{H,t+1} - \iota_H \pi^*_{H,t}\right) \\
&= \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H} \left[ \hat{w}^*_t - a^*_t - \frac{\hat{\theta}^*_{H,t}}{(\theta_H - 1)} \right] + \beta E_t \left(\pi^*_{H,t+1} - \iota_H \pi^*_{H,t}\right)
\end{aligned}$$

# 6 Solving the Model under Learning

This section provides an overview of the general approach to solving DSGE models under non-rational expectations. We derive the representation of aggregate dynamics

$$A_0 z_t = \hat{E}_t \sum_{s=1}^{10} A_s \sum_{T=t}^{\infty} \lambda_s^{-(T-t)} z_{T+1} + A_{11} z_{t-1} + A_{12} \varepsilon_t \qquad (7)$$

where the vector $z_t$ collects all model variables; the vector $\varepsilon_t$ collects exogenous innovations; the discounts $\lambda_s$ for $s \in 1, ..., 10$ index the model's set of unstable eigenvalues; $A_i$, for $i \in 1, ..., 12$, coefficient matrices; and

$$\hat{E}_t = \int_0^1 \hat{E}^i_t di$$

average beliefs. This representation holds for arbitrary beliefs, including rational expectations. Dynamics depend on a set of projections into the indefinite future, reflecting the intertemporal decision problems solved by households and firms. The projected variables are those macroeconomic objects taken as given and beyond the control of each decision maker. We then close the model with a theory of belief formation to evaluate expectations and derive the state-space representation to exploit in estimation.

The solution procedure starts by solving for the optimal intertemporal decisions of each type of agent in the model. There are five blocks of forward-looking agents: domestic households, domestic goods producing firms, domestic imported goods producing firms, foreign households and foreign goods producers. Taking expectations as given, we solve for optimal decisions as a function of expectations about variables that are exogenous to each agent's decision problem. Aggregation of these decisions together with the remaining equilibrium conditions which don't involve expectations — such as market clearing conditions, definitions and policy assumptions — yields the system (7).

**Decision Rule 1: Households** We want to write the optimal decision of household $i$ in the form

$$\Psi_0^{HH} E_t^i \begin{bmatrix} \hat{w}_{t+1}^*(i) \\ \hat{c}_{t+1}(i) \\ \lambda_{t+1}(i) \\ b_{t+1}^*(i) \\ \hat{c}_t(i) \\ b_t(i) \\ b_t^*(i) \end{bmatrix} = \Psi_1^{HH} \begin{bmatrix} \hat{w}_t^*(i) \\ \hat{c}_t(i) \\ \lambda_t(i) \\ b_t^*(i) \\ \hat{c}_{t-1}(i) \\ b_{t-1}(i) \\ b_{t-1}^*(i) \end{bmatrix} + \delta_t$$

Let's rearrange the first order conditions one by one.

$$\delta_t = \begin{bmatrix} \delta_t^w \\ \delta_t^c \\ \delta_t^\lambda \\ \delta_t^{b^*} \\ 0 \\ \delta_t^{b_{lag}} \\ 0 \end{bmatrix}$$

But this representation has a singular $\Psi_0^{HH}$. Let's proceed in a different way, by substituting out for the FBC as in the analytical results.

1. Real Wage. Optimal wage setting gives

$$\begin{aligned} \xi_w \beta E_t^j \hat{w}_{t+1}^*(j) &= \hat{w}^*(j) - \xi_w \beta E_t^j \left(\pi_{t+1}^w - \hat{\gamma}_t^z - \iota_w \pi_t\right) \\ &\quad - \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}} \left[\hat{\varphi}_t + \phi_t^{-1}\left(\hat{y}_{H,t} - a_t\right) - \hat{w}_t^r - \hat{\lambda}_t(j) - \frac{\hat{\theta}_{w,t}}{(\theta_w - 1)}\right] \\ &= \hat{w}^*(j) + \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\hat{\lambda}_t(j) - \xi_w\beta E_t^j\left(\pi_{t+1}^w - \hat{\gamma}_t^z - \iota_w\pi_t\right) \\ &\quad - \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}} \left[\hat{\varphi}_t + \phi^{-1}\left(\hat{y}_{H,t} - a_t\right) - \hat{w}_t^r - \frac{\hat{\theta}_{w,t}}{(\theta_w - 1)}\right] \\ &= \hat{w}^*(j) + \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\hat{\lambda}_t(j) + \delta_t^w \end{aligned}$$

where

$$\delta_t^w = -\xi_w\beta E_t^j\left(\pi_{t+1}^w - \hat{\gamma}_{z,t} - \iota_w\pi_t\right) - \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_t + \phi^{-1}\left(\hat{y}_{H,t} - a_t\right) - \hat{w}_t^r - \frac{\hat{\theta}_{w,t}}{(\theta_w - 1)}\right]$$

2. Consumption. First order condition with respect to consumption provide

$$\begin{aligned} \beta b\frac{\sigma}{1-b}E_t\hat{c}_{t+1}(i) - \frac{\sigma}{1-b}\left(1+\beta b^2\right)\hat{c}_t(i) &= (1-\beta b)\hat{\lambda}_t(i) - \hat{\xi}_t - \frac{b\sigma}{1-b}\hat{c}_{t-1}(i) + \beta b E_t\hat{\xi}_{t+1} \\ &= (1-\beta b)\hat{\lambda}_t(i) - \frac{b\sigma}{1-b}\hat{c}_{t-1}(i) + \delta_t^c \end{aligned}$$

where

$$\delta_t^c = \beta b E_t \hat{\xi}_{t+1} - \hat{\xi}_t.$$

3. Marginal utility: $\hat{\lambda}_t(i)$

$$\begin{aligned} E_t^i\left(\hat{\lambda}_{t+1}(i)\right) - \chi^*\left[(1+\beta)\tilde{b}_t^*(i) - \beta E_t^i \tilde{b}_{t+1}^*(i)\right] &= \hat{\lambda}_t(i) - R_t^* - \chi^*\tilde{b}_{t-1}^*(i) \\ &\quad - E_t^i\left(-\hat{\gamma}_{z,t} - \pi_{t+1} + \log S_{t+1} - \log S_t\right) \\ &= \hat{\lambda}_t(i) - \chi^*\tilde{b}_{t-1}^*(i) + \delta_t^\lambda \end{aligned}$$

where

$$\delta_t^\lambda = -R_t^* - E_t^i\left(-\hat{\gamma}_{z,t} - \pi_{t+1} + \log S_{t+1} - \log S_t\right) + \phi^{R^*}\left(\frac{c_F}{y}\right)^{-1}\tilde{b}_t^*(i) + \phi^{R^*}\varepsilon_t^{R^*}$$

We assume that individual households take the risk premium as exogenous and hence, $\phi^{R^*}\left(\frac{c_F}{y}\right)^{-1}\tilde{b}_t^*(i)$ enters $\delta_t^\lambda$.

4.Debt holdings (FBC): $b_t^*$

$$P^B\tilde{b}_t(i) + P^{B^*}\tilde{b}_t^*(i) + \frac{c}{y}\hat{c}_t(i) = \beta^{-1}\left(P^B\tilde{b}_{t-1}(i) + P^B\tilde{b}_{t-1}^*(i)\right) + \eta_t$$

where

$$\eta_t = \left(\frac{c_F}{y}\alpha - \left(1-\frac{g}{y}\right)(1-\alpha)\right)\log\left(\frac{P_{F,t}}{P_{H,t}}\right) + \hat{y}_{H,t} - \frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right) - \tilde{g}_t - \frac{c_F}{y}q_t$$

Now rewrite as

$$\beta\tilde{b}_{t+1}(i) = \left(\tilde{b}_t(i) + \tilde{b}_t^*(i)\right) + \gamma_z\eta_{t+1} - \beta\tilde{b}_{t+1}^*(i) - \gamma_z\frac{c}{y}\hat{c}_{t+1}(i)$$

and substitute into the domestic bondholding Euler equation

$$E_t^i\hat{\lambda}_{t+1}(i) - \chi\left[(1+\beta)\tilde{b}_t(i) - \beta E_t^i\tilde{b}_{t+1}(i)\right] = \hat{\lambda}_t(i) - R_t + \hat{\gamma}_{z,t} + E_t^i\pi_{t+1} - \chi\tilde{b}_{t-1}(i)$$

to give

$$E_t^i\hat{\lambda}_{t+1}(i) - \chi\left[\beta\tilde{b}_t(i) - \left(\tilde{b}_t^*(i) + \gamma_z\eta_{t+1} - \beta\tilde{b}_{t+1}^*(i) - \gamma_z\frac{c}{y}\hat{c}_{t+1}(i)\right)\right] = \hat{\lambda}_t(i) - R_t + \hat{\gamma}_{z,t} + E_t^i\pi_{t+1} - \chi\tilde{b}_{t-1}(i)$$

or

$$E_t^i\hat{\lambda}_{t+1}(i) - \chi\left[\beta\tilde{b}_t(i) - \tilde{b}_t^*(i) - \gamma_z\eta_{t+1} + \beta\tilde{b}_{t+1}^*(i) + \gamma_z\frac{c}{y}\hat{c}_{t+1}(i)\right] = \hat{\lambda}_t(i) - R_t + \hat{\gamma}_{z,t} + E_t^i\pi_{t+1} - \chi\tilde{b}_{t-1}(i)$$

which simplifies to

$$E_t^i\hat{\lambda}_{t+1}(i) - \chi\left[\beta\tilde{b}_t(i) - \tilde{b}_t^*(i) + \beta\tilde{b}_{t+1}^*(i) + \gamma_z\frac{c}{y}\hat{c}_{t+1}(i)\right] = \hat{\lambda}_t(i) - \chi\tilde{b}_{t-1}(i) + \delta_t^{b^*}$$

where

$$\begin{aligned} \delta_t^{b^*} &= -R_t + \hat{\gamma}_{z,t} + E_t^i\pi_{t+1} - \chi\gamma_z E_t\eta_{t+1} \\ &= -R_t + \hat{\gamma}_{z,t} + E_t^i\pi_{t+1} \\ &\quad -\chi\gamma_z\left(\left(\frac{c_F}{y}\alpha - \left(1-\frac{g}{y}\right)(1-\alpha)\right)\log\left(\frac{P_{F,t+1}}{P_{H,t+1}}\right) + \hat{y}_{H,t+1} - \frac{c_F}{y}\log\left(\frac{Z_{t+1}^*}{Z_{t+1}}\right) - \tilde{g}_{t+1} - \frac{c_F}{y}q_{t+1}\right). \end{aligned}$$

5. The flow budget constraint can be re-written as

$$\beta\tilde{b}_t(i) = -\gamma_z \frac{c}{y}\hat{c}_t(i) - \beta\tilde{b}^*_t(i) + \tilde{b}_{t-1}(i) + \tilde{b}^*_{t-1}(i) + \delta_t^{b_{lag}}$$

$$\delta_t^{b_{lag}} = \gamma_z \eta_t = \gamma_z \left( \left( \frac{c_F}{y}\alpha - \left(1 - \frac{g}{y}\right)(1-\alpha) \right) \log\left(\frac{P_{F,t}}{P_{H,t}}\right) + \hat{y}_{H,t} - \frac{c_F}{y}\log\left(\frac{Z^*_t}{Z_t}\right) - \tilde{g}_t - \frac{c_F}{y} q_t \right)$$

Let

$$\delta_t = \begin{bmatrix} \delta_t^w \\ \delta_t^c \\ \delta_t^\lambda \\ \delta_t^{b^*} \\ 0 \\ \delta_t^{b_{lag}} \\ 0 \end{bmatrix}$$

Then the system is of the form

$$\Psi_0^{HH} E_t^i \begin{bmatrix} \hat{w}^*_{t+1}(i) \\ \hat{c}_{t+1}(i) \\ \lambda_{t+1}(i) \\ b^*_{t+1}(i) \\ \hat{c}_t(i) \\ b_t(i) \\ b^*_t(i) \end{bmatrix} = \Psi_1^{HH} \begin{bmatrix} \hat{w}^*_t(i) \\ \hat{c}_t(i) \\ \lambda_t(i) \\ b^*_t(i) \\ \hat{c}_{t-1}(i) \\ b_{t-1}(i) \\ b^*_{t-1}(i) \end{bmatrix} + \delta_t$$

where

$$\Psi_0^{HH} = \begin{bmatrix} \xi_w\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta b\frac{\sigma}{1-b} & 0 & 0 & -\frac{\sigma}{1-b}\left(1+\beta b^2\right) & 0 & 0 \\ 0 & 0 & 1 & \chi^*\beta & 0 & 0 & -\chi^*(1+\beta) \\ 0 & -\gamma_z\chi\frac{c}{y} & 1 & -\chi\beta & 0 & -\chi\beta & \chi \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\Psi_1^{HH} = \begin{bmatrix} 1 & 0 & \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-\beta b) & 0 & -\frac{b\sigma}{1-b} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\chi^* \\ 0 & 0 & 1 & 0 & 0 & -\chi & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{c}{y} & 0 & -\beta & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The solution can be written

$$\begin{bmatrix} \hat{c}_t(i) \\ \hat{w}^*_t(i) \\ \lambda_t(i) \\ b^*_t(i) \end{bmatrix} = \bar{D}_k \begin{bmatrix} \hat{c}_{t-1}(i) \\ b_{t-1}(i) \\ b^*_{t-1}(i) \end{bmatrix} + \bar{D}_{\delta 1} \sum_{T=t}^{\infty} \Lambda^{-(T-t)} \bar{D}_{\delta 2} E_t \delta_T$$

where $\bar{D}_k$ $(5 \times 3)$, $\bar{D}_{\delta 1}$ $(5 \times 3)$and $\bar{D}_{\delta 2}$ $(3 \times 8)$ have elements that are composites of model primitives. Note that because the final three elements of $\delta_t$ are zero we can just delete the final three columns of $\bar{D}_{\delta 2}$. The code getdecision.m implements these calculations, takings as inputs $\Psi_0^{HH}$,$\Psi_1^{HH}$ and $n_k$ where the latter denotes the number of household level state variables here equal to 3.

We are not quite done. We have to rearrange to ensure all contemporaneous terms are on the left hand side for the canonical representation. We also have to aggregate and apply market clearing conditions. We now do this, working equation by equation. Note we can write the system as

$$\begin{aligned}
\begin{bmatrix} \hat{c}_t(i) \\ \hat{w}_t^*(i) \\ \lambda_t(i) \\ b_t^*(i) \end{bmatrix} &= \bar{D}_k \begin{bmatrix} \hat{c}_{t-1}(i) \\ b_{t-1}(i) \\ b_{t-1}^*(i) \end{bmatrix} + D_{\lambda_1} \sum_{T=t}^{\infty} \lambda_1^{-(T-t)} E_t \bar{\delta}_T \\
&\quad + D_{\lambda 2} \sum_{T=t}^{\infty} \lambda_2^{-(T-t)} E_t \bar{\delta}_T + D_{\lambda_3} \sum_{T=t}^{\infty} \lambda_3^{-(T-t)} E_t \bar{\delta}_T + D_{\lambda_4} \sum_{T=t}^{\infty} \lambda_4^{-(T-t)} E_t \bar{\delta}_T
\end{aligned} \tag{8}$$

where

$$D_{\lambda_1} = \bar{D}_{\delta 1} \begin{bmatrix} \bar{D}_{\delta 2}^1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; D_{\lambda_2} = \bar{D}_{\delta 1} \begin{bmatrix} 0 \\ \bar{D}_{\delta 2}^2 \\ 0 \\ 0 \end{bmatrix}; D_{\lambda_3} = \bar{D}_{\delta 1} \begin{bmatrix} 0 \\ 0 \\ \bar{D}_{\delta 2}^3 \\ 0 \end{bmatrix}; D_{\lambda_4} = \bar{D}_{\delta 1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{D}_{\delta 2}^4 \end{bmatrix}$$

where $\bar{D}_{\delta 2}^i$ denotes the $i$th row of the matrix $\bar{D}_{\delta 2}$.

Write the solution in the following form (dropping the household indicator)

$$\begin{aligned}
\hat{w}_t^* - \sum_{i=1}^{4} \sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{0,1}(\lambda_i, \delta_j) &= \bar{D}_k^1 \begin{bmatrix} \hat{c}_{t-1}(i) \\ b_{t-1}(i) \\ b_{t-1}^*(i) \end{bmatrix} + \sum_{i=1}^{4} \sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{e,1}(\lambda_i, \delta_j) \\
\hat{c}_t - \sum_{i=1}^{4} \sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{0,2}(\lambda_i, \delta_j) &= \bar{D}_k^2 \begin{bmatrix} \hat{c}_{t-1}(i) \\ b_{t-1}(i) \\ b_{t-1}^*(i) \end{bmatrix} + \sum_{i=1}^{4} \sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{e,2}(\lambda_i, \delta_j) \\
\hat{\lambda}_t - \sum_{i=1}^{4} \sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{0,3}(\lambda_i, \delta_j) &= \bar{D}_k^3 \begin{bmatrix} \hat{c}_{t-1}(i) \\ b_{t-1}(i) \\ b_{t-1}^*(i) \end{bmatrix} + \sum_{i=1}^{4} \sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{e,3}(\lambda_i, \delta_j) \\
\tilde{b}_t^* - \sum_{i=1}^{4} \sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{0,4}(\lambda_i, \delta_j) &= \bar{D}_k^4 \begin{bmatrix} \hat{c}_{t-1}(i) \\ b_{t-1}(i) \\ b_{t-1}^*(i) \end{bmatrix} + \sum_{i=1}^{4} \sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{e,4}(\lambda_i, \delta_j)
\end{aligned}$$

where $\bar{D}_k^i$ is the ith row of $\bar{D}_k$ and $j \in \{c, \lambda, w, b^*\}$ indexes the collections of exogenous terms $\delta_j$ appearing in the original system. The new terms $P_t^{0,l}(\lambda_i, \delta_j)$ and $P_t^{e,l}(\lambda_i, \delta_j)$ capture the contemporaneous and expectational variables respectively, and $l$ indexes the row of the system ( 8) and $i$ indexes the terms attached to each discount factor. NOTE: Domestic debt will be in zero supply. So eliminate to give

$$\begin{aligned}
\hat{w}_t^* - \sum_{i=1}^{4}\sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{0,1}(\lambda_i,\delta_j) &= \bar{D}_k(1,1)\,\hat{c}_{t-1}(i) + \bar{D}_k(1,3)\,b_{t-1}^*(i) + \sum_{i=1}^{4}\sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{e,1}(\lambda_i,\delta_j) \\
\hat{c}_t - \sum_{i=1}^{4}\sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{0,2}(\lambda_i,\delta_j) &= \bar{D}_k(2,1)\,\hat{c}_{t-1}(i) + \bar{D}_k(2,3)\,b_{t-1}^*(i) + \sum_{i=1}^{4}\sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{e,2}(\lambda_i,\delta_j) \\
\hat{\lambda}_t - \sum_{i=1}^{4}\sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{0,3}(\lambda_i,\delta_j) &= \bar{D}_k(3,1)\,\hat{c}_{t-1}(i) + \bar{D}_k(3,3)\,b_{t-1}^*(i) + \sum_{i=1}^{4}\sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{e,3}(\lambda_i,\delta_j) \\
\tilde{b}_t^* - \sum_{i=1}^{4}\sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{0,4}(\lambda_i,\delta_j) &= \bar{D}_k(4,1)\,\hat{c}_{t-1}(i) + \bar{D}_k(4,3)\,b_{t-1}^*(i) + \sum_{i=1}^{4}\sum_{j}^{\{c,\lambda,w,b^*,b_{lag}\}} P_t^{e,4}(\lambda_i,\delta_j)
\end{aligned}$$

The matlab code loops over the indexes $i$ and $l$ so we need only deal with reorganizing the terms in the deltas as follows. Reminder: they are:

$$\begin{aligned}
\delta_t^w &= -\xi_w\beta E_t^j\left(\pi_{t+1}^w - \hat{\gamma}_{z,t} - \iota_w\pi_t\right) - \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_t + \phi^{-1}\left(\hat{y}_{H,t} - a_t\right) - \hat{w}_t^r - \frac{\hat{\theta}_{w,t}}{(\theta_w - 1)}\right] \\
\delta_t^c &= \beta b E_t\hat{\xi}_{t+1} - \hat{\xi}_t = \left(\beta b\rho_\xi - 1\right)\hat{\xi}_t \\
\delta_t^\lambda &= \hat{P}_t^{B^*} - E_t^i\left(-\hat{\gamma}_{z,t} - \pi_{t+1} + \log S_{t+1} - \log S_t\right) \\
\delta_t^{b^*} &= \hat{P}_t^B + \hat{\gamma}_{z,t} + E_t^i\pi_{t+1} + \chi\gamma_z\rho_g\tilde{g}_t \\
&\quad -\chi\gamma_z\left(\left(\frac{c_F}{y}\alpha - \left(1-\frac{g}{y}\right)(1-\alpha)\right)\log\left(\frac{P_{F,t+1}}{P_{H,t+1}}\right) + \hat{y}_{H,t+1} - \frac{c_F}{y}\log\left(\frac{Z_{t+1}^*}{Z_{t+1}}\right) - \frac{c_F}{y}q_{t+1}\right) \\
\delta_t^{b_{lag}} &= \gamma_z\left(\left(\frac{c_F}{y}\alpha - \left(1-\frac{g}{y}\right)(1-\alpha)\right)\log\left(\frac{P_{F,t}}{P_{H,t}}\right) + \hat{y}_{H,t} - \frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right) - \tilde{g}_t - \frac{c_F}{y}q_t\right)
\end{aligned}$$

Contemporaneous terms:

$$\begin{aligned}
P_t^{0,l}(\lambda_i,\delta_w) &= D_{\lambda_i}(l,1)\left(\xi_w\beta\left(\hat{\gamma}_t^z + \iota_w\pi_t\right) - \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_t + \phi^{-1}\left(\hat{y}_{H,t} - a_t\right) - \hat{w}_t^r - \frac{\hat{\theta}_{w,t}}{(\theta_w - 1)}\right]\right) \\
P_t^{0,l}(\lambda_i,\delta_c) &= D_{\lambda_i}(l,2)\left(\beta\rho_\xi b - 1\right)\hat{\xi}_t \\
P_t^{0,l}(\lambda_i,\delta_\lambda) &= D_{\lambda_i}(l,3)\left(\hat{P}_t^{B^*} + \hat{\gamma}_{z,t} + \log S_t\right) \\
P_t^{0,l}(\lambda_i,\delta_{b^*}) &= D_{\lambda_i}(l,4)\left(\hat{P}_t^B + \hat{\gamma}_{z,t} + \chi\gamma_z\rho_g\tilde{g}_t\right) \\
P_t^{0,l}\left(\lambda_i,\delta_{b_{lag}}\right) &= D_{\lambda_i}(l,5)\,\gamma_z\left(\left(\frac{c_F}{y}\alpha - \left(1-\frac{g}{y}\right)(1-\alpha)\right)\log\left(\frac{P_{F,t}}{P_{H,t}}\right) + \hat{y}_{H,t} - \frac{c_F}{y}\log\left(\frac{Z_t^*}{Z_t}\right) - \tilde{g}_t - \frac{c_F}{y}q_t\right)
\end{aligned}$$

and the discounted terms:

$$P_t^{e,l}(\lambda_i,\delta_w) = \lambda_i^{-1}D_{\lambda_i}(l,1)\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left(\begin{array}{c} -\xi_w\beta\left(\lambda_i\pi_{T+1}^w-\hat{\gamma}_{T+1}^z-\iota_w\pi_{T+1}\right)- \\ \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_{T+1}+\phi^{-1}\left(\hat{y}_{H,T+1}-a_{T+1}\right)-\hat{w}_{T+1}^r-\frac{\hat{\theta}_{w,T+1}}{(\theta_w-1)}\right]\end{array}\right)$$

$$P_t^{e,l}(\lambda_i,\delta_c) = \lambda_i^{-1}D_{\lambda_i}(l,2)\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left(\beta\rho_\xi b-1\right)\hat{\xi}_{T+1}$$

$$P_t^{e,l}(\lambda_i,\delta_\lambda) = D_{\lambda_i}(l,3)\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left(\lambda_i^{-1}\hat{P}_{T+1}^{B^*}-E_t^i\left(-\lambda_i^{-1}\hat{\gamma}_{z,T+1}-\pi_{T+1}+\log S_{T+1}-\lambda_i^{-1}\log S_{T+1}\right)\right)$$

$$P_t^{e,l}(\lambda_i,\delta_{b^*}) = D_{\lambda_i}(l,4)\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left(\begin{array}{c}\lambda_i^{-1}\hat{P}_{T+1}^{B}+\lambda_i^{-1}\hat{\gamma}_{z,T+1}+\pi_{T+1}+\lambda_i^{-1}\chi\gamma_z\rho_g g_{T+1} \\ -\chi\gamma_z\left(\begin{array}{c}\left(\frac{c_F}{y}\alpha-\left(1-\frac{g}{y}\right)(1-\alpha)\right)\log\left(\frac{P_{F,t+1}}{P_{H,t+1}}\right) \\ +\hat{y}_{H,t+1}-\frac{c_F}{y}\log\left(\frac{Z_{t+1}^*}{Z_{t+1}}\right)-\frac{c_F}{y}q_{t+1}\end{array}\right)\end{array}\right)$$

$$P_t^{e,l}\left(\lambda_i,\delta_{b_{lag}}\right) = \lambda_i^{-1}D_{\lambda_i}(l,4)\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left(\gamma_z\left(\begin{array}{c}\left(\frac{c_F}{y}\alpha-\left(1-\frac{g}{y}\right)(1-\alpha)\right)\log\left(\frac{P_{F,T+1}}{P_{H,T+1}}\right) \\ +\hat{y}_{H,T+1}-\frac{c_F}{y}\log\left(\frac{Z_{T+1}^*}{Z_{T+1}}\right)-\tilde{g}_{T+1}-\frac{c_F}{y}q_{T+1}\end{array}\right)\right)$$

### 6.1 Decision Rule 2: Firms – Domestic Goods

For domestic goods supply we have

$$\pi_{H,t}-\iota_H\pi_{H,t-1}=\frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H}E_t\sum_{T=t}^{\infty}(\xi_H)^{T-t}\left[\begin{array}{c}\frac{\xi_H\beta}{1-\xi_H\beta}\left(\pi_{H,T+1}-\iota_H\pi_{H,T}\right) \\ +\left(\hat{w}_T+(1-\alpha)\log\left(\frac{P_{F,T}}{P_{H,T}}\right)-a_T\right)-\frac{\hat{\theta}_{H,T}}{(\theta_H-1)}\end{array}\right]$$

which can be rearranged as

$$\begin{aligned}\pi_{H,t}-\iota_H\pi_{H,t-1} &= \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H}\left(\hat{w}_t+(1-\alpha)\log\left(\frac{P_{F,t}}{P_{H,t}}\right)-a_T-\frac{\hat{\theta}_{H,t}}{(\theta_H-1)}\right)-(1-\xi_H)\beta\iota_H\pi_{H,t} \\ & \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H}E_t\sum_{T=t}^{\infty}(\xi_H\beta)^{T-t}\left[\begin{array}{c}\frac{\xi_H\beta}{1-\xi_H\beta}\left(\pi_{H,T+1}-\iota_H\xi_H\beta\pi_{H,T+1}\right) \\ +\xi_H\beta\left(\hat{w}_{T+1}+(1-\alpha)\log\left(\frac{P_{F,T}}{P_{H,T}}\right)-a_{T+1}-\frac{\hat{\theta}_{H,T+1}}{(\theta_H-1)}\right)\end{array}\right]\end{aligned}$$

or

$$\begin{aligned}(1+(1-\xi_H)\beta\iota_H)\pi_{H,t}- &= \iota_H\pi_{H,t-1}+\frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H}\left(\hat{w}_t+(1-\alpha)\log\left(\frac{P_{F,t}}{P_{H,t}}\right)-a_T-\frac{\hat{\theta}_{H,t}}{(\theta_H-1)}\right)+ \\ & \frac{(1-\xi_H)(1-\xi_H\beta)}{\xi_H}E_t\sum_{T=t}^{\infty}(\xi_H\beta)^{T-t}\left[\begin{array}{c}\frac{\xi_H\beta(1-\iota_H\xi_H\beta)}{1-\xi_H\beta}\pi_{H,T+1} \\ +\xi_H\beta\left(\begin{array}{c}\hat{w}_{T+1}+(1-\alpha)\log\left(\frac{P_{F,T+1}}{P_{H,T+1}}\right) \\ -a_{T+1}-\frac{\hat{\theta}_{H,T+1}}{(\theta_H-1)}\end{array}\right)\end{array}\right]\end{aligned}$$

### 6.2 Decision Rule 3: Firms – Imported Goods

For imported goods inflation we have

$$\pi_{F,t}-\iota_F\pi_{F,t-1}=\frac{(1-\xi_F)(1-\xi_F\beta)}{\xi_F}E_t\sum_{T=t}^{\infty}(\xi_F\beta)^{T-t}\left[\frac{\xi_F\beta}{1-\xi_F\beta}\left(\pi_{F,T+1}-\iota_F\pi_{F,T}\right)+\psi_T-\frac{\hat{\theta}_{F,T}}{(\theta_F-1)}\right]$$

which can be re-arranged as

$$\begin{aligned}(1+(1-\xi_F)\beta\iota_F)\pi_{F,t} &= \iota_F\pi_{F,t-1}+\frac{(1-\xi_F)(1-\xi_F\beta)}{\xi_F}\left(\psi_t-\frac{\hat{\theta}_{F,t}}{(\theta_F-1)}\right)- \\ &\quad +\frac{(1-\xi_F)(1-\xi_F\beta)}{\xi_F}E_t\sum_{T=t}^{\infty}(\xi_F\beta)^{T-t}\left[\begin{array}{c}\frac{\xi_F\beta(1-\iota_F\xi_F\beta)}{1-\xi_F\beta}\pi_{F,T+1} \\ +\xi_H\beta\left(\psi_{T+1}-\frac{\hat{\theta}_{F,T+1}}{(\theta_F-1)}\right)\end{array}\right]\end{aligned}$$

## 6.3 Decision Rule 4: Foreign Households

We want to write the optimal decision of foreign household $i$ in the form

$$\Psi_0^{HH}E_t^i\left[\begin{array}{c}\hat{w}_{t+1}^{**}(i) \\ \hat{c}_{t+1}^{*}(i) \\ \lambda_{t+1}^{*}(i) \\ \hat{c}_t^{*}(i) \\ b_t^{*}(i)\end{array}\right]=\Psi_1^{HH}\left[\begin{array}{c}\hat{w}_t^{**}(i) \\ \hat{c}_t^{*}(i) \\ \lambda_t^{*}(i) \\ \hat{c}_{t-1}^{*}(i) \\ b_{t-1}^{*}(i)\end{array}\right]+\delta_t$$

Let's rearrange the first order conditions one by one.

$$\delta_t=\left[\begin{array}{c}\delta_t^w \\ \delta_t^c \\ \delta_t^\lambda \\ 0 \\ \delta_t^{b^*}\end{array}\right]$$

The first order conditions are

1. Optimal reset wage: $w_t^{**}$and wage inflation $\pi_t^{w^*}$

$$\begin{aligned}\hat{w}^{**}(j) &= \xi_w\beta E_t^j\left(\pi_{t+1}^{w^*}-\hat{\gamma}_t^{z^*}-\iota_w\pi_t^*\right)+\frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_t^*+\phi^{-1}N_t^*-\hat{w}_t^{r^*}-\hat{\lambda}_t^*(j)-\frac{\hat{\theta}_{w,t}^*}{(\theta_w-1)}\right]+\xi_w\beta E_t^j\hat{w}_{t+1}^{**}(j) \\ w_t^{**} &= \frac{\xi_w}{1-\xi_w}\left(\pi_t^{w^*}-\iota_w\pi_{t-1}^*-\hat{\gamma}_{t-1}^*\right)\end{aligned}$$

or

$$\begin{aligned}\xi_w\beta E_t^j\hat{w}_{t+1}^{**}(j) &= \hat{w}^{**}(j)-\xi_w\beta E_t^j\left(\pi_{t+1}^{w^*}-\hat{\gamma}_t^{z^*}-\iota_w\pi_t^*\right)-\frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_t^*+\phi^{-1}N_t^*-\hat{w}_t^{r^*}-\hat{\lambda}_t^*(j)-\frac{\hat{\theta}_{w,t}^*}{(\theta_w-1)}\right] \\ &= \hat{w}^{**}(j)+\frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\hat{\lambda}_t^*(j)+\delta_t^{w^{**}} \\ \delta_t^{w^{**}} &= -\xi_w\beta E_t^j\left(\pi_{t+1}^{w^*}-\hat{\gamma}_t^{z^*}-\iota_w\pi_t^*\right)-\frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_t^*+\phi^{-1}N_t^*-\hat{w}_t^{r^*}-\frac{\hat{\theta}_{w,t}^*}{(\theta_w-1)}\right]\end{aligned}$$

3. Marginal Utility: $c_t^*(i)$

$$(1-\beta b)\,\hat{\lambda}_t^*(i) = \hat{\xi}_t^* + \frac{b\sigma}{1-b}\hat{c}_{t-1}^*(i) - \frac{\sigma}{1-b}\left(1+\beta b^2\right)\hat{c}_t^*(i) - \beta b E_t \hat{\xi}_{t+1}^* + \beta b \frac{\sigma}{1-b} E_t \hat{c}_{t+1}^*(i)$$

or

$$\begin{aligned}
\beta b \frac{\sigma}{1-b} E_t \hat{c}_{t+1}^*(i) &= (1-\beta b)\,\hat{\lambda}_t^*(i) + \frac{\sigma}{1-b}\left(1+\beta b^2\right)\hat{c}_t^*(i) - \frac{b\sigma}{1-b}\hat{c}_{t-1}^*(i) - (1-\beta b\rho_\xi)\,\hat{\xi}_t^* \\
&= (1-\beta b)\,\hat{\lambda}_t^*(i) + \frac{\sigma}{1-b}\left(1+\beta b^2\right)\hat{c}_t^*(i) - \frac{b\sigma}{1-b}\hat{c}_{t-1}^*(i) + \delta_t^{c^*} \\
\delta_t^{c^*} &= (\beta b\rho_\xi - 1)\,\hat{\xi}_t^*
\end{aligned}$$

4. Bond holdings: $\lambda_t^*$

$$\hat{\lambda}_t^*(i) - R_t^* = E_t^i\left(\hat{\lambda}_{t+1}^*(i) - \hat{\gamma}_{z,t}^* - \pi_{t+1}^*\right)$$

or

$$\begin{aligned}
E_t^i \hat{\lambda}_{t+1}^*(i) &= \hat{\lambda}_t^*(i) - R_t^* + \hat{\gamma}_{z,t}^* + E_t^i \pi_{t+1}^* = \hat{\lambda}_t^*(i) + \delta_t^{\lambda^*} \\
\delta_t^{\lambda^*} &= R_t^* + \hat{\gamma}_{z,t}^* + E_t^i \pi_{t+1}^*
\end{aligned}$$

5. Flow budget constraint: $b_t^*(i)$ (will assume in zero supply)

$$\begin{aligned}
P^{B^*}\tilde{b}_t^*(i) &= \beta^{-1}\left(P^B \tilde{b}_{t-1}^*(i)\right) - \frac{c}{y}\hat{c}_t^*(i) + \frac{c}{y}\hat{c}_t^* \\
&= \beta^{-1}\left(P^B \tilde{b}_{t-1}^*(i)\right) - \frac{c}{y}\hat{c}_t^*(i) + \delta_t^{b^*} \\
\delta_t^{b^*} &= \frac{c}{y}\hat{c}_t^*
\end{aligned}$$

Hence we have

$$\Psi_0^{HH} = \begin{bmatrix} \xi_w\beta & 0 & 0 & 0 & 0 \\ 0 & \beta b\frac{\sigma}{1-b} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & P^{B^*} \end{bmatrix} \quad \text{and} \quad \Psi_1^{HH} = \begin{bmatrix} 1 & 0 & \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}} & 0 & 0 \\ 0 & \frac{\sigma}{1-b}\left(1+\beta b^2\right) & (1-\beta b) & -\frac{b\sigma}{1-b} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{c}{y} & 0 & 0 & \beta^{-1}P^{B^*} \end{bmatrix}$$

where terms, exogenous to the household, are collected in the vector

$$\delta_t = \begin{bmatrix} \delta_t^{w^*} \\ \delta_t^{c^*} \\ \delta_t^{\lambda^*} \\ 0 \\ \delta_t^{b^*} \end{bmatrix} = \begin{bmatrix} -\xi_w\beta E_t^j\left(\pi_{t+1}^{w^*} - \hat{\gamma}_t^{z^*} - \iota_w\pi_t^*\right) - \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_t^* + \phi^{-1}\left(\frac{c}{y}c_t^* + \tilde{g}_t^* - a_t^*\right) - \hat{w}_t^{r^*} - \frac{\hat{\theta}_{w,t}^*}{(\theta_w-1)}\right] \\ (\beta b\rho_\xi - 1)\,\hat{\xi}_t^* \\ -R_t^* + \hat{\gamma}_{z,t}^* + E_t^i\pi_{t+1}^* \\ 0 \\ \frac{c}{y}\hat{c}_t^* \end{bmatrix}$$

The solution to this system is

$$\begin{bmatrix} \hat{w}_t^{**}(i) \\ \hat{c}_t^*(i) \\ \lambda_t^*(i) \end{bmatrix} = \bar{D}_k \begin{bmatrix} \hat{c}_{t-1}^*(i) \\ b_{t-1}^*(i) \end{bmatrix} + \bar{D}_{\delta 1}\sum_{T=t}^{\infty}\Lambda^{-(T-t)}\bar{D}_{\delta 2}\hat{E}_t^i\delta_T$$

where the fourth column of $\bar{D}_{\delta 2}$ should be eliminated as it is multiplied by zero (the fourth element of the vector $\delta_t$). We can rewrite as before in the form

$$\begin{bmatrix} \hat{w}_t^{**}(i) \\ \hat{c}_t^{*}(i) \\ \lambda_t^{*}(i) \end{bmatrix} = \bar{D}_k \begin{bmatrix} \hat{c}_{t-1}^{*}(i) \\ b_{t-1}^{*}(i) \end{bmatrix} + D_{\lambda_1} \sum_{T=t}^{\infty} \lambda_1^{-(T-t)} E_t \bar{\delta}_T \tag{9}$$
$$+ D_{\lambda 2} \sum_{T=t}^{\infty} \lambda_2^{-(T-t)} E_t \bar{\delta}_T + D_{\lambda_3} \sum_{T=t}^{\infty} \lambda_3^{-(T-t)} E_t \bar{\delta}_T$$

where

$$D_{\lambda_1} = \bar{D}_{\delta 1} \begin{bmatrix} \bar{D}_{\delta 2}^1 \\ 0 \\ 0 \end{bmatrix}; D_{\lambda_2} = \bar{D}_{\delta 1} \begin{bmatrix} 0 \\ \bar{D}_{\delta 2}^2 \\ 0 \end{bmatrix}; D_{\lambda_3} = \bar{D}_{\delta 1} \begin{bmatrix} 0 \\ 0 \\ \bar{D}_{\delta 2}^3 \end{bmatrix}$$

where $\bar{D}_{\delta 2}^i$ denotes the $i$th row of the matrix $\bar{D}_{\delta 2}$.

Write the solution in the following form (dropping the household indicator and imposing the equilibrium condition that debt is in zero supply, with all households having an equal share)

$$\hat{w}_t^{**} - \sum_{i=1}^{3} \sum_{j}^{\{c^*,\lambda^*,w^{**},b^*\}} P_t^{0,1}(\lambda_i, \delta_j) = \bar{D}_k(1,1)\hat{c}_{t-1}(i) + \sum_{i=1}^{3} \sum_{j}^{\{c^*,\lambda^*,w^{**},b^*\}} P_t^{e,1}(\lambda_i, \delta_j)$$
$$\hat{c}_t - \sum_{i=1}^{3} \sum_{j}^{\{c^*,\lambda^*,w^{**},b^*\}} P_t^{0,2}(\lambda_i, \delta_j) = \bar{D}_k(2,1)\hat{c}_{t-1}(i) + \sum_{i=1}^{3} \sum_{j}^{\{c^*,\lambda^*,w^{**},b^*\}} P_t^{e,2}(\lambda_i, \delta_j)$$
$$\hat{\lambda}_t - \sum_{i=1}^{3} \sum_{j}^{\{c^*,\lambda^*,w^{**},b^*\}} P_t^{0,3}(\lambda_i, \delta_j) = \bar{D}_k(3,1)\hat{c}_{t-1}(i) + \sum_{i=1}^{3} \sum_{j}^{\{c^*,\lambda^*,w^{**},b^*\}} P_t^{e,3}(\lambda_i, \delta_j)$$

where $j = \{c^*, \lambda^*, w^{**}, b^*\}$ index the collections of exogneous terms in $\delta_t$ appearing in the original system. The matlab code loops over the indexes $i$ and $l$ so we need only deal with reorganizing the terms in the deltas as follows. Reminder: they are:The matlab code loops over the indexes $i$ and $l$ so we need only deal with reorganizing the terms in the deltas as follows. Reminder: they are:

$$\delta_t = \begin{bmatrix} \delta_t^{w^*} \\ \delta_t^{c^*} \\ \delta_t^{\lambda^*} \\ 0 \\ \delta_t^{b^*} \end{bmatrix} = \begin{bmatrix} -\xi_w \beta E_t^j \left(\pi_{t+1}^{w^*} - \hat{\gamma}_t^{z^*} - \iota_w \pi_t^*\right) - \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}} \left[\hat{\varphi}_t^* + \phi^{-1}\left(\frac{c}{y}c_t^* + \tilde{g}_t^* - a_t^*\right) - \hat{w}_t^{r^*} - \frac{\hat{\theta}_{w,t}^*}{(\theta_w - 1)}\right] \\ (\beta b \rho_\xi - 1)\hat{\xi}_t^* \\ \hat{P}_t^{B^*} + \hat{\gamma}_{z,t}^* + E_t^i \pi_{t+1}^* \\ 0 \\ \frac{c}{y}\hat{c}_t^* \end{bmatrix}$$

Contemporaneous terms:

$$P_t^{0,l}(\lambda_i, \delta_w) = D_{\lambda_i}(l,1)\left(\xi_w \beta E_t^j \left(\hat{\gamma}_t^{z^*} + \iota_w \pi_t^*\right) - \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}} \left[\hat{\varphi}_t^* + \phi^{-1}\left(\frac{c}{y}c_t^* + \tilde{g}_t^* - a_t^*\right) - \hat{w}_t^{r^*} - \frac{\hat{\theta}_{w,t}^*}{(\theta_w - 1)}\right]\right)$$
$$P_t^{0,l}(\lambda_i, \delta_c) = D_{\lambda_i}(l,2)(\beta\rho_\xi b - 1)\hat{\xi}_t^*$$
$$P_t^{0,l}(\lambda_i, \delta_\lambda) = D_{\lambda_i}(l,3)\left(-R_t^* + \hat{\gamma}_{z,t}^*\right)$$
$$P_t^{0,l}(\lambda_i, \delta_{b^*}) = D_{\lambda_i}(l,4)\frac{c}{y}\hat{c}_t^*$$

and the discounted terms:

$$\begin{aligned}
P_t^{e,l}\left(\lambda_i,\delta_w\right) &= \lambda_i^{-1}D_{\lambda_i}\left(l,1\right)\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left(\begin{array}{c} -\xi_w\beta\left(\lambda_i\pi_{T+1}^{w^*}-\hat{\gamma}_{z,T+1}^*-\iota_w\pi_{T+1}^*\right)- \\ \frac{1-\xi_w\beta}{1+\theta_w\phi^{-1}}\left[\hat{\varphi}_{T+1}^*+\phi^{-1}\left(\frac{c}{y}c_{T+1}^*+\tilde{g}_{T+1}^*-a_{T+1}^*\right)-\hat{w}_{T+1}^{r^*}-\frac{\hat{\theta}_{w,T+1}^*}{(\theta_w-1)}\right]\end{array}\right) \\
P_t^{0,l}\left(\lambda_i,\delta_c\right) &= \lambda_i^{-1}D_{\lambda_i}\left(l,2\right)\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left(\beta\rho_\xi b-1\right)\hat{\xi}_{T+1}^* \\
P_t^{0,l}\left(\lambda_i,\delta_\lambda\right) &= \lambda_i^{-1}D_{\lambda_i}\left(l,3\right)\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\left(-R_{T+1}^*+\hat{\gamma}_{z,T+1}^*+\lambda_i\pi_{t+1}^*\right) \\
P_t^{e,l}\left(\lambda_i,\delta_{b^*}\right) &= \lambda_i^{-1}D_{\lambda_i}\left(l,4\right)\sum_{T=t}^{\infty}\lambda_i^{-(T-t)}\frac{c}{y}\hat{c}_{T+1}^*
\end{aligned}$$

### 6.4 Decision Rule 5: Foreign Firms

For foreign home goods supply we have

$$\pi_{H,t}^*-\iota_H\pi_{H,t-1}^*=\frac{\left(1-\xi_H\right)\left(1-\xi_H\beta\right)}{\xi_H}E_t\sum_{T=t}^{\infty}\left(\xi_H\right)^{T-t}\left[\frac{\xi_H\beta}{1-\xi_H\beta}\left(\pi_{H,T+1}^*-\iota_H\pi_{H,T}^*\right)+\left(\hat{w}_T^*-a_T^*\right)-\frac{\hat{\theta}_{H,T}^*}{(\theta_H-1)}\right]$$

which can be rearranged as

$$\begin{aligned}
\pi_{H,t}^*-\iota_H\pi_{H,t-1}^* &= \frac{\left(1-\xi_H\right)\left(1-\xi_H\beta\right)}{\xi_H}\left(\hat{w}_t^*-a_t^*-\frac{\hat{\theta}_{H,t}^*}{(\theta_H-1)}\right)-\left(1-\xi_H\right)\beta\iota_H\pi_{H,t}^* \\
& \quad \frac{\left(1-\xi_H\right)\left(1-\xi_H\beta\right)}{\xi_H}E_t\sum_{T=t}^{\infty}\left(\xi_H\beta\right)^{T-t}\left[\begin{array}{c}\frac{\xi_H\beta(1-\iota_H\xi_H\beta)}{1-\xi_H\beta}\pi_{H,T+1}^* \\ +\xi_H\beta\left(\hat{w}_{T+1}^*-a_{T+1}^*-\frac{\hat{\theta}_{H,T+1}^*}{(\theta_H-1)}\right)\end{array}\right]
\end{aligned}$$

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