Online Appendix to "A Fiscal Theory of Monetary Policy with Partially Repaid Long-Term Debt"

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Appendix A. Model solution algebra

This Appendix sets out the algebra to solve the model (8)-(20). I express the model in the form

$$y_{t+1} = By_t + C\varepsilon_{t+1} + D\delta_{t+1} \tag{A.1}$$

where y is a vector of variables, ε are the structural shocks, and δ are expectational errors in the equations that only tie down expectations. (The general case has a leading term Ay_{t+1}, but we do not need that here.) We eigenvalue decompose the transition matrix B, we solve unstable roots forward and stable roots backward to determine the expectational errors δ as a function of the structural shocks ε. Then, we can compute impulse-response functions. This is the standard Blanchard and Kahn (1980) method, applied to this problem.

We can send the model as is to the computer, but we save some time with preliminary simplifications. First, as discussed above (26)-(35), we have $v = v^*$ and $\pi = \pi^*$ in equilibrium, so we can eliminate the starred variables. Second, since surpluses and debt do not enter utility or pricing decisions, we can simplify by first solving for $\{\pi_t, x_t, r_t^n\}$ given $\left\{\varepsilon_t^s, \varepsilon_t^i\right\}$ and then calculating surpluses and debt from (32), (33), (35).

Adding δ shocks in place of expectations and rearranging the equations we now have

$$\begin{aligned}
x_{t+1} &= x_t + \sigma i_t - \sigma\pi_{t+1} + \delta_{x,t+1} + \sigma\delta_{\pi,t+1} \\
\beta\pi_{t+1} &= \pi_t - \kappa x_t + \beta\delta_{\pi,t+1} \\
i_t &= \theta_{i\pi}\pi_t + \theta_{ix}x_t + u_t^i \\
\delta_{\pi,t+1} &= -\beta_s\varepsilon_{t+1}^s - \beta_i\varepsilon_{t+1}^i
\end{aligned}$$

$$
\begin{aligned}
r^n_{t+1} &= i_t + \delta_{r^n,t+1} \\
\omega q_{t+1} &= q_t + r^n_{t+1} \\
u^i_{t+1} &= \rho_i u^i_t + \varepsilon^i_{t+1}.
\end{aligned}
$$

Eliminating i_t, $\delta_{\pi,t}$, r^n_{t+1},

$$
\begin{aligned}
x_{t+1} &= \left(1 + \frac{\kappa\sigma}{\beta} + \sigma\theta_{ix}\right) x_t + \sigma\left(\theta_{i\pi} - \frac{1}{\beta}\right)\pi_t + \sigma u^i_t + \delta_{x,t+1} \\
\pi_{t+1} &= -\frac{\kappa}{\beta}x_t + \frac{1}{\beta}\pi_t - \beta_s\varepsilon^s_{t+1} - \beta_i\varepsilon^i_{t+1} \\
\omega q_{t+1} &= \theta_{ix}x_t + \theta_{i\pi}\pi_t + q_t + u^i_t + \delta_{r^n,t+1} \\
u^i_{t+1} &= \rho_i u^i_t + \varepsilon^i_{t+1}.
\end{aligned}
$$

In matrix form,

$$
\begin{aligned}
\begin{bmatrix} x_{t+1} \\ \pi_{t+1} \\ q_{t+1} \\ u^i_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 + \sigma\theta_{ix} + \sigma\kappa/\beta & \sigma\theta_{i\pi} - \sigma/\beta & 0 & \sigma \\ -\kappa/\beta & 1/\beta & 0 & 0 \\ \theta_{ix}/\omega & \theta_{i\pi}/\omega & 1/\omega & 1/\omega \\ 0 & 0 & 0 & \rho_i \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \\ q_t \\ u^i_t \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ -\beta_i & -\beta_s \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^i_{t+1} \\ \varepsilon^s_{t+1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{x,t+1} \\ \delta_{r^n,t+1} \end{bmatrix}.
\end{aligned}
$$

Now, we solve the model as

$$
\begin{aligned}
y_{t+1} &= By_t + \ C\varepsilon_{t+1} + D\delta_{t+1} \\
y_{t+1} &= Q\Lambda Q^{-1}y_t + C\varepsilon_{t+1} + D\delta_{t+1} \\
Q^{-1}y_{t+1} &= \Lambda Q^{-1}y_t + Q^{-1}C\varepsilon_{t+1} + Q^{-1}D\delta_{t+1} \\
z_{t+1} &= \Lambda z_t + Q^{-1}C\varepsilon_{t+1} + Q^{-1}D\delta_{t+1}. \qquad \text{(A.2)}
\end{aligned}
$$

Let G_f select rows with eigenvalues greater than or equal to one, and G_b select rows with eigenvalues less than one. For example, if the first and third eigenvalues are greater than or equal to one,

$$
G_f = \begin{bmatrix} 1 & 0 & 0 & 0 & ... \\ 0 & 0 & 1 & 0 & ... \end{bmatrix},
$$

$$G_b = \begin{bmatrix} 0 & 1 & 0 & 0 & ... \\ 0 & 0 & 0 & 1 & ... \end{bmatrix}.$$

The z_t corresponding to eigenvalues greater than one must be zero, so we can find the expectational errors δ_{t+1} in terms of the structural shocks ε_{t+1},

$$\begin{aligned} 0 &= G_f Q^{-1} C \varepsilon_{t+1} + G_f Q^{-1} D \delta_{t+1} \\ \delta_{t+1} &= -\left(G_f Q^{-1} D\right)^{-1} G_f Q^{-1} C \varepsilon_{t+1}. \end{aligned}$$

For this approach to work there must be as many rows of G_f as columns of δ, i.e. as many eigenvalues greater or equal to one as there are expectational errors. Substituting in (A.2), we have the evolution of the transformed z variables,

$$z_{t+1} = \Lambda z_t + Q^{-1} \left[I - D \left(G_f Q^{-1} D\right)^{-1} G_f Q^{-1} \right] C \varepsilon_{t+1},$$

and then the original variables obey

$$y_t = Q z_t.$$

For computation it is better to force the elements of z_t that should be zero to be exactly zero. Machine zeros ($1e - 14$) multiplied by explosive eigenvalues eventually explode. Thus, I find the non-zero z only by simulating forward the nonzero elements of z,

$$G_b z_{t+1} = G_b \Lambda z_t + G_b Q^{-1} \left[C - D \left(G_f Q^{-1} D\right)^{-1} G_f Q^{-1} C \right] \varepsilon_{t+1}.$$

Appendix B. Surplus process estimates

This section presents direct estimates of the surplus process. Table B.2 presents three vector autoregressions involving surpluses and debt. Here, v_t is the log market value of US federal debt divided by consumption, scaled by the consumption/GDP ratio. I divide by consumption to focus on variation in the debt rather than cyclical variation in GDP. π is the log GDP deflator, g_t is log consumption growth, r_t^n is the nominal return on the government bond portfolio,

i_t is the three month treasury bill rate and y_t is the 10 year government bond yield. I infer the surplus s from the linearized identity allowing growth,

$$\rho v_{t+1} = v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} - s_{t+1}.$$

Cochrane (2020a) describes the data and VAR in more detail.

	s_t	v_t	π_t	g_t	r^n_t	i_t	y_t	$\sigma(\varepsilon)$	$\sigma(s)$
VAR $s_{t+1} =$	0.35	0.043	-0.25	1.37	-0.32	0.50	-0.04	4.75	6.60
std. err.	(0.09)	(0.022)	(0.31)	(0.45)	(0.16)	(0.46)	(0.58)		
$v_{t+1} =$	-0.24	0.98	-0.29	-2.00	0.28	-0.72	1.60		
std. err.	(0.12)	(0.03)	(0.43)	(0.61)	(0.27)	(0.85)	(1.04)		
Small VAR $s_{t+1} =$	0.55	0.027						5.46	6.60
std. err.	(0.07)	(0.016)							
$v_{t+1} =$	-0.54	0.96							
std. err.	(0.11)	(0.02)							
AR(1) $s_{t+1} =$	0.55							5.55	6.60
std. err.	(0.07)								

Table B.2: Surplus forecasting regressions. Variables are s = surplus, v = debt/GDP, π = inflation, g = growth, i = 3 month rate, y = 10 year yield. Sample 1947-2018.

The first group of regressions in Table B.2 presents the surplus and value regressions in the full VAR. The surplus is moderately persistent (0.35). Most importantly, the surplus responds to the value of the debt (0.043). This coefficient is measured with a t statistic of barely 2, using simple OLS standard errors. However, this point estimate confirms estimates such as Bohn (1998). Bohn includes additional variables in the regression, which one may interpret as estimates of the θ terms of the policy rule, and which soak up a good deal of residual variance. For this reason, and by using longer samples, Bohn finds much stronger statistical significance. Debt is very persistent (0.98), and higher surpluses pay down debt (-0.24).

The second group of estimates presents a smaller VAR consisting of only surplus and debt. The coefficients are similar to those of surplus and debt in the larger VAR, and we will see that this smaller VAR contains most of the message of the larger VAR.

The third estimate is a simple AR(1). Though the small VAR and AR(1) have the same coefficient 0.55 of the surplus on the lagged surplus, nearly the same R^2, and a barely significant coefficient on debt, we will see how the specifications differ crucially on long-run properties.

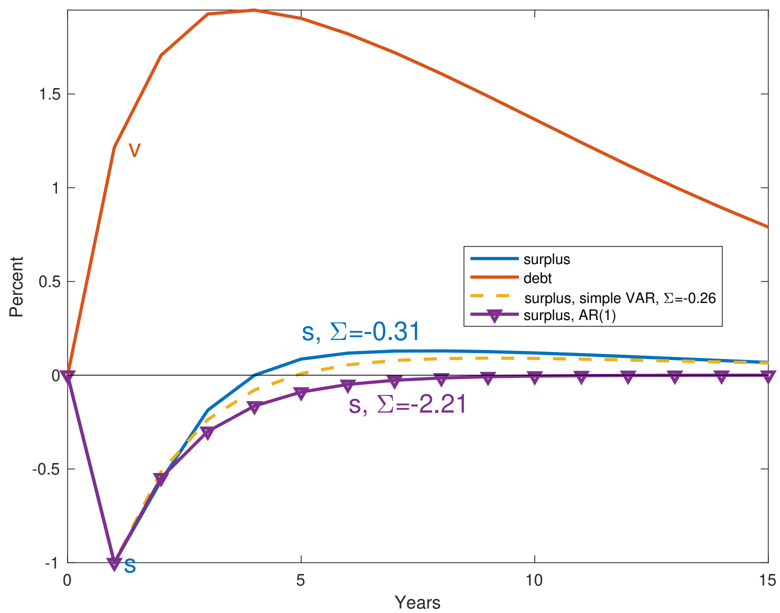

Figure B.5: Responses to 1% deficit shocks from VARs. "$\sum =$" gives the sum of the responses.

Figure B.5 presents responses of these VARs to a 1% deficit shock at time 1. I allow all variables to move contemporaneously to the deficit shock. The central point shows up right away: *The VAR shows an s-shaped surplus moving average.* The initial 1.0% deficit is followed by two more periods of deficit, for a cumulative 1.75% deficit. But then the surplus response turns positive. The many small positive surpluses chip away at the debt, until the sum of surpluses in response to the deficit shock is only $-a(1) = \sum_{j=0}^{\infty} s_{1+j} = -0.31\%$.

Mechanically, the surplus response function comes from the coefficient by which the surplus responds to the value of debt. The value of debt jumps up initially when surplus jumps down. Shocks to the surplus and value of debt are strongly negatively correlated, itself below a piece of evidence for an s-shaped response: When the government runs a deficit, the value of debt rises, which can only happen if people expect future repayment. Surpluses then respond to the greater value of debt, and slowly bring down the value of debt. Thus, the s-shaped surplus response estimate is robust and intuitive, as the ingredients come from the negative sign of the regression of surplus on debt, the persistent debt response, and the pattern that higher surpluses bring down the value of debt.

The simple VAR shows almost exactly the same surplus response as the full VAR, emphasizing how the response comes just from these intuitive features of that VAR. The point estimate of the sum of coefficients in the simple VAR is smaller, $a(1) = 0.26$. (The sums of responses are negative in the plot because the shock is negative) The simple VAR surplus response crosses that of the full VAR and continues to be larger past the right end of the graph.

The AR(1) response looks almost the same – but it does not rise above zero. It would be very hard to tell univariate and VAR surplus responses apart based on autocorrelations or short-run forecasting ability emphasized in statistical tests. But the long-run implications are dramatically different. For the AR(1), we have $a(1) = 2.21$. Where a simpleminded constant discount rate model, fed the VAR-estimated surplus process, predicts 0.26%-0.31% inflation in response to a 1% fiscal shock, the AR(1) predicts 2.28% inflation. The volatility of real one-period bonds is entirely unexpected inflation, so the AR(1) also predicts dramatically higher bond return volatility.

Leaving the value of debt out of the VAR is not a specification choice, and not to be decided by the usual specification tests. Leaving the value of debt out of the VAR, and then using a present value formula, is a classic econometric mistake. Equation (5),

$$v_t = E\left(\sum_{j=0}^{\infty} \rho^j s_{t+1+j} | \Omega_t\right)$$

where Ω_t denotes consumer/investor information sets only implies

$$v_t = E\left(\sum_{j=0}^{\infty} \rho^j s_{t+1+j} | I_t\right)$$

where I_t is the VAR information set $I_t \subset \Omega_t$, if $v_t \in I_t$.

The true moving average representation is "non-invertible" in the sense of Fernández-Villaverde et al. (2007). It cannot be recovered by a VAR that excludes the value of debt.

Appendix C. Basic policy rule regressions

I report here basic policy rule regressions. I do not report them in the paper or use their values, since I do not attempt the hard topic of identification. Model right hand variables are correlated with model shocks, so OLS regressions are not valid. I present the regressions to show the data, and to give reassurance that a fiscal policy rule which loads on output and (to a lesser extent) on inflation is not unreasonable, and to examine the size of the correlations.

Table C.3 presents regressions of the interest rate and surplus on inflation and the output gap. Figure C.6 presents the interest rate, output gap and inflation data underlying the monetary policy rule regressions, and Figure C.7 presents the surplus, output gap and inflation data underlying the surplus policy rule regressions. The data are annual, and the same as used in Cochrane (2020a).

I start with an interest rate regression in part to frame the contrast with surplus policy rule regressions in the same data set. The OLS regressions show a small 0.19 output gap response and a large 0.98 inflation response just below one. In simple regressions such as this, the inflation response is not greater than one. Figure C.6 shows that the inflation response is, of course, driven by the rise and fall of inflation in the 1970s and 1980s. The single regression coefficients

$$i_t = a + \rho i_{t-1} + b x_t + c\pi_t + u_t^i$$

	ρ	b	c	ρ_u	R^2
OLS		0.19	0.98	0.72	0.52
s.e.		(0.16)	(0.21)		
Single OLS		0.13		0.90	0.01
s.e.		(0.23)			
Single OLS			0.97	0.69	0.50
s.e.			(0.21)		
$1-\rho L$		0.29	0.48		0.29
s.e.		(0.10)	(0.24)		
With lag	0.81	0.30	0.22	0.16	0.86
s.e.	(0.06)	(0.06)	(0.11)		
$b/(1-\rho)$		1.55	1.15		

$$s_t = a + \rho s_{t-1} + b x_t + c\pi_t + u_t^s$$

	ρ	b	c	ρ_u	R^2
OLS		1.62	-0.38	0.37	0.37
s.e.		(0.34)	(0.35)		
Single OLS		1.64		0.38	0.35
s.e.		(0.32)			
Single OLS			-0.49	0.53	0.03
s.e.			(0.44)		
$1-\rho L$		1.45	-0.24		0.26
s.e.		(0.33)	(0.37)		
With lag	0.39	1.27	-0.38	-0.06	0.50
s.e.	(0.10)	(0.27)	(0.27)		
$b/(1-\rho)$		2.06	-0.62		

Table C.3: Policy rule regressions. i= interest rate, x = GDP gap, s = surplus/GDP. Sample 1949-2018.

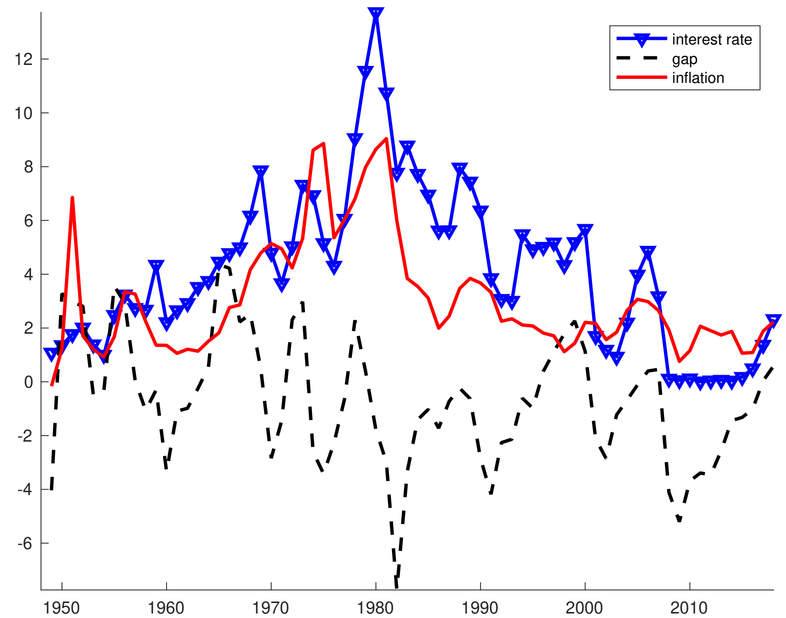

Figure C.6: Interest rate, output gap, and inflation

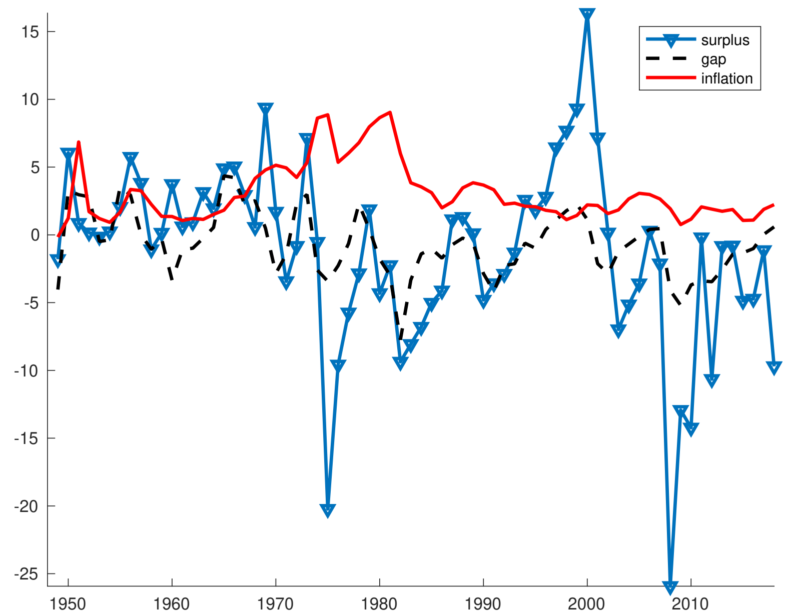

Figure C.7: Surplus, output gap, and inflation are just about the same as the multiple regression coefficients, with the output gap providing very little explanatory power.

One may wish to focus on the business cycle frequencies. The next two rows do that, and address serial correlation of the error, in two different ways. Using the error serial correlation ρ_i of the OLS regression, the regression labeled "$1-\rho L$" runs $(1-\rho L)i_t$ on $(1-\rho L)x_t+(1-\rho L)\pi_t$. This specification mirrors that of the policy rule, (12) and (19), and it is how one would estimate that rule by GLS in the presence of serially correlated residuals. Here, the main effect is to lower the inflation response to about 0.5. The nearly unit inflation response of the simple regression does reflect the low frequency rise and fall of inflation rather than business cycle movement.

Adding a lagged interest rate, in the last regression, estimates a partial adjustment model, common in the monetary policy shock literature. The coefficients are reduced, but the implied long run coefficients are larger. One needs this sort of model to produce a coefficient greater than one on inflation, as Clarida, Galí, and Gertler (2000) famously found. (They also use instruments and additional refinements.) The standard error is large, half the size of the coefficient. Stationary data do not easily produce a coefficient that leads to explosive behavior.

Overall, these regressions reflect the great uncertainty and sensitivity to specification typical of the literature. For example, the difference between autocorrelated residuals and lagged dependent variables is subtle, but makes a dramatic change in the result here.

The OLS regression estimate of the surplus rule in the second panel shows most of all a strong association with the output gap. This association stands out in Figure C.7. It is clear at business cycle frequencies and also in the long dip of potential GDP (and, not reported, unemployment) in the 1970s and 1980s. The tables are turned. Here the output gap is the strong correlation, and the inflation coefficient is insignificant and results in small marginal R^2.

This surplus is the ratio of surplus to value of the debt, or equivalently (surplus/GDP) to (value/GDP). Thus, the coefficient that a 1% rise in GDP gap results in a 1.62 percentage point rise in surplus means, if debt/GDP = 0.5, a 0.81 percentage point rise in surplus/GDP ratio.

One expects the coefficient of surplus on inflation to be positive, due to an imperfectly indexed tax code. The point estimate is -0.38, though insignificant. One can see in Figure C.7 that the 1970s, with high inflation, had lower surpluses. This observation however reinforces the central weak point of such regressions. The negative correlation of surpluses with inflation is may well reflect the response of inflation to surplus shocks, not the rule.

Any serious estimation of policy rules, which this is not, must surmount the identification problem seriously. To measure the interest rate or surplus policy rules, we must find movements in inflation and output gap which are not correlated with the interest rate or surplus disturbances u_t^i and u_t^s. This is a different task than the usual one, of measuring directly the economy's response to monetary or fiscal policy shocks. There, one must find movements in u_t^i and u_t^s that are not correlated with changing expectations of future inflation, output, etc.

Identification is not a hopeless task. The Romer and Romer (1989) approach could look for such shocks. Romer and Romer looked for shocks that were a response to inflation, but not to output, in order to measure the response of output to such shocks. We need to measure the systematic part of policy, not the economy's response to policy. So, we either need narrative measurements of the systematic component, or we can use the monetary shock to measure the fiscal response function. Likewise Ramey (2011) pioneered the use of military spending as an exogenous shock to u_t^s. We can use this to measure the monetary response function.

The structural or narrative approach may be much more fruitful for the fiscal response function than it is for the monetary response function. Much of the strong response of surpluses to output, and the response we wish to measure to inflation, are generated by the tax code and automatic stabilizers. Those can be modeled to generate θ_s parameters. Additional fiscal decisions are measurable too, in acts of Congress.

Identification and estimation within the structure of a model may also be fruitful. The task is not as hopeless as it seems from the Cochrane (2011) critique of monetary policy rule estimation in new-Keynesian models. That paper concerned the difficulties of measuring off-equilibrium responses from data in an equilibrium, which really is hard. The θ responses here are all relations between variables that we do see in equilibrium.