Online Appendix to "Indeterminacy and Imperfect Information"*

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A Three Motivating Examples: Further Details

A.1 NK Model with Observed Output and Inflation

We assume that the central bank's information set is $Z_t = \{\pi_t, y_t\}$, which implies that $\pi_{t|t} = \pi_t$ and $y_{t|t} = y_t$. However, the central bank still has to project $x_{t|t}$, $\overline{y}_{t|t}$, and $u_{t|t}$. We derive these projections from the conditioned-down system, namely the solution of the model under the central bank's information set. This is isomorphic to the full-information solution. Specifically, as noted in equations (5) – (7) of the paper:

$$\pi_{t|t} = \frac{\phi_x + \sigma}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t} - \frac{\kappa\sigma}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_{t|t}, \tag{5}$$

$$x_{t|t} = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t} - \frac{\sigma}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_{t|t}, \tag{6}$$

$$y_{t|t} = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t} + \frac{\phi_x + \kappa\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_{t|t}. \tag{7}$$

Together with $\pi_{t|t} = \pi_t$ and $y_{t|t} = y_t$, this equation system can be solved for the projections as functions of the variables in the central bank's information set:

$$\begin{aligned}
\overline{y}_{t|t} &= \frac{\phi_x + \sigma}{\phi_x} y_t + \frac{\phi_\pi}{\phi_x} \pi_t, \\
u_{t|t} &= \frac{\kappa\sigma}{\phi_x} y_t + \frac{\phi_x + \kappa\phi_\pi}{\phi_x} \pi_t, \\
x_{t|t} &= -\frac{\sigma}{\phi_x} y_t - \frac{\phi_\pi}{\phi_x} \pi_t.
\end{aligned}$$

This expression can be used to find a 'reduced-form' policy rule by substituting out $x_{t|t}$:

$$i_t = -\sigma y_t.$$

Using this rule in the original equation system results in the following system:

$$\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \kappa x_t + u_t, \\
0 &= E_t x_{t+1} + E_t \pi_{t+1}.
\end{aligned}$$

It can be quickly verified that this system is singular, that is, underdetermined. Consequently, and in addition, the solution is indeterminate since the second equation, which is derived from the Euler-equation after imposing the reduced-form policy rule, only pins down (linear combinations of) expectations of the output gap and inflation and not their actual values. In contrast, the first equation is an unstable difference equation ($0 < \beta < 1$) and determines inflation dynamics. Using the forecast-error decomposition from Sims (2002), we can write:

$$\begin{aligned}\pi_t &= \beta\pi_{t+1} + \kappa x_t + u_t - \beta\eta^{\pi}_{t+1} = \\ &= \kappa x_t + u_t + \sum\nolimits_{j=1}^{\infty} \beta^j \left(\kappa x_{t+j} + u_{t+j} - \eta^{\pi}_{t+j}\right).\end{aligned}$$

Stability requires that $\kappa x_{t+j} + u_{t+j} - \eta^{\pi}_{t+j} = 0, \forall j$, so that the solution for the forecast error $\eta^{\pi}_t = \pi_t - E_{t-1}\pi_t = \kappa x_t + u_t$. Consequently, $\pi_t = \kappa x_t + u_t$. Since $E_t x_{t+1} = 0$, we can specify a general solution $x_t = \gamma_u u_t + \gamma_{\overline{y}}\overline{y}_t + \gamma_b b_t$, following Farmer, Khramov, and Nicolò (2015). $b_t \sim_{iid} N(0,1)$ is a normalized sunspot or belief shock, while γ_u, $\gamma_{\overline{y}}$, and γ_b are the respective loadings on the disturbances, which have to be consistent with a rational expectations equilibrium. The set of potential solutions to the simple NK model under imperfect information can thus be written as:

$$\begin{aligned}\pi_t &= (1+\kappa\gamma_u)\,u_t + \kappa\gamma_{\overline{y}}\overline{y}_t + \kappa\gamma_b b_t, \\ y_t &= \gamma_u u_t + (1+\gamma_{\overline{y}})\,\overline{y}_t + \gamma_b b_t, \\ x_t &= \gamma_u u_t + \gamma_{\overline{y}}\overline{y}_t + \gamma_b b_t.\end{aligned}$$

As the central bank's information set is nested inside the public's information set, the model's expectational linear-difference system can be conditioned down onto the central bank's information. As a result, any equilibrium has to be consistent with central bank projections, which we refer to as the projection condition. In the present example, the projection condition leads to $\pi_t = \pi_{t|t} = g_{\pi u} u_{t|t} + g_{\pi\bar{y}}\bar{y}_{t|t}$ and $x_t = x_{t|t} = g_{xu}u_{t|t} + g_{x\bar{y}}\bar{y}_{t|t}$, where $g_{\pi u}$, $g_{\pi\bar{y}}$, g_{xu}, $g_{x\bar{y}}$ are known from the full-information solutions $\pi_t = g_{\pi u}u_t + g_{\pi\bar{y}}\bar{y}_t$ and $x_t = g_{xu}u_t + g_{x\bar{y}}\bar{y}_t$. Specifically, the full-information coefficient values are:

$$\begin{aligned} g_{xu} &= -\frac{\phi_\pi}{\sigma+\kappa\phi+\phi_x}, & \qquad g_{\pi u} &= 1+\kappa g_{xu}, \\ g_{x\bar{y}} &= -\frac{\sigma}{\sigma+\kappa\phi+\phi_x}, & \qquad g_{\pi\bar{y}} &= \kappa g_{x\bar{y}}, \end{aligned}$$

To derive the set of all admissible solutions, we find it useful to rewrite the imperfect-information solution in terms of deviations from the full-info case:

$$\begin{aligned} x_t &= g_{xu}u_t + g_{x\bar{y}}\bar{y}_t + \delta_t \\ \pi_t &= g_{\pi u}u_t + g_{\pi\bar{y}}\bar{y}_t + \kappa\delta_t \\ y_t &= g_{xu}u_t + (1+g_{x\bar{y}})\bar{y}_t + \delta_t \end{aligned}$$

with

$$\delta_t = \tilde{\gamma}_u u_t + \tilde{\gamma}_{\bar{y}} \bar{y}_t + \gamma_b b_t \,,$$
$$\tilde{\gamma}_u \equiv \gamma_u - g_{xu} \,,$$
$$\tilde{\gamma}_{\bar{y}} \equiv \gamma_{\bar{y}} - g_{x\bar{y}} \,.$$

The projection condition (for all variables) then reduces to $\delta_{t|t} = 0$. The full-information solution corresponds to $\delta_t = 0$, which trivially fulfills the projection condition.

To satisfy the projection condition, $\delta_{t|t} = E(\delta_t|\pi^t y^t) = 0$, we require δ_t to be uncorrelated with π_t and y_t at all times and at all lags. Since δ_t, π_t and y_t are iid in this specific example, the following covariance conditions are necessary and sufficient:

$$\mathrm{Cov}\,(\delta_t, \pi_t) = 0\,, \qquad \text{and} \quad \mathrm{Cov}\,(\delta_t, y_t) = 0\,. \tag{A-1}$$

By enforcing an orthogonality, that is, zero covariance condition, we can simplify the variance-covariance matrix of inflation and output in the Kalman gains without solving for a matrix inverse. In general, this is not possible and hinges on the specific assumptions for the information set. Given the $i.i.d.$ property of all variables involved, signal extraction involves the following Kalman gain, K:

$$\delta_{t|t} = K \begin{bmatrix} \pi_t \\ y_t \end{bmatrix}, \qquad \text{with} \quad K = \begin{bmatrix} \mathrm{Cov}\,(\delta_t, \pi_t) \\ \mathrm{Cov}\,(\delta_t, y_t) \end{bmatrix} \mathrm{Var}\left(\begin{bmatrix} \pi_t \\ y_t \end{bmatrix} \right)^{-1}. \tag{A-2}$$

The two equations of the projection condition (A-1) yield two equations in three unknowns:

$$\mathrm{Cov}\,(\delta_t, \pi_t) = \tilde{\gamma}_u \cdot (\kappa g_{xu} + 1)\, \sigma_u^2 + \tilde{\gamma}_{\bar{y}} \cdot (\kappa g_{x\bar{y}})\, \sigma_{\bar{y}}^2 + \kappa\, \mathrm{Var}\,(\delta_t) \stackrel{!}{=} 0\,, \tag{A-3}$$
$$\mathrm{Cov}\,(\delta_t, y_t) = \tilde{\gamma}_u \cdot g_{xu} \sigma_u^2 + \tilde{\gamma}_{\bar{y}} \cdot (\kappa g_{x\bar{y}} + 1)\, \sigma_{\bar{y}}^2 + \mathrm{Var}\,(\delta_t) \stackrel{!}{=} 0\,, \tag{A-4}$$

with

$$\mathrm{Var}\,(\delta_t) = \tilde{\gamma}_u^2 \sigma_u^2 + \tilde{\gamma}_{\bar{y}}^2 \sigma_{\bar{y}}^2 + \gamma_b^2\,. \tag{A-5}$$

Equations (A-3) and (A-4) are two *non-linear* equations in the three unknowns $\tilde{\gamma}_u$, $\tilde{\gamma}_{\bar{y}}$, γ_b. Alternatively, we can consider $\mathrm{Var}\,(\delta_t)$ a yet to be determined unknown, so that equations (A-3) and (A-4) are two linear equations in the three unknowns $\tilde{\gamma}_u$, $\tilde{\gamma}_{\bar{y}}$, and $\mathrm{Var}\,(\delta_t)$. These equations are straightforward to solve and lead to expressions for $\tilde{\gamma}_u$, $\tilde{\gamma}_{\bar{y}}$ as functions of parameters and $\mathrm{Var}\,(\delta_t)$.

With those solutions in hand, equation (A-5) becomes a quadratic equation in $\mathrm{Var}\,(\delta_t)$, with a single unknown parameter, γ_b^2. Intuitively, γ_b^2 reflects the extent to which belief shocks matter, while $\mathrm{Var}\, delta_t$ measures the amount of fluctuations in the output gap due to imperfect information. The quadratic equation for $\mathrm{Var}\,(\delta_t)$ has the form

$$a \cdot \mathrm{Var}\,(\delta_t)^2 - \mathrm{Var}\,(\delta_t) = -\gamma_b^2$$

with $a > 0$ reflecting the solution to (A-3) and (A-4). As noted in the main text above, we can then derive that

$$\gamma_b^2 < \frac{1}{4 \cdot a}\,, \quad \text{with} \quad a = \left[\frac{\kappa^2}{\sigma_u^2} + \frac{1}{\sigma_{\bar{y}}^2}\right] \cdot \left(\frac{\sigma + \kappa\phi_\pi + \phi_x}{\phi_x}\right)^2 . \tag{10}$$

To see this, let

$$\boldsymbol{v}_t = \begin{bmatrix} u_t \\ \bar{y}_t \end{bmatrix}, \qquad \boldsymbol{\gamma} = \begin{bmatrix} \tilde{\gamma}_u \\ \tilde{\gamma}_{\bar{y}} \end{bmatrix}, \qquad \boldsymbol{\psi} = \begin{bmatrix} \kappa \\ 1 \end{bmatrix}, \qquad \boldsymbol{G} = \begin{bmatrix} g_{\pi,u} & g_{\pi,\bar{y}} \\ g_{y,u} & g_{y,\bar{y}} \end{bmatrix}, \qquad \boldsymbol{\Omega} = \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_{\bar{y}}^2 \end{bmatrix},$$

so that we have

$$\delta_t = \boldsymbol{\gamma}'\boldsymbol{v}_t + \gamma_b b_t\,, \qquad\qquad \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} = \boldsymbol{G}\boldsymbol{v}_t + \begin{bmatrix} \kappa \\ 1 \end{bmatrix} \delta_t\,,$$

and the covariance restrictions emanating from the projection condition in (A-2) can be restated as

$$\mathrm{Cov}\left(\delta_t, \begin{bmatrix} \pi_t \\ y_t \end{bmatrix}\right) = \boldsymbol{\gamma}'\boldsymbol{\Omega}\boldsymbol{G}' + \mathrm{Var}\,(\delta_t)\boldsymbol{\psi}\,.$$

To set the covariances between δ_t and inflation and output to zero, respectively, we obtain

$$\boldsymbol{\gamma}' = -\,\mathrm{Var}\,(\delta_t)\boldsymbol{\psi}'\left(\boldsymbol{G}^{-1}\right)'\boldsymbol{\Omega}^{-1}\,.$$

From $\mathrm{Var}\,(\delta_t) = \boldsymbol{\gamma}'\Omega\boldsymbol{\gamma} + \gamma_b^2$ we can deduce that $a = \boldsymbol{\alpha}'\boldsymbol{\alpha}$ with:

$$\boldsymbol{\alpha} = \boldsymbol{\Omega}^{-1/2}\boldsymbol{G}^{-1}\boldsymbol{\psi}$$

and straightforward algebra yields the result in (10).

A.2 Extension to a Taylor rule with interest-rate smoothing

As an example of letting policy respond also to past conditions, consider adding interest-rate smoothing to the Taylor rule. In the perfect-information case the rule can then be written as follows:

$$i_t = \phi_i i_{t-1} + (1 - \phi_i)(\phi_\pi \pi_t + \phi_x x_t) = (1 - \phi_i) \sum_{k=0}^{\infty} \phi_i^k (\phi_\pi \pi_{t-k} + \phi_x x_{t-k}) \tag{A-6}$$

As shown above, interest-rate smoothing can also be viewed as monetary policy responding to lagged outcomes. Woodford (2003) offers a prominent discussion of likely benefits for policy design from such a dependency of policy rules on lagged policy rates, and similar rules are also discussed regularly in the Federal Reserve Board's Monetary Policy Report to the U.S. Congress.

As before, we maintain the standard assumption that inflation and output are perfectly observed by the policymaker, while the output gap is not directly observed. The policy maker has perfect recall, so that her information set reflects the entire history of current and past values of inflation and output, π_t and y_t. The imperfect-information version of the augmented Taylor rule in (A-6) then becomes:

$$i_t = \phi_i i_{t-1} + (1 - \phi_i)(\phi_\pi \pi_t + \phi_x x_{t|t})$$

The projection condition requires the policymaker's optimal projections to be consistent with decision rules known from the full-information equilibrium, which in turn are consistent with equilibrium conditions evaluated at the policymaker's projections for states and outcomes. In particular, the policymaker's interest rate setting has to be consistent with the (projected) Fisher equation:[1]

$$i_t = r_{t|t} + \pi_{t+1|t} = \sigma(y_{t+1|t} - y_{t|t}) + \pi_{t+1|t}$$

With interest-rate smoothing, the lagged interest rate becomes a state variable (under full and imperfect information), and full-information decision rules take the form $\pi_t = g_{\pi,u} u_t + g_{\pi,\bar{y}} \bar{y}_t + g_{\pi,i} i_{t-1}$ (and similarly for other variables), with coefficients obtained via matrix methods, such as those in Klein (2000). Of note, the resulting values for $g_{\pi,u}$, $g_{y,u}$ etc. are different than those derived for the example without interest-rate smoothing in Appendix A.1.

With u_t and $\bar{y}_t$ being $i.i.d.$, and i_t known to the policymaker, it follows that $\pi_{t+1|t} = g_{\pi,i}\, i_t$. With

[1] The same logic applies in case of the example without interest-rate smoothing described in Appendix A.1. In the absence of endogenous state variables, $i.i.d.$ shocks and observable output, that example leads to $\pi_{t+1|t} = 0$ and $r_{t|t} = -\sigma y_{t|t} = -\sigma y_t = r_t$.

y_t directly observed as well, we can conclude that $y_{t+1|t} = g_{y,i}\, y_t$ and thus that

$$i_t = -\frac{\sigma}{1 - \sigma g_{y,i} - g_{\pi,i}} \cdot y_t \,. \tag{A-7}$$

The same result can be obtained by following the steps described for the case without interest rate smoothing in Appendix A.1:

a. Invert the full-information decision rules for inflation and output to back out the policymaker's projections $u_{t|t}$ and $\bar{y}_{t|t}$ as linear combination of inflation, output and lagged interest rate, which are all known to the policymaker.

b. Construct an output gap projection from the projection condition, $x_{t|t} = g_{x,u} u_{t|t} + g_{x,\bar{y}} \bar{y}_{t|t} + g_{x,i} i_{t-1}$.

c. Finally, when the output gap projection is plugged into the Taylor rule, we obtain (A-7) as effective Taylor rule that takes into account the effects of the projection condition.

An algebraic derivation via this alternative approach is complex, but the approach via an inversion of decision rules can be used to numerically verify the solution in (A-7).

An important upshot of (A-7) is that interest-rate smoothing does not undo the indeterminacy found earlier. Since the effective Taylor rule in (A-7) responds only to output, and not to inflation (nor to lagged policy rates), the resulting dynamic system falls in the class of simple outcome-based policy rules studied by Bullard and Mitra (2002), and fails their necessary and sufficient criterion for determinacy.[2]

Despite the apparent isomorphism in terms of evaluating the determinacy criterion of the effective Taylor rule in the augmented model with interest-rate smoothing, the resulting equilibria are richer than in the case of a simple Taylor rule. While the lagged interest rate plays no rule in checking for the effective Taylor rule's determinacy criterion, we still require that (projected) outcomes satisfy the projection condition as applied to the full-information outcomes of the augmented model, which feature a dependence of i_{t-1}. Specifically, equilibrium outcomes need to be characterized by

$$\begin{aligned}
y_t &= g_{y,u}\, u_{t|t} + g_{y,\bar{y}}\, \bar{y}_{t|t} + g_{y,i}\, i_{t-1} \\
\pi_t &= g_{\pi,u}\, u_{t|t} + g_{\pi,\bar{y}}\, \bar{y}_{t|t} + g_{\pi,i}\, i_{t-1} \\
i_t &= g_{i,u}\, u_{t|t} + g_{i,\bar{y}}\, \bar{y}_{t|t} + g_{i,i}\, i_{t-1}
\end{aligned}$$

[2]The condition from Bullard and Mitra (2002) for our case is $\kappa(\bar{\phi}_\pi - 1) + (1-\beta)]bar\phi_y > 0$. It is never satisfied by the effective Taylor rule in (A-7), which has response coefficients of $\bar{\phi}_\pi = 0$ and $\bar{\phi}_y < 0$.

where "$g_{\cdot,\cdot}$" are coefficients known from the full-information solution of the NK model with the interest-rate smoothing rule in (A-6).[3]

Based on our solution for the simple Taylor-rule model in Appendix A.1, the $i.i.d.$ nature of u_t and $\bar{y}_t$ and since the only endogenous state variable, i_{t-1} is observed, we deduce that equilibrium outcomes are characterized as the sum of their full-information outcomes, plus a univariate, $i.i.d.$ process, δ_t, that measures the effects of indeterminacy:

$$\begin{aligned}
\delta_t &= \tilde{\gamma}_u u_t + \tilde{\gamma}_{\bar{y}} \bar{y}_t + \gamma_b b_t \\
y_t &= \quad\ \delta_t + g_{y,u}\, u_t + g_{y,\bar{y}}\, \bar{y}_t + g_{y,i}\, i_{t-1} \\
\pi_t &= \psi_\pi\, \delta_t + g_{\pi,u}\, u_t + g_{\pi,\bar{y}}\, \bar{y}_t + g_{\pi,i}\, i_{t-1} \\
i_t &= \psi_i\, \delta_t\ + g_{i,u}\, u_t + g_{i,\bar{y}}\, \bar{y}_t + g_{i,i}\, i_{t-1}
\end{aligned}$$

with

$$\begin{aligned}
& \psi_\pi = \kappa + \beta g_{\pi,i} \psi_i\,, && \psi_i = (1-\phi_i)(\phi_\pi \psi_\pi + \phi_x)\,, \\
\Leftrightarrow \quad & \psi_\pi = \frac{\kappa + \beta g_{\pi,i}(1-\phi_i)\phi_x}{1-\beta g_{\pi,i}(1-\phi_i)\phi_\pi}\,, && \psi_i = (1-\phi_i)\frac{\phi_x + \kappa\,\phi_\pi}{1-\beta g_{\pi,i}(1-\phi_i)\phi_\pi}
\end{aligned}$$

The variable δ_t is expressed in units of output, and the coefficients ψ_i and ψ_π reflect the effects of δ_t on the policy rate and inflation.

As in case of the simple Taylor rule discussed in in Appendix A.1, the projection condition requires that $\delta_{t|t} = 0$ leading the covariance restrictions $\text{Cov}\,(\pi_t, \delta_t) = 0$ and $\text{Cov}\,(y_t, \delta_t) = 0$.[4] Proceeding analogously to the simple case, the covariance restrictions lead to one degree of freedom for the three parameters γ_u, $\gamma_{\bar{y}}$ and γ_b (albeit evaluated at coefficient values $g_{\cdot,\cdot}$ applicable in the model with interest rate smoothing).

The problem of solving for $\tilde{\gamma}_u$, $\tilde{\gamma}_{\bar{y}}$, and γ_b is isomorphic to the case without interest-rate smoothing described in Appendix A.1, but using the coefficients $g_{\pi,u}$, $g_{\pi,\bar{y}}$, $g_{y,u}$, etc. that characterize the full-information solution in the case of the augmented Taylor rule in (A-6), and with $\boldsymbol{\psi} = \begin{bmatrix}\psi_\pi & 1\end{bmatrix}'$. A crucial difference to the simple case is that through the dependence of policy rates on lagged rates, the contemporaneous effects of δ_t now lead to the propagation of belief shocks to current and future outcomes for inflation and output.

[3]The full-information decision rules of the NK model with the interest-rate smoothing rule in (A-6) differ, of course, from those derived in closed form for the case of a simple rule as derived in Appendix A.1. They reflect the solution to a second-order difference equation, which is straightforward to obtain using matrix methods such as in Blanchard and Kahn (1980), King and Watson (1998), Klein (2000), or Sims (2002).

[4]Note: The dependence of π_t and y_t on i_{t-1} has no effect on these covariance restrictions, since δ_t is iid.

A.3 NK Model with Observed Inflation Only

We assume that the central bank's information set is $Z_t = \{\pi_t\}$, which implies that $\pi_{t|t} = \pi_t$. Moreover, we assume that $\overline{y}_t \equiv 0$, so that there are no shocks to potential output and $r_t = 0$. Consequently, the central bank has to project $x_{t|t}$ and $u_{t|t}$. The unique rational expectations solution under full information is:

$$\pi_t = \frac{\phi_x + \sigma}{\phi_x + \sigma + \kappa\phi_\pi} u_t\,, \qquad x_t = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_t.$$

Similarly, the projection condition is:

$$\pi_{t|t} = \frac{\phi_x + \sigma}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t}\,, \qquad x_{t|t} = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t}.$$

A.3.1 Overview of Solution Approach

Since $\pi_{t|t} = \pi_t$, we can write $x_{t|t} = -\frac{\phi_\pi}{\phi_x+\sigma}\pi_t$ and use this to substitute out the projected output gap from the policy rule:

$$i_t = \frac{\sigma\phi_\pi}{\phi_x + \sigma}\pi_t = \overline{\phi}_\pi \pi_t.$$

and results in a liner-difference system for inflation and output gap of the following form:

$$\begin{bmatrix} \pi_{t+1} \\ x_{t+1} \end{bmatrix} = \frac{1}{\sigma\beta}\begin{bmatrix} \sigma & -\sigma\kappa \\ \beta\bar{\phi} - 1 & \kappa + \beta\sigma \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \boldsymbol{\eta}_{t+1} + \begin{bmatrix} -\frac{1}{\beta} \\ \frac{1}{\sigma\beta} \end{bmatrix} u_t, \tag{A-8}$$

The 'reduced-form' policy rule implies a unique equilibrium if $\overline{\phi}_\pi > 1$, or equivalently, $\phi_\pi > 1 + \frac{1}{\sigma}\phi_x$. Under the latter restriction, the solution to the imperfect information economy is identical to its full-information counterpart.

If $\phi_\pi < 1 + \frac{1}{\sigma}\phi_x$, the equilibrium of the imperfect information model is indeterminate. Mechanically, this implies that only one root of the dynamic system in (A-8) is unstable, while the other is stable, which results in endogenous persistence as discussed in Lubik and Schorfheide (2003). Closed-form solutions are hard to obtain for this system, but the remainder of this appendix describes a numerical solution.

A.3.2 Details of Solution Framework

The dynamic system in (A-8) has the following general format:

$$\boldsymbol{y}_{t+1} = \boldsymbol{B}\, \boldsymbol{y}_t + \boldsymbol{\eta}_{t+1} + \boldsymbol{D}\, \boldsymbol{v}_t \tag{A-9}$$

where $\boldsymbol{\eta}_t$ is a vector of yet-to-be-determined endogenous forecast errors, and $\boldsymbol{y}_t$ is a vector of jump variables. The projection condition is given by $\boldsymbol{y}_{t|t} = \boldsymbol{G}\ \boldsymbol{v}_{t|t}$ where $\boldsymbol{G}$ reflects the full-information solution.

MSV equilibrium: Given our assumptions about parameter values, the full information solution is determinate, and we can recover $\boldsymbol{G}$ also as solution to the minimal-state-variable (MSV) equilibrium to the system in (A-9). When all elements of $\boldsymbol{y}_{t+1}$ are jump variables, and provided that $|\boldsymbol{B}| \neq 0$, the MSV approach leads to

$$\boldsymbol{y}_t = -\boldsymbol{B}^{-1}\boldsymbol{D}\ \boldsymbol{v}_t\,, \qquad \text{with} \quad -\boldsymbol{B}^{-1}\boldsymbol{D} = \boldsymbol{G}\,, \tag{A-10}$$

and $\boldsymbol{G}$ collecting the coefficients of the full-information solution.[5] The reason is that the projection condition used to rewrite Taylor rule ensures that the MSV solutions under full information carries over also the system in (A-9). As a corollary, the MSV solution trivially satisfies the projection condition as well. The remainder of this section derives a set of additional REE that are stable, satisfy the projection condition and solve (A-9).

Canonical variables and other equilibria: Henceforth, we assume that the determinacy condition does not hold, so that $\boldsymbol{B}$ has one stable and one unstable root so that (A-10) is not unique unless the projection condition pins down $\boldsymbol{\eta}_t$ as a unique function of fundamental shocks. Moreover, since all roots have different moduli, it is assured that all are real.[6]

Consider the Schur decomposition of $\boldsymbol{B} = \boldsymbol{Z}'\boldsymbol{T}\boldsymbol{Z}$ where $\boldsymbol{T}$ is block upper triangular and $\boldsymbol{Z}$ is an orthonormal matrix, so that $\boldsymbol{Z}'\boldsymbol{Z} = \boldsymbol{I}$. Let the Schur decomposition be ordered, such that the stable roots of $\boldsymbol{B}$ come first and we have the following transformation into canonical variables:

$$\begin{aligned} & \begin{bmatrix} \chi_t \\ \gamma_t \end{bmatrix} = \boldsymbol{Z}\boldsymbol{y}_t\,, \\ \implies \quad & \begin{bmatrix} \chi_{t+1} \\ \gamma_{t+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} \\ \boldsymbol{0} & \boldsymbol{T}_{22} \end{bmatrix} \begin{bmatrix} \chi_t \\ \gamma_t \end{bmatrix} + \tilde{\boldsymbol{\eta}}_{t+1} + \boldsymbol{Z}\boldsymbol{D}\boldsymbol{v}_t \end{aligned} \tag{A-11}$$

where $\tilde{\boldsymbol{\eta}}_{t+1} = \boldsymbol{Z}\boldsymbol{\eta}_{t+1}$ (we will denote appropriate partitions of $\tilde{\boldsymbol{\eta}}_t$ by $\tilde{\boldsymbol{\eta}}_t^{\chi}$ and $\tilde{\boldsymbol{\eta}}_t^{\gamma}$). Crucially, γ_t is the unstable canonical variable (vector), since all roots of $\boldsymbol{T}_{22}$ are outside the unit circles; conversely, χ_t is the stable canonical variable. (In our specific example, γ_t, χ_t, $\boldsymbol{T}_{11}$, $\boldsymbol{T}_{12}$, and $\boldsymbol{T}_{22}$ are scalars.)

[5] If $|\boldsymbol{B}| = 0$, we can use the MSV solution of the original full-information economy, which still applies.

[6] Since $\boldsymbol{B}$ is real, complex eigenvalues would arise only in complex-conjugate pairs, each sharing the same modulus.

In terms of canonical variables, the benchmark equilibrium then implies

$$\begin{bmatrix} \boldsymbol{\chi}_t \\ \boldsymbol{\gamma}_t \end{bmatrix} = -\boldsymbol{T}^{-1}\boldsymbol{Z}\boldsymbol{D}\,\boldsymbol{v}_t = \begin{bmatrix} -\boldsymbol{T}_{11}^{-1}\tilde{\boldsymbol{D}}_\chi + \boldsymbol{T}_{11}^{-1}\boldsymbol{T}_{12}\boldsymbol{T}_{22}^{-1}\tilde{\boldsymbol{D}}_\gamma \\ -\boldsymbol{T}_{22}^{-1}\tilde{\boldsymbol{D}}_\gamma \end{bmatrix} \boldsymbol{v}_t$$

$$= \begin{bmatrix} \tilde{\boldsymbol{G}}_\chi \\ \tilde{\boldsymbol{G}}_\gamma \end{bmatrix} \boldsymbol{v}_t$$

with $\tilde{\boldsymbol{D}}_\chi$ and $\tilde{\boldsymbol{D}}_\gamma$ being appropriate partitions of $\tilde{\boldsymbol{D}} \equiv \boldsymbol{Z}\boldsymbol{D} = \begin{bmatrix} \tilde{\boldsymbol{D}}_\chi \\ \tilde{\boldsymbol{D}}_\gamma \end{bmatrix}$.

For any stable solution of (A-11), the unstable canonical variables must be solved forward, and we obtain the benchmark outcomes for $\boldsymbol{\gamma}_t$ that apply in any stable equilibrium:

$$\boldsymbol{\gamma}_t = \tilde{\boldsymbol{\eta}}_t^\gamma = \tilde{\boldsymbol{G}}_\gamma\,\boldsymbol{v}_t\,. \tag{A-12}$$

Translated into canonical variables, the projection condition requires that the projected canonical variables are equal to the projected benchmark outcomes:

$$\begin{bmatrix} \boldsymbol{\chi}_{t|t} \\ \boldsymbol{\gamma}_{t|t} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{G}}_\chi \\ \tilde{\boldsymbol{G}}_\gamma \end{bmatrix} \boldsymbol{v}_{t|t} \tag{A-13}$$

and with (A-12) the projection condition *for the unstable variables* $\boldsymbol{\gamma}_t$ is trivially fulfilled *for any stable equilibrium.*

Stable canonical variables and projection condition: It remains to be seen whether the projection condition is sufficient to rule out indeterminate outcomes for the stable variables. It will be useful to focus on deviations of $\boldsymbol{\chi}_t$ from the benchmark outcomes, and see under which conditions the deviations are zero. Collecting terms we have:

$$\begin{aligned} \boldsymbol{\delta}_{t+1} &\equiv \boldsymbol{\chi}_{t+1} - \tilde{\boldsymbol{G}}_\chi\,\boldsymbol{v}_{t+1} = \boldsymbol{T}_{11}\boldsymbol{\delta}_t + \tilde{\boldsymbol{\eta}}_{t+1}^\delta\,, \\ \text{with} \quad \tilde{\boldsymbol{\eta}}_{t+1}^\delta &= \tilde{\boldsymbol{\eta}}_{t+1}^\chi - \tilde{\boldsymbol{G}}_\chi\boldsymbol{v}_t = \tilde{\boldsymbol{\Gamma}}_v\boldsymbol{v}_{t+1} + \tilde{\boldsymbol{\Gamma}}_b\boldsymbol{b}_{t+1} \end{aligned} \tag{A-14}$$

where $\boldsymbol{b}_t \sim N(0, I)$ is a (vector of) non-fundamental belief shock(s). $\tilde{\boldsymbol{\Gamma}}_v$ and $\tilde{\boldsymbol{\Gamma}}_b$ are (yet to be determined) shock loadings.

The central bank's filtering problem is characterized by the state equation (A-14) for $\boldsymbol{\delta}_t$, together with the following measurement equation:

$$\boldsymbol{z}_t = \boldsymbol{Z}^{z,\chi}\boldsymbol{\delta}_t + \boldsymbol{G}_z\boldsymbol{v}_t \tag{A-15}$$

where $\boldsymbol{z}_t = \pi_t$ in case of one example of Section 2, and $\boldsymbol{z}_t = \boldsymbol{y}_t$ in case of the other example of Section 2, and $\boldsymbol{Z}^{z,\chi}$ and $\boldsymbol{G}_z$ refer to the corresponding partitions of $\boldsymbol{Z}'$ and $\boldsymbol{G}$. In addition, let $\text{Var}(\boldsymbol{v}_t) = \boldsymbol{\Omega}_v$.

We seek to find parameters for the shock loadings of $\boldsymbol{\delta}_t$, that is $\tilde{\boldsymbol{\Gamma}}_v$ and $\tilde{\boldsymbol{\Gamma}}_b$, that satisfy the projection condition. With (A-15) the projection reduces to the following:

$$\boldsymbol{\delta}_{t|t} = \boldsymbol{0} \quad \Leftrightarrow \quad \text{Cov}\left(\boldsymbol{\delta}_t, \boldsymbol{z}_t | \boldsymbol{z}^{t-1}\right) = \text{Cov}\left(\boldsymbol{\delta}_t, \boldsymbol{z}_t\right) = \boldsymbol{0}\,. \tag{A-16}$$

The conventional determinacy requirement is to solve unstable canonical variables forward. Our imperfect-information framework adds the condition that projections of stable canonical variables have to be equal to their projected solution in the benchmark (or MSV) equilibrium. To preview the derivation: Let $\boldsymbol{\Sigma} = \text{Var}\left(\boldsymbol{\delta}_t | \boldsymbol{z}^t\right)$ and note that the projection condition (A-16) implies that $\text{Var}\left(\boldsymbol{\delta}_t | \boldsymbol{z}^t\right) = \text{Var}\left(\boldsymbol{\delta}_t\right)$. We then use the covariance restriction (A-16) to solve for $\tilde{\boldsymbol{\Gamma}}_v$ as a function of $\boldsymbol{\Sigma}$ and $\tilde{\boldsymbol{\Gamma}}_b$ (and other fixed model parameters). Substituting out $\tilde{\boldsymbol{\Gamma}}_v$ yields a quadratic equation for $\boldsymbol{\Sigma}$ which merely places an upper bound on $\tilde{\boldsymbol{\Gamma}}_b$, but without uniquely determining the belief shock. In particular, the belief shock loadings need not be zero. In detail:

$$\begin{aligned} \text{Cov}\left(\boldsymbol{\delta}_t, \boldsymbol{z}_t\right) = \boldsymbol{\Sigma}(\boldsymbol{Z}^{z,\chi})' + \tilde{\boldsymbol{\Gamma}}_v \boldsymbol{\Omega}_v \boldsymbol{G}_z' \qquad & \stackrel{!}{=} \boldsymbol{0} \\ \Rightarrow \tilde{\boldsymbol{\Gamma}}_v = -\boldsymbol{\Sigma}(\boldsymbol{Z}^{z,\chi})'(\boldsymbol{G}_z')^{-1}\boldsymbol{\Omega}_v^{-1} & \end{aligned} \tag{A-17}$$

provided that $\boldsymbol{\Omega}_v \boldsymbol{G}_z'$ is square and invertible.[7] In addition, with $\boldsymbol{\Sigma} = \text{Var}\left(\boldsymbol{\delta}_t | \boldsymbol{z}^t\right) = \text{Var}\left(\boldsymbol{\delta}_t\right)$ and (A-14) we have the following expression for $\boldsymbol{\Sigma}$, from which we can substitute out $\tilde{\boldsymbol{\Gamma}}_v$ based on (A-17):

$$\begin{aligned} \boldsymbol{\Sigma} &= \boldsymbol{T}_{11}\boldsymbol{\Sigma}\,\boldsymbol{T}_{11}' + \tilde{\boldsymbol{\Gamma}}_v \boldsymbol{\Omega}_v \tilde{\boldsymbol{\Gamma}}_v' + \tilde{\boldsymbol{\Gamma}}_b \tilde{\boldsymbol{\Gamma}}_b' = \boldsymbol{T}_{11}\boldsymbol{\Sigma}\,\boldsymbol{T}_{11}' + \boldsymbol{\Sigma}\boldsymbol{\Psi}\boldsymbol{\Sigma} + \tilde{\boldsymbol{\Gamma}}_b \tilde{\boldsymbol{\Gamma}}_b' \\ \text{with} \quad \boldsymbol{\Psi} &= (\boldsymbol{Z}^{z,\chi})'(\boldsymbol{G}_z')^{-1}\boldsymbol{\Omega}_v^{-1}\boldsymbol{G}_z^{-1}\boldsymbol{Z}^{z,\chi} \end{aligned} \tag{A-18}$$

where $\boldsymbol{\Psi}$ and $\boldsymbol{T}_{11}$ are known and we seek to characterize the set of all $\tilde{\boldsymbol{\Gamma}}_b$ that admit a finite, symmetric and positive definite solution to $\boldsymbol{\Sigma}$: One solution that always exists, is $\tilde{\boldsymbol{\Gamma}}_b = \boldsymbol{0}$ which recovers the MSV solution (with $\boldsymbol{\Sigma} = \boldsymbol{0}$, $\tilde{\boldsymbol{\Gamma}}_v = \boldsymbol{0}$ and $\boldsymbol{\delta}_t = \boldsymbol{0}$). However, with $\boldsymbol{\Psi} \neq \boldsymbol{0}$, we have to worry about non-existence of an appropriate solution of $\boldsymbol{\Sigma}$ for arbitrary $\tilde{\boldsymbol{\Gamma}}_b$.

Putting all together In case of indeterminacy, outcomes are additive in two components, the MSV solution and the process for $\boldsymbol{\delta}$. Added persistence arising from indeterminacy is thus characterized

[7] Since $\boldsymbol{\Omega}_v$ is square and invertible, $\boldsymbol{\Omega}_v \boldsymbol{G}_z'$ being square and invertible requires $\boldsymbol{G}_z$ to be square and invertible, which in turn implies that there are as many fundamentals drivers (column of $\boldsymbol{G}_z$) as observables to the central bank (rows of $\boldsymbol{G}_z$).

by the stable root of the "effective" Taylor rule system, captured by $\boldsymbol{T}_{11}$.

$$\boldsymbol{\chi}_t = \tilde{\boldsymbol{G}}_\chi \boldsymbol{v}_t + \boldsymbol{\delta}_t\,, \qquad \boldsymbol{\gamma}_t = \tilde{\boldsymbol{G}}_\gamma \boldsymbol{v}_t\,, \qquad \Longrightarrow \quad \boldsymbol{y}_t = \boldsymbol{Z}' \begin{bmatrix} \boldsymbol{\chi}_t \\ \boldsymbol{\gamma}_t \end{bmatrix} = \boldsymbol{G}\boldsymbol{v}_t + \boldsymbol{Z}^\chi \boldsymbol{\delta}_t$$

where $\boldsymbol{Z}^\chi$ corresponds to the first block of columns of $\boldsymbol{Z}'$ that is associated with $\boldsymbol{\chi}_t$.[8]

Analytical solution in case of the earlier examples: In our examples, we have only one stable root of $\boldsymbol{B}$, so that $\boldsymbol{\chi}_t$ is a scalar, and so are $\boldsymbol{\delta}$, $\boldsymbol{T}_{11}$, $\boldsymbol{\Psi}$, $\boldsymbol{\Sigma}$, and $\tilde{\boldsymbol{\Gamma}}_b$, which allows us to simplify further, and obtain:

$$a \cdot \boldsymbol{\Sigma}^2 - b \cdot \boldsymbol{\Sigma} = -\tilde{\boldsymbol{\Gamma}}_b^2 \qquad \text{with} \quad a = \boldsymbol{\Psi} > 0 \qquad \text{and} \quad b = 1 - \boldsymbol{T}_{11}^2 > 0\,.$$

which has the following non-negative solution, provided $\tilde{\boldsymbol{\Gamma}}_b^2 < 0.25 \cdot b^2/a$:

$$\Rightarrow \quad \boldsymbol{\Sigma} = \frac{1}{2}\frac{b}{a} \pm \sqrt{\left(\frac{1}{4}\frac{b^2}{a^2}\right) - \frac{\tilde{\boldsymbol{\Gamma}}_b^2}{a}}$$

A.4 A Fisher Economy with Measurement Error

A.4.1 Equilibrium with an Endogenous Signal

This section presents a formal statement of the equilibrium in the Fisher economy with a noisy inflation signal and serially correlated shocks. Exposition in the main text and in the additional material presented below assumes that the shocks are $i.i.d.$ for tractability. We assume that the central bank observes the inflation rate with measurement error ν_t such that $Z_t = \pi_t + \nu_t$. We present the analysis in terms of the projection equation for the real rate :

$$r_{t|t} = r_{t|t-1} + \kappa_r \left(\pi_t - \pi_{t|t-1} + \nu_t\right).$$

This leads to the full equation system:

$$\begin{aligned} \pi_t &= \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \eta_t, \\ r_{t|t} &= (\rho + \kappa_r)\, r_{t-1|t-1} + \kappa_r r_{t-1} + \kappa_r \nu_t + \kappa_r \eta_t, \\ r_t &= \rho r_{t-1} + \varepsilon_t. \end{aligned} \tag{A-19}$$

[8] Recall that $\tilde{\boldsymbol{G}}_\chi$ and $\tilde{\boldsymbol{G}}_\gamma$ are the appropriate partitions of $\tilde{\boldsymbol{G}} = \boldsymbol{Z}\boldsymbol{G}$.

The key coefficient is $(\rho + \kappa_r)$ on the lagged real rate projection, which determines the stability properties of the system but is not sufficient to determine equilibrium in contrast with the full information case. In addition, real rate projections depend on the endogenous forecast error η_t. As a result, the solution for κ_r depends on the equilibrium law of motion for π_t.

To solve the model, we first derive the endogenous Kalman gain and the associated forecast error variance. We then derive the projection condition and assess consistency with the proposed equilibrium paths. The steady-state Kalman gain is $\kappa_r = cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) / var(\widetilde{Z}_t)$, where $\widetilde{r}_t = r_t - r_{t|t-1}$ and $\widetilde{Z}_t = \widetilde{\pi}_t + \nu_t$. As before, we decompose the endogenous forecast error $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_\nu \nu_t + \gamma_b b_t$. It can be quickly verified that $cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) = -\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2$. The negative sign in this expression reflects the inverse relationship between inflation and the real rate when the signal is endogenous. Using $\widetilde{\pi}_t = -\left(\widetilde{r}_{t-1} - \widetilde{r}_{t-1|t-1}\right) + \eta_t$, we find that $var(\widetilde{Z}_t)$ can be expressed as $var(\widetilde{Z}_t) = \Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2$.

We can now derive the following expression for the Kalman gain:

$$\kappa_r = \frac{-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2}. \tag{A-20}$$

Although the forecast error variance Σ still needs to be determined as a function of the structural parameters, we can make two observations already. First, the gain κ_r can be negative for small enough γ_ε, that is, $\kappa_r < 0$ if $\gamma_\varepsilon < \rho\Sigma/\sigma_\varepsilon^2$. Second, existence of a steady-state Kalman filter implies that $|\rho + \kappa_r| < 1$ as long as $\Sigma > 0$, as shown in Proposition 1 below. We return to a discussion of the case where $r_t = r_{t|t}$, so that $\Sigma = 0$, later in this section. In the next step, we compute the projection error variance $\Sigma = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right)$. Using $var\left(\widetilde{r}_t\right) = \rho^2\Sigma + \sigma_\varepsilon^2$ and $var\left(\widetilde{r}_{t|t}\right) = \kappa_r cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ we can derive the following Riccati equation, which is quadratic in Σ:

$$\Sigma = \rho^2\Sigma + \sigma_\varepsilon^2 - \frac{\left(-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2\right)^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2}. \tag{A-21}$$

Finally, an equilibrium has to obey the restrictions imposed by central bank projections, namely $\pi_{t|t} = \frac{1}{\phi - \rho} r_{t|t}$. This implies a covariance restriction of projection errors which differs from the exogenous signal case because of different information sets. Specifically, we have that $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = \frac{1}{\phi-\rho} cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ or alternatively $(\phi - \rho)\, cov\left(\widetilde{\pi}_t, \widetilde{\pi}_t + \nu_t\right) = cov\left(\widetilde{r}_t, \widetilde{\pi}_t + \nu_t\right)$. After some rearranging we can write this expression as:

$$\gamma_\nu (1 + \gamma_\nu) = -\frac{\phi}{\phi - \rho} \frac{\Sigma}{\sigma_\nu^2} - \frac{\gamma_b^2}{\phi - \rho} \frac{\sigma_b^2}{\sigma_\nu^2} + \frac{\left[1 - (\phi - \rho)\gamma_\varepsilon\right]\gamma_\varepsilon}{\phi - \rho} \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}. \tag{A-22}$$

The condition places a quadratic restriction on all three innovation loadings γ. This can imply that there are no or multiple solution to this equation, and thus for the overall equilibrium, for a given parameterization of the model. We summarize our findings in the following proposition.

PROPOSITION 1 (Limited Information REE in the Fisher Economy) *The set of stationary rational expectations equilibria (REE) in the model (A-19) under LIRE with signal $Z_t = \pi_t + \nu_t$ is characterized by the following dynamic equations:*

$$\begin{aligned}
\pi_t &= \frac{\phi}{\phi-\rho} r_{t-1|t-1} - r_{t-1} + \gamma_\varepsilon \varepsilon_t + \gamma_\nu \nu_t + \gamma_b b_t, && \text{(A-23)}\\
r_{t|t} &= (\rho + \kappa_r)\, r_{t-1|t-1} - \kappa_r r_{t-1} + \kappa_r \gamma_\varepsilon \varepsilon_t + \kappa_r (1+\gamma_\nu)\, \nu_t + \kappa_r \gamma_b b_t, && \text{(A-24)}\\
r_t &= \rho r_{t-1} + \varepsilon_t,
\end{aligned}$$

With $\Sigma > 0$, we have:

$$|\rho + \kappa_r| < 1 \tag{A-25}$$

$$\Sigma = \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + 4\beta}\right), \tag{A-26}$$

$$\alpha = (1 + 2\rho\gamma_\varepsilon)\, \sigma_\varepsilon^2 - (1-\rho)^2 \left(\gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 + (1+\gamma_\nu)^2 \sigma_\nu^2\right), \tag{A-27}$$

$$\beta = \left(\gamma_b^2 + (1+\gamma_\nu)^2 \sigma_\nu^2\right) \sigma_\varepsilon^2, \tag{A-28}$$

$$\kappa_r = \frac{-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 + (1+\gamma_\nu)^2 \sigma_\nu^2}, \tag{A-20}$$

$$\gamma_\nu (1+\gamma_\nu) = -\frac{\phi}{\phi-\rho}\frac{\Sigma}{\sigma_\nu^2} - \frac{1}{\phi-\rho}\frac{\sigma_b^2}{\sigma_\nu^2} + \left(\frac{1}{\phi-\rho} - \gamma_\varepsilon\right)\gamma_\varepsilon \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}. \tag{A-22}$$

Proof. *Equations* (A-26), (A-27) *and* (A-28) *follow directly from solving the quadratic equation for $\Sigma > 0$ in* (A-21)*. The expression for the Kalman gain κ_r in* (A-20) *and the restrictions from the projection condition in* (A-22) *restate earlier results. The requirement that with $\Sigma > 0$ we must have $|\rho + \kappa_r| < 1$ is an application of Theorem 2 in Appendix B.4. In this specific example, the result that $|\rho + \kappa_r| < 1$ can be derived as follows: Consider the candidate value $\kappa_r = 0$ for the Kalman gain; in this case we would have $\Sigma = \mathrm{Var}(r_t)$. The optimal Kalman gain seeks to minimize Σ and the optimal value of Σ must thus be (weakly) smaller than $\mathrm{Var}(r_t) = \sigma_\varepsilon^2/(1-\rho^2)$ and finite. With the optimal Kalman gain, projections are given by* (A-24) *and the process for the projection errors $r_t^* = r_t - r_{t|t}$ is $r_t^* = (\rho+\kappa_r) r_{t-1}^* + \varepsilon_t - \kappa_r(\eta_t + \nu_t)$. Recall that $\Sigma \equiv \mathrm{Var}(r_t^*)$. We can thus conclude that for $0 < \Sigma < \infty$ the optimal Kalman gain must be such that $|\rho + \kappa_r| < 1$.* ■

The Proposition describes the set of solutions under indeterminacy. With $|\rho + \kappa_r| < 1$ the equation system has only stable roots and therefore lacks a restriction to determine the endogenous forecast error uniquely. As in the case of an exogenous signal, the projection condition that ensures internal consistency of central bank and private sector expectation formation restricts the set of possible equilibria. Specifically, an equilibrium with $\Sigma > 0$ does not exist when no innovation loadings can be found to ensure existence of a steady-state Kalman filter that is consistent with the projection condition. Moreover, the set of solutions is restricted over the parameter space by the nonlinear Riccati equation for the forecast error variance, by non-negativity constraints on variances and by ruling out complex solutions.

In our framework, feedback between filtering and model solution is central to equilibrium determination. Filtering depends on the information set, the result of which affects equilibrium outcomes and the content of the information set. This fixed-point problem has been noted before, at least as early as Sargent (1991). We go beyond this insight by showing that equilibrium determination is substantially different from the standard linear RE case. A solution may not exist even when the root-counting criterion for existence of an equilibrium indicates a sufficient number of stable roots in standard full information settings. While the root-counting approach for given κ_r could indicate non-existence, uniqueness or indeterminacy, it is the second-moment restrictions due to the less informed agent's filtering problem that determine equilibrium. In that sense, indeterminacy is generic in this environment since existence of a stable Kalman filter introduces a stable root into the dynamic system. At the same time, the second-moment restrictions resulting from the projection condition restrain belief shock loadings, which stands in stark contrast to the case of indeterminacy in a full-information scenario.

A.4.2 Indeterminacy in the Fisher Example

We show that $|\phi\kappa_\pi| < 1$ in the simple example of Section 2 with $i.i.d.$ real rate shocks. The projection condition requires $\phi\kappa_\pi = \kappa_r$, where κ_r is the Kalman gain in the central bank's signal extraction for the real rate, $r_{t|t} = \kappa_r\left(\pi_t + \nu_t\right)$. Optimal signal extraction implies $|\kappa_r| < 1$ for any equilibrium where r_t cannot be perfectly inferred by the central bank.[9]

In the $i.i.d.$ case we have $\rho = 0$ and $r_t = \varepsilon_t$, and the central bank's real-rate projections are given by $r_{t|t} = \kappa_r\left(\pi_t + \nu_t\right)$ with $\kappa_r = \operatorname{Cov}\left(r_t, \pi_t + \nu_t \,|\, Z^{t-1}\right) / \operatorname{Var}\left(\pi_t + \nu_t \,|\, Z^{t-1}\right)$. Furthermore, recall the equilibrium process for inflation:

$$\pi_{t+1} = -\left(r_t - r_{t|t}\right) + \gamma_\varepsilon \varepsilon_{t+1} + \gamma_\nu \nu_{t+1} + \gamma_b b_{t+1} \tag{A-29}$$

We now show that $|\kappa_r| < 1$ for any γ_ε, γ_ν, and γ_b. This carries over to $|\phi\kappa_\pi| < 1$ when the

[9]When r_t cannot be perfectly inferred by the central bank we have $\operatorname{Var}\left(r_{t|t}\right) < \operatorname{Var}\left(r_t\right)$.

projection condition is satisfied so that $\phi\kappa_\pi = \kappa_r$. Define the following variance ratio:

$$R_\varepsilon^2 \equiv \frac{\mathrm{Var}\left(\varepsilon_{t|t}\right)}{\mathrm{Var}\left(\varepsilon_t\right)} \qquad \text{with} \quad 0 \le R_\varepsilon^2 \le 1. \tag{A-30}$$

$$= \frac{\mathrm{Cov}\left(\varepsilon_t, \pi_t + \nu_t\right)^2}{\mathrm{Var}\left(\pi_t + \nu_t\right) \cdot \mathrm{Var}\left(\varepsilon_t\right)}$$

$$= \frac{\gamma_\varepsilon^2 \cdot \sigma_\varepsilon^2}{\mathrm{Var}\left(\pi_t + \nu_t\right)} = \gamma_\varepsilon \cdot \kappa_r, \tag{A-31}$$

where the second line follows from optimal signal extraction and the last line from (A-29). Recall further that $r_t = \varepsilon_t$ in the iid case. Based on (A-29) and (A-30) we also have

$$\mathrm{Var}\left(\pi_t + \nu_t\right) = \left(1 - R_\varepsilon^2\right) \cdot \sigma_\varepsilon^2 + \gamma_\varepsilon^2 \cdot \sigma_\varepsilon^2 + (1 + \gamma_\nu)^2 \cdot \sigma_\nu^2 + \gamma_b^2$$

and (A-31) can be rewritten as

$$R_\varepsilon^2 \cdot \left(\left(1 - R_\varepsilon^2\right) \cdot \sigma_\varepsilon^2 + (1 + \gamma_\nu)^2 \cdot \sigma_\nu^2 + \gamma_b^2\right) = \left(1 - R_\varepsilon^2\right) \cdot \left(\gamma_\varepsilon^2 \cdot \sigma_\varepsilon^2\right) \tag{A-32}$$

First, consider the case of $R_\varepsilon^2 < 1$ so that the real rate is not perfectly revealed in equilibrium. Equation (A-32) then implies the following:

$$\gamma_\varepsilon^2 = R_\varepsilon^2 \cdot \left(1 + \frac{(1 + \gamma_\nu)^2 \cdot \sigma_\nu^2 + \gamma_b^2}{(1 - R_\varepsilon^2) \cdot \sigma_\varepsilon^2}\right) \ge R_\varepsilon^2.$$

With (A-31) we have $\kappa_r^2 = R_\varepsilon^2 \cdot (R_\varepsilon^2/\gamma_\varepsilon^2) < 1$ and thus $|\kappa_r| < 1$. The inequality is strict, since $R_\varepsilon^2 < 1$ in this case.

A.4.3 A Full Revelation Equilibrium

In the current example we can also construct an equilibrium with $|\kappa_r| = |\phi| > 1$ and where $R_\varepsilon^2 = 1$. In this case, (A-32) requires $\gamma_\nu = -1$ and $\gamma_b = 0$, from which it follows that $\pi_t = \bar{g} \cdot r_t - \nu_t$ with $\bar{g} = 1/\phi$ and thus $\kappa_r = \phi$. This equilibrium generates an unstable root in the linear RE system characterizing the economy since we have $\kappa_\pi = 1$ and thus $|\phi \cdot \kappa_\pi| > 1$. As before the unstable root reflects the endogenous Kalman gain.

Although this equilibrium shares features of the unique equilibrium under full information, we classify it as a belonging to the set of multiple equilibria under indeterminacy. In that sense it is similar to the continuity solution under indeterminacy as in Lubik and Schorfheide (2003), which is classified as a sunspot equilibrium without sunspots, where agents coordinate their beliefs on behavior that does not differ from its determinate counterpart. Nevertheless, it is just one of many equilibria that agents can coordinate on.

This equilibrium is identical in outcomes to what is generated by the policy rule described in (A-42). The construction of this alternative equilibrium straightforwardly applies also in the case where the real rate is not $i.i.d.$ but follows an AR(1) process.

A.4.4 An Alternative Derivation

We can derive existence and determinacy conditions in an alternative manner, using stability properties of lag polynomials. [10]

We consider the following system:

$$i_t = \phi\pi_{t|t} = \mathbb{E}_t\left[\pi_{t+1}\right] + r_t$$
$$\pi_{t|t} = \left(1-\kappa_\pi\right)\rho\pi_{t-1|t-1} + \kappa_\pi\left(\pi_t + \nu_t\right).$$

The central bank's projection of inflation can be expressed as:

$$\pi_{t|t} = \frac{1}{1-\left(1-\kappa_\pi\right)\rho L}\kappa_\pi\left(\pi_t+\nu_t\right),$$

which then leads to:

$$\phi\frac{1}{1-\left(1-\kappa_\pi\right)\rho L}\kappa_\pi\left(\pi_t+\nu_t\right) = \mathbb{E}_t\left[\pi_{t+1}\right] + r_t$$

We can now state the dynamics for inflation directly as a moving average process (in contrast to the exposition in the rest of the paper):

$$\pi_t = g(L)\epsilon_t + h(L)\nu_t + d(L)b_t, \tag{A-33}$$

where $d(L)b_t$ denotes the dynamics driven by the sunspot shock b_t. Using the Hansen-Sargent prediction formula and the method of undetermined coefficients, it can be shown that:

$$\begin{aligned} g(L) &= \frac{-\frac{L(1-(1-\kappa_\pi)\rho L)}{1-\rho L}+g_0(1-(1-\kappa_\pi)\rho L)}{1-(\phi\kappa_\pi+(1-\kappa_\pi)\rho)L} \\ h(L) &= \frac{\phi\kappa_\pi L+h_0(1-(1-\kappa_\pi)\rho L)}{1-(\phi\kappa_\pi+(1-\kappa_\pi)\rho)L} \\ d(L) &= \frac{d_0(1-(1-\kappa_\pi)\rho L)}{1-(\phi\kappa_\pi+(1-\kappa_\pi)\rho)L} \end{aligned}$$

A unique equilibrium arises if $\left|\phi\kappa_\pi + \left(1-\kappa_\pi\right)\rho\right| > 1$. This obtains because in this case $g(L), h(L)$ and $d(L)$ contain an explosive root, so that one can set g_0 and h_0 akin to solving the unstable equation forward. This leads to $d_0 = 0$ and the sunspot component is eliminated. On the other hand, if $\left|\phi\kappa_\pi + \left(1-\kappa_\pi\right)\rho\right| < 1$, then there is no restriction on g_0, h_0 and d_0 (except that they

[10] We thank a referee for proposing this derivation of the solution for the Fisher model with persistent shocks.

have to be consistent with the Kalman filter), and hence we have multiple equilibria.

A.4.5 Closed-Form Solutions with An Alternative Information Set

We present a variant of the Fisher economy with a particular information set that enables us to derive a number of results in closed form.The model combines an exogenous AR(1) process for the real rate with a Fisher equation and a Taylor rule that responds to the central bank's inflation projection:

$$\begin{aligned} i_t &= r_t + E_t\pi_{t+1}\,, & i_t &= \phi\,\pi_{t|t}\,, & |\phi| &> 1\,, \\ r_t &= \rho\, r_{t-1} + \varepsilon_t & \varepsilon_t &\sim iidN(0, \sigma_\varepsilon^2)\,, & |\rho| &< 1\,, \end{aligned}$$

and the projection condition requires $\pi_{t|t} = r_{t|t}/(\phi - \rho)$. As before, we express the endogenous forecast error as a linear combination of fundamental and belief shocks. Collecting terms yields the following characterization of the inflation process:

$$\pi_{t+1} = -(r_t - r_{t|t}) + \frac{\rho}{\phi - \rho} r_t + \eta_{t+1} \tag{A-34}$$

$$\begin{aligned} \text{with} \quad \eta_{t+1} &\equiv \pi_{t+1} - E_t\pi_{t+1} \\ &= \gamma_\varepsilon \varepsilon_{t+1} + \gamma_\nu \nu_{t+1} + \gamma_b b_{t+1}\,, \qquad b_{t+1} \sim iidN(0,1)\,, \end{aligned} \tag{A-35}$$

We now assume that the central bank's information set is characterized by a bivariate signal, which includes a perfect reading of the real rate and a noisy signal of current inflation:

$$\boldsymbol{Z}_t = \begin{bmatrix} r_t \\ \pi_t + \nu_t \end{bmatrix} \qquad \text{with} \quad \nu_t \sim iidN(0, \sigma_\nu^2) \tag{A-36}$$

$$\Rightarrow \quad r_{t|t} = r_t \quad \Rightarrow \quad \pi_{t|t} = \frac{1}{\phi - \rho} r_t\,, \tag{A-37}$$

In light of (A-37), the inflation dynamics specified in (A-34) simplify to

$$\pi_{t+1} = \frac{\rho}{\phi - \rho} r_t + \eta_{t+1}\,. \tag{A-38}$$

Before determining the shock loadings γ_ε, γ_ν, and γ_b of the endogenous forecast error η_t, we can already note that one-step-ahead expectations of inflation, $E_t\pi_{t+1} = r_t/(\phi - \rho)$, are identical to the full-information case so that the effects of indeterminacy will be limited to changes in the amplification of shocks, without consequences for inflation persistence.

In light of (A-38), we can conclude that the history of $\boldsymbol{Z}_t$ spans the same information content as the history of

$$\boldsymbol{W}_t = \begin{bmatrix} \varepsilon_t \\ (1+\gamma_\nu)\,\nu_t + \gamma_b\, b_t \end{bmatrix}.$$

$\boldsymbol{W}^t$ spans $\boldsymbol{Z}^t$ since, with $|\rho| < 1$, ε^t spans r^t, and since observing $(1+\gamma_\nu)\,\nu_t + \gamma_b\, b_t$ adds the same information to the span of ε^t as observing $\pi_t + \nu_t$. The signal vector $\boldsymbol{W}_t$ consists of two mutually orthogonal elements that are serially uncorrelated over time. As a result, projections onto $\boldsymbol{W}^t$ can be decomposed into the sum of projections onto its individual elements.

The projection condition (A-37) then requires $\eta_{t|t} = \varepsilon_{t|t}/(\phi - \rho)$, and with $\varepsilon_{t|t} = \varepsilon_t$, we can conclude that

$$\gamma_\varepsilon = \frac{1}{\phi - \rho}\,. \tag{A-39}$$

In addition we have $\pi_t^* \equiv \pi_t - \pi_{t|t} = \gamma_\nu\, \nu_t + \gamma_b\, b_t$, and thus $E\left(\gamma_\nu\, \nu_t + \gamma_b\, b_t \,\middle|\, \boldsymbol{Z}^t\right) = 0$, which implies the following restriction on γ_ν and γ_b:

$$\operatorname{Cov}\left(\gamma_\nu\, \nu_t + \gamma_b\, b_t \,\middle|\, (1+\gamma_\nu)\,\nu_t + \gamma_b\, b_t\right) = \gamma_\nu(1+\gamma_\nu)\sigma_\nu^2 + \gamma_b^2 = 0\,. \tag{A-40}$$

$$\Rightarrow \quad \gamma_\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\gamma_b^2}{\sigma_\nu^2}} \tag{A-41}$$

Real solutions to (A-41) require $|\gamma_b| < 0.5\,\sigma_\nu$ leading to a continuum of solutions with $\gamma_\nu \in [-1, 0]$. While there is a unique solution for the shock loading γ_ε, as given in (A-39), there are multiple solutions for γ_ν and γ_b as characterized by (A-41).

Equilibrium dynamics of inflation are described by (A-34) and (A-35) together with (A-39) and (A-41). Collecting terms, inflation evolves according to:

$$\pi_t = \frac{1}{\phi - \rho} r_t + \gamma_\nu\, \nu_t + \gamma_b\, b_t$$

where γ_ν and γ_b are restricted by (A-41). In light of the projection condition (A-40), the variance of inflation is given by

$$\operatorname{Var}(\pi_t) = \left(\frac{1}{\phi - \rho}\right)^2 \operatorname{Var}(r_t) + |\gamma_\nu|\sigma_\nu^2\,.$$

Since γ_ν is bounded by one in absolute value, there is an upper bound for the inflation variance equal to $\operatorname{Var}(r_t)/(\phi - \rho)^2 + \sigma_\nu^2$.

A key feature of this example is that belief shock loadings are not generally zero, and non-fundamental belief shocks can affect equilibrium outcomes. In addition, there is an upper bound on belief shock loadings, which stems from the projection condition and is unique to our imperfect information framework. When indeterminacy arises in full-information models, there are no such bounds on the scale with which belief shocks can affect economic outcomes.

In the absence of measurement error on inflation, $\sigma_\nu = 0$, the outcomes in our simplified example collapse to the full-information solution $\pi_t = r_t/(\phi - \rho)$ and equilibria are continuous with respect to the full-information case as σ_ν approaches zero. For any measurement error variance, the range of possible equilibria includes the case where outcomes are identical to the full-information case, with $\gamma_\nu = \gamma_b = 0$.

A.4.6 Determinacy without Optimal Projections

In the model with imperfect information indeterminacy can arise from the interplay between the effects of policy on inflation and the policymaker's use of optimal projections. An alternative policy would be for the policymaker to respond directly to the noisy signal Z_t, rather than an optimal projection based on the signal. This alternative setup obviates the role of the endogenous Kalman gain κ_π. Consider the policy rule:

$$i_t = \phi\left(\pi_t + \nu_t\right) . \tag{A-42}$$

When $|\phi| > 1$, policy responds to actual inflation by more than one-for-one; the Taylor principle applies and results in a unique equilibrium, given by $\pi_t = \bar{g}\, r_t - \nu_t$. While the equilibrium is unique, inflation is subject to additional fluctuations, at least compared to the full-information case, which should be undesirable from the perspective of the policymaker. This is due to the presence of the measurement error ν_t. In fact, the projection condition guarantees that any of the multiple equilibria obtained under the projection-based Taylor rule imply a (weakly) lower variance of inflation than under the determinate policy rule in (A-42):

$$\mathrm{Var}\left(\pi_t\right) \leq \bar{g}^2\ \mathrm{Var}\left(r_t\right) + \mathrm{Var}\left(\nu_t\right) \tag{A-43}$$

The projection condition implies $\mathrm{Var}\left(\pi_{t|t}\right) = \bar{g}^2\ \mathrm{Var}\left(r_t\right)$, and the optimality of projections leads to $\mathrm{Var}\left(\pi_{t|t}\right) \leq \bar{g}^2\ \mathrm{Var}\left(r_{t|t}\right)$. Moreover, the definition of the signal implies $\pi_t - \pi_{t|t} = -(\nu_t - \nu_{t|t})$, since $Z_{t|t} = Z_t$, so that $\mathrm{Var}\left(\pi_t - \pi_{t|t}\right) = \mathrm{Var}\left(\nu_t - \nu_{t|t}\right)$. We obtain $\mathrm{Var}\left(\pi_t\right) = \mathrm{Var}\left(\pi_{t|t}\right) + \mathrm{Var}\left(\pi_t - \pi_{t|t}\right) \leq \bar{g}^2\ \mathrm{Var}\left(r_t\right) + \mathrm{Var}\left(\nu_t\right)$. Furthermore, recall that $\mathrm{Var}\left(\nu_t\right) = \sigma_\nu^2$ and, in the iid case, $\mathrm{Var}\left(r_t\right) = \sigma_\varepsilon^2$.

B General Framework

In this section we discuss the general framework laid out in Section 3 in more detail.

B.1 Full Information Equilibrium in the General Case

The general framework nests the case of full information, where $\boldsymbol{S}_{t+h|t} = E_t \boldsymbol{S}_{t+h}\ \forall h \geq 0$. This holds, for example, when $\boldsymbol{H} = \boldsymbol{I}$ such that $\boldsymbol{Z}_t = \boldsymbol{S}_t$. The full-information system can easily be solved using standards methods such as those found in King and Watson (1998), Klein (2000) or Sims (2002). We stack all variables, including the policy control, into a vector $\boldsymbol{\mathcal{S}}_t$ that is partitioned into a vector of $N_i + N_x$ backward-looking variables, $\boldsymbol{\mathcal{X}}_t$, and a vector of $N_y + N_i$ forward-looking variables, $\boldsymbol{\mathcal{Y}}_t$.[11] We begin by restating some equations from the main part of the paper:

$$\boldsymbol{\mathcal{S}}_t = \begin{bmatrix} \boldsymbol{\mathcal{X}}_t \\ \boldsymbol{\mathcal{Y}}_t \end{bmatrix}, \qquad \text{where} \quad \boldsymbol{\mathcal{X}}_t = \begin{bmatrix} \boldsymbol{i}_{t-1} \\ \boldsymbol{X}_t \end{bmatrix} \qquad \boldsymbol{\mathcal{Y}}_t = \begin{bmatrix} \boldsymbol{Y}_t \\ \boldsymbol{i}_t \end{bmatrix}. \tag{16}$$

Using $\boldsymbol{\mathcal{S}}'_t = \begin{bmatrix} \boldsymbol{i}'_{t-1} & \boldsymbol{S}'_t & \boldsymbol{i}'_t \end{bmatrix}$, the dynamics of the system under full information are then characterized by the following expectational difference equation:

$$\underbrace{\begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{I} + \hat{\boldsymbol{J}} & 0 \\ 0 & \boldsymbol{\Phi}_J & 0 \end{bmatrix}}_{\boldsymbol{\mathcal{J}}} E_t \boldsymbol{\mathcal{S}}_{t+1} = \underbrace{\begin{bmatrix} 0 & 0 & \boldsymbol{I} \\ 0 & \boldsymbol{A} + \hat{\boldsymbol{A}} & \boldsymbol{A}_i \\ -\boldsymbol{\Phi}_i & -\boldsymbol{\Phi}_H & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{\mathcal{A}}} \boldsymbol{\mathcal{S}}_t \tag{17}$$

Existence and uniqueness of a solution to this system depend on the properties of the matrices $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$. Throughout this paper we focus on environments where a unique full-information solution exists and therefore impose the following assumption known from Klein (2000) and King and Watson (1998).[12]

ASSUMPTION 1 (Unique full-information solution) *The set of generalized eigenvalues of the system matrices $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$ as defined in* (17) *has $N_i + N_x$ roots inside the unit circle and $N_y + N_i$ roots outside the unit circle. Moreover, there is some complex number z such that $|\boldsymbol{\mathcal{J}}\, z - \boldsymbol{\mathcal{A}}| \neq 0$.*

The solution can, for instance, be computed using the numerical methods of Klein (2000), and

[11] The presence of the lagged policy control among the backward-looking variables in $\boldsymbol{\mathcal{X}}_t$ serves to handle the case of interest-rate smoothing, $\boldsymbol{\Phi}_i \neq 0$, and can otherwise be omitted.

[12] This is akin to a root-counting criterion where the number of explosive eigenvalues matches the number of forward-looking variables. Sims (2002) offers a slight generalization of this criterion. In the case of simple monetary policy models, the root-counting condition is satisfied by requiring that the central bank's interest-rate rule satisfies the Taylor principle, that is, by responding more than one-to-one to fluctuations in inflation.

has the following form:

$$E_t \boldsymbol{\mathcal{X}}_{t+1} = \boldsymbol{\mathcal{P}} \, \boldsymbol{\mathcal{X}}_t \,, \qquad \boldsymbol{\mathcal{Y}}_t = \boldsymbol{\mathcal{G}} \, \boldsymbol{\mathcal{X}}_t \,, \qquad \boldsymbol{\mathcal{G}} = \begin{bmatrix} \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix} , \tag{18}$$

where $\boldsymbol{\mathcal{P}}$ is a stable matrix with all eigenvalues inside the unit circle. In the full-information version of our framework certainty equivalence holds; that is, the decision-rule coefficients $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{G}}$ do not depend on the shock variances encoded in $\boldsymbol{B}_{x\varepsilon}$ nor on the measurement loadings $\boldsymbol{H}$.[13] Equilibrium dynamics in the full-information case are then summarized by:

$$\boldsymbol{\mathcal{S}}_{t+1} = \bar{\boldsymbol{\mathcal{T}}} \boldsymbol{\mathcal{S}}_t + \bar{\boldsymbol{\mathcal{H}}} \boldsymbol{\varepsilon}_{t+1} \,, \qquad \bar{\boldsymbol{\mathcal{T}}} = \begin{bmatrix} \boldsymbol{\mathcal{P}} & \boldsymbol{0} \\ \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{P}} & \boldsymbol{0} \end{bmatrix} , \qquad \bar{\boldsymbol{\mathcal{H}}} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \\ \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix} \boldsymbol{B}_{x\varepsilon} \,, \tag{19}$$

where $\bar{\boldsymbol{\mathcal{T}}}$ is stable because $\boldsymbol{\mathcal{P}}$ is, and the endogenous forecast errors are given by $\boldsymbol{\eta}_t = \boldsymbol{\mathcal{G}}_{yx} \, \boldsymbol{B}_{x\varepsilon} \, \boldsymbol{\varepsilon}_t$.

To construct the coefficient matrices $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{P}}$ we can follow Klein (2000) and perform a complex generalized Schur decomposition of $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$. The Schur decomposition yields a pair of unitary matrices, $\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{Z}}$, as well as upper triangular matrices $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{T}}$, such that $\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{T}}$.[14] The result is the following triangular system in the so-called canonical variables $\boldsymbol{\mathcal{X}}^*_{t+1}$ and $\boldsymbol{\mathcal{Y}}^*_{t+1}$:

$$\underbrace{\begin{bmatrix} \boldsymbol{\mathcal{U}}_{11} & \boldsymbol{\mathcal{U}}_{12} \\ \boldsymbol{0} & \boldsymbol{\mathcal{U}}_{22} \end{bmatrix}}_{\boldsymbol{\mathcal{U}}} E_t \begin{bmatrix} \boldsymbol{\mathcal{X}}^*_{t+1} \\ \boldsymbol{\mathcal{Y}}^*_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{\mathcal{T}}_{11} & \boldsymbol{\mathcal{T}}_{12} \\ \boldsymbol{0} & \boldsymbol{\mathcal{T}}_{22} \end{bmatrix}}_{\boldsymbol{\mathcal{T}}} \begin{bmatrix} \boldsymbol{\mathcal{X}}^*_t \\ \boldsymbol{\mathcal{Y}}^*_t \end{bmatrix} , \tag{A-44}$$

where

$$\begin{bmatrix} \boldsymbol{\mathcal{X}}_t \\ \boldsymbol{\mathcal{Y}}_t \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{\mathcal{Z}}_{11} & \boldsymbol{\mathcal{Z}}_{12} \\ \boldsymbol{\mathcal{Z}}_{21} & \boldsymbol{\mathcal{Z}}_{22} \end{bmatrix}}_{\boldsymbol{\mathcal{Z}}} \begin{bmatrix} \boldsymbol{\mathcal{X}}^*_t \\ \boldsymbol{\mathcal{Y}}^*_t \end{bmatrix} \tag{A-45}$$

Without loss of generality, the triangular system can always be ordered such that $\boldsymbol{\mathcal{U}}_{11}^{-1}\boldsymbol{\mathcal{T}}_{11}$ is a stable matrix, and $\boldsymbol{\mathcal{U}}_{22}^{-1}\boldsymbol{\mathcal{T}}_{22}$ is an unstable matrix, which designates $\boldsymbol{\mathcal{X}}^*_t$ and $\boldsymbol{\mathcal{Y}}^*_t$ to be stable and unstable canonical variables.

Crucially, our Assumption 1 of a unique stable solution in the full-information case means that the number of stable canonical variables in the vector $\boldsymbol{\mathcal{X}}^*_t$ is identical to the number of backward-

[13]The imperfect information setup could include measurement errors as part of the vector of backward-looking variables, $\boldsymbol{X}_t$. They would affect endogenous variables of the system only via $\boldsymbol{H}$, which does not play a role in the full-information solution. Conceptually, there is no harm including measurement errors in $\boldsymbol{X}_t$, and the corresponding columns of $\boldsymbol{\mathcal{G}}_{yx}$ are equal to zero in this case.

[14]In general, $\boldsymbol{\mathcal{Q}}$, $\boldsymbol{\mathcal{Z}}$, $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{T}}$ are complex.

looking variables in $\boldsymbol{\mathcal{X}}_t$ (and the length of the unstable canonical variables' vector $\boldsymbol{\mathcal{Y}}_t^*$ is identical to the length of $\boldsymbol{\mathcal{Y}}_t$). A stable solution of the system then requires the familiar prescription to solve the unstable canonical variables forward and the stable canonical variables backward. The result is the following unique solution:

$$\boldsymbol{\mathcal{Y}}_t^* = \mathbf{0} \tag{A-46}$$

$$\implies \quad \boldsymbol{\mathcal{X}}_t = \boldsymbol{\mathcal{Z}}_{11}\boldsymbol{\mathcal{X}}_t^*$$

$$\implies \quad E_t\boldsymbol{\mathcal{X}}_{t+1} = \underbrace{\boldsymbol{\mathcal{Z}}_{11}\boldsymbol{\mathcal{U}}_{11}^{-1}\boldsymbol{\mathcal{T}}_{11}\boldsymbol{\mathcal{Z}}_{11}^{-1}}_{\boldsymbol{\mathcal{P}}}\boldsymbol{\mathcal{X}}_t \qquad \text{and} \quad \boldsymbol{\mathcal{Y}}_t = \underbrace{\boldsymbol{\mathcal{Z}}_{21}\boldsymbol{\mathcal{Z}}_{11}^{-1}}_{\boldsymbol{\mathcal{G}}}\boldsymbol{\mathcal{X}}_t \tag{A-47}$$

The defining feature of the unique full-information equilibrium is that non-fundamental disturbances, such as sunspot or belief shocks, do not affect equilibrium outcomes, as discussed, for example, by Lubik and Schorfheide (2003). Hence, the equilibrium is determinate. This is no longer the case under imperfect information.

B.2 Expectation Formation under Imperfect Information

We proceed by making a sequence of assumptions that allow us to map the general framework under asymmetric information into a standard linear rational expectations setting that can be solved and studied with familiar methods. To ensure that a Kalman filter delivers an exact representation of the true conditional expectations, we are interested in linear equilibria, driven by normally distributed disturbances. Therefore, we make the following assumption:

ASSUMPTION 2 (Jointly normal forecast errors) *The endogenous forecast errors are a linear combination of the N_ε exogenous errors, $\boldsymbol{\varepsilon}_t$, and N_y so-called belief shocks, $\mathbf{b}_t$, that are mean zero and uncorrelated with $\boldsymbol{\varepsilon}_t$:*

$$\boldsymbol{\eta}_t = \boldsymbol{\Gamma}_\varepsilon\boldsymbol{\varepsilon}_t + \boldsymbol{\Gamma}_b\boldsymbol{b}_t \tag{A-48}$$

Moreover, exogenous shocks and belief shocks are generated from a joint standard normal distribution,

$$\boldsymbol{w}_t \equiv \begin{bmatrix}\boldsymbol{\varepsilon}_t \\ \boldsymbol{b}_t\end{bmatrix} \sim N\left(\begin{bmatrix}0 \\ 0\end{bmatrix}, \begin{bmatrix}\boldsymbol{I} & 0 \\ 0 & \boldsymbol{I}\end{bmatrix}\right)$$

The matrix of belief shock loadings $\boldsymbol{\Gamma}_b$ need not have full rank so that linear combinations of endogenous and exogenous forecast errors might be perfectly correlated.

As a corollary of Assumption 2, exogenous and endogenous forecast errors are joint normally distributed as well:

$$\begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\eta}_t \end{bmatrix} \sim N\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{I} & \boldsymbol{\Gamma}_\varepsilon' \\ \boldsymbol{\Gamma}_\varepsilon & \boldsymbol{\Omega}_\eta \end{bmatrix} \right), \qquad \text{with} \quad \boldsymbol{\Omega}_\eta = \boldsymbol{\Gamma}_b \boldsymbol{\Gamma}_b' + \boldsymbol{\Gamma}_\varepsilon \boldsymbol{\Gamma}_\varepsilon' .$$

In this specification of the endogenous forecast error we follow Lubik and Schorfheide (2003) and Farmer, Khramov, and Nicolò (2015). In a determinate equilibrium, $\boldsymbol{\Gamma}_b$ is the zero matrix, while under indeterminacy it allows for sunspot shocks, or belief shocks, to affect endogenous forecast errors and expectations.[15] As in the full-information literature, Assumption 2 sees endogenous forecast errors as a linear combination of exogenous fundamental shocks and non-fundamental belief shocks. In addition, the belief shocks are assumed to be normally distributed, which is not entirely innocuous in that it restricts the stochastic nature of the equilibrium outcomes. While this assumption is immaterial for the solution under full information, it allows us to derive an exact Kalman-filter representation of expectation formation under imperfect information. Extension to non-Gaussian shocks are beyond the scope of the paper, but constitute important further research.

We now define and characterize a class of stationary, linear and time-invariant equilibria, where expectations are represented by a Kalman filter and where we treat the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ as given. In a subsequent step, we turn to solution methods that determine values for $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ consistent with these equilibria.

DEFINITION 1 (Stationary, linear, time-invariant equilibrium) *In a stationary, linear, and time-invariant equilibrium, backward- and forward-looking variables, $\boldsymbol{Y}_t$ and $\boldsymbol{X}_t$, and the instrument, $\boldsymbol{i}_t$, are stationary and their equilibrium dynamics satisfy the expectational difference system described by* (12) *and* (13)*.*[16] *All expectations are rational, and the imperfectly informed agent's information set is described by* (15)*. In addition, Assumption 2 holds, which means that the forecast errors of the forward-looking variables are a linear combination of fundamental shocks and belief shocks, with time-invariant loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, as in* (A-48) *and all shocks are joint normally distributed.*

We now show that in such an equilibrium the policymaker's conditional expectations can be represented by a Kalman filter. The policymaker observes the measurement equation (15), and the

[15] In a linear equilibrium, where (A-48) holds, the belief shocks affect outcomes only via the product $\boldsymbol{\Gamma}_b \, \boldsymbol{b}_t$. Thus, without loss of generality, $\boldsymbol{b}_t$ can be normalized to have a variance-covariance matrix equal to the identity matrix.

[16] Linear equilibria considered in this paper are driven by normally distributed shocks, leading to normally distributed outcomes, such that covariance stationarity also implies strict stationarity. Hence, we will not distinguish between both concepts and merely refer to stationarity.

state equation of the filtering problem is given by:

$$\boldsymbol{S}_{t+1} + \hat{\boldsymbol{J}}\boldsymbol{S}_{t+1|t} = \boldsymbol{A}\boldsymbol{S}_t + \hat{\boldsymbol{A}}\boldsymbol{S}_{t|t} + \boldsymbol{A}_i\, \boldsymbol{i}_t + \boldsymbol{B}\boldsymbol{w}_{t+1}\,, \qquad \text{with} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \\ \boldsymbol{\Gamma}_\varepsilon & \boldsymbol{\Gamma}_b \end{bmatrix}, \tag{A-49}$$

which combines the expectational difference equation (12) with the implications of Assumption 2 for the endogenous forecast errors. The appearance of projections $\boldsymbol{S}_{t+1|t}$ and $\boldsymbol{S}_{t|t}$ in (A-49) lends this state equation a slightly non-standard format. However, when expressed in terms of innovations, the filtering problem can be cast in the canonical "ABCD" form, studied, among others, by Fernández-Villaverde, Rubio-Ramírez, Sargent, and Watson (2007):[17]

$$\tilde{\boldsymbol{S}}_{t+1} = \boldsymbol{A}\left(\tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}\right) + \boldsymbol{B}\boldsymbol{w}_{t+1}\,, \tag{A-50}$$

$$\tilde{\boldsymbol{Z}}_{t+1} = \boldsymbol{C}\left(\tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}\right) + \boldsymbol{D}\boldsymbol{w}_{t+1}\,, \tag{A-51}$$

$$\text{with} \quad \boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}\,, \quad \boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}\,, \quad \text{since} \quad \tilde{\boldsymbol{Z}}_{t+1} = \boldsymbol{H}\tilde{\boldsymbol{S}}_{t+1}\,. \tag{A-52}$$

To ensure a well-behaved filtering problem, we impose the following assumption on the shocks to the policymaker's measurement vector $\boldsymbol{D}\boldsymbol{w}_t = \boldsymbol{Z}_t - E_{t-1}\boldsymbol{Z}_t$.

ASSUMPTION 3 (Non-degenerate shocks to the signal equation) *The variance-covariance matrix of the shocks to the measurement equation has full rank:* $|\boldsymbol{D}\boldsymbol{D}'| \neq 0$.

A necessary condition for Assumption 3 to hold is that the signal vector does not have more elements than the sum of endogenous and exogenous forecast errors: $N_z \leq N_\varepsilon + N_y \leq N_x + N_y$.

Existence of a steady-state Kalman filter requires certain conditions on $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ to hold, known in the literature as "observability" and "unit-circle controllability," formally defined in the appendix. As shown in (A-49), $\boldsymbol{B}$ depends on the yet to be determined shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$, which are endogenous in our framework. To ensure the existence of a steady-state filter, the following assumption restricts the set of potential equilibria that we consider. But, as discussed further below, these conditions impose only weak restrictions on the shock loadings in $\boldsymbol{B}$.

ASSUMPTION 4 (Sufficient condition for existence of a steady-state Kalman filter) *The equilibrium shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_t$ are such that $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ are detectable and unit-circle controllable as stated in Definition 5 of Appendix B.4.*

A central component of our framework is that the less informed agent observes data imperfectly, but employs optimal signal extraction, which is provided by the Kalman filter in this linear Gaussian

[17] The innovations form is obtained by projecting both sides of (A-49) onto $\boldsymbol{Z}^t$ and subtracting these projections from (A-49). Note that the control $\boldsymbol{i}_t$ is always in the policymaker's information set. The innovations form given by (A-50) and (A-51) is identical to the innovations form of a state-space system with $\boldsymbol{S}_{t+1} = \boldsymbol{A}\boldsymbol{S}_t + \boldsymbol{B}\boldsymbol{w}_{t+1}$ in place of (A-49) while maintaining (15) as measurement equation, as noted also by Baxter, Graham, and Wright (2011).

environment. The optimal projections $\boldsymbol{S}_{t|t}$ can be decomposed into the sum of previous period's forecasts $\boldsymbol{S}_{t|t-1}$ and an update that is linear in the innovations in measurement vector. When a steady-state filter exists, a constant Kalman gain relates the projected innovations in the state vector, $\tilde{\boldsymbol{S}}_{t|t}$, to innovations in the measurement vector, $\tilde{\boldsymbol{Z}}_t$:

$$\boldsymbol{S}_{t|t} = \boldsymbol{S}_{t|t-1} + \tilde{\boldsymbol{S}}_{t|t} \quad \text{with} \quad \tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t\,, \quad \text{and} \quad \boldsymbol{K} = \mathrm{Cov}\left(\tilde{\boldsymbol{S}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\mathrm{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1}. \tag{A-53}$$

The Kalman gain matrix, $\boldsymbol{K}$, is given by the solution of a standard Riccati equation involving $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{H}$. We detail the derivation in the appendix.

We focus on the case when a steady-state Kalman filter exists, which allows us to represent the policymaker's conditional expectations as a recursive linear system with time-invariant coefficients. We formally state these results in the following two propositions.

PROPOSITION 2 (Existence of steady-state Kalman filter) *When Assumptions 3 and 4 hold in a linear, stationary, and time-invariant equilibrium, a steady-state Kalman filter exists that describes the projection of innovations in the state vector,* $\tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t$ *with a constant Kalman gain* $\boldsymbol{K}$ *as in* (A-53)*. The variance-covariance matrix of projection residuals is constant,* $\mathrm{Var}\left(\boldsymbol{S}_t|\boldsymbol{Z}^t\right) = \mathrm{Var}\left(\boldsymbol{S}_t^*\right) = \Sigma^*$*. Existence of a steady-state Kalman filter ensures that innovations* $\tilde{\boldsymbol{S}}_t = \boldsymbol{S}_t - \boldsymbol{S}_{t|t-1}$ *and residuals* $\boldsymbol{S}_t^* = \boldsymbol{S}_t - \boldsymbol{S}_{t|t}$ *are stationary as are innovations to the measurement equation,* $\tilde{\boldsymbol{Z}}_t$*.*

Proof. See Theorem 2 of Appendix B.4. The stationarity of $\tilde{\boldsymbol{Z}}_t = \boldsymbol{H}\tilde{\boldsymbol{S}}_t$ follows then from the stationarity of $\tilde{\boldsymbol{S}}_t$. ■

Given the existence of a Kalman filter, equation (A-53) describes the optimal update from forecasts, $\boldsymbol{S}_{t|t-1}$, to current projections, $\boldsymbol{S}_{t|t}$. What remains to be characterized is the transition equation from projections $\boldsymbol{S}_{t|t}$ to forecasts, $\boldsymbol{S}_{t+1|t}$. This relationship is restricted by the linear difference equations (12) and the rule (13) for the policy instrument $\boldsymbol{i}_t$. Since the instrument can depend on its own lagged value via $\boldsymbol{\Phi}_i \neq \boldsymbol{0}$ in (13), we construct the transition from $\boldsymbol{S}_{t|t}$ to $\boldsymbol{S}_{t+1|t}$ based on the vector $\boldsymbol{\mathcal{S}}_t$, which includes $\boldsymbol{S}_t$ and $\boldsymbol{i}_{t-1}$, as defined in (16).

We derive the transition equation from projections to forecasts by noting that the structure of the dynamic system conditioned down onto the policymaker's information takes a familiar form. That is, conditioning down (12) and (13) onto $\boldsymbol{Z}^t$ yields a system of expectational linear difference equations in $\boldsymbol{\mathcal{S}}_t$ that is akin to the full-information system shown in (17), except for the use of policymaker projections in lieu of full-information expectations:

$$\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{S}}_{t|t} \tag{A-54}$$

with $\boldsymbol{\mathcal{J}}$, and $\boldsymbol{\mathcal{A}}$ as defined in (17) above. In a stationary equilibrium consistent with Definition 1, $\boldsymbol{\mathcal{S}}_t$ is stationary, and so is its projection $\boldsymbol{\mathcal{S}}_{t|t}$. When the equilibrium is linear and time-invariant, the projections $\boldsymbol{\mathcal{S}}_{t|t}$ must follow

$$\boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{T}}\,\boldsymbol{\mathcal{S}}_{t|t}\ , \qquad \text{with} \quad (\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{T}} - \boldsymbol{\mathcal{A}})\,\boldsymbol{\mathcal{S}}_{t|t} = 0 \qquad \text{for some stable matrix } \boldsymbol{\mathcal{T}}. \tag{A-55}$$

We can already state another key result in this section. In a linear equilibrium with normally distributed shocks and with a given linear transformation from projections $\boldsymbol{\mathcal{S}}_{t|t}$ into forecasts $\boldsymbol{\mathcal{S}}_{t+1|t}$, the Kalman filter represents conditional expectations of $\boldsymbol{\mathcal{S}}_t$ (and thus also $\boldsymbol{S}_t$). We establish below that in a stationary and linear imperfect information equilibrium that conforms to Definition 1 $\boldsymbol{\mathcal{T}}$ must be identical to its full-information counterpart $\bar{\boldsymbol{\mathcal{T}}}$ as described in equation (19).

PROPOSITION 3 (Kalman filter represents conditional expectations)
In a linear, time-invariant stationary equilibrium, when the conditions for Proposition 2 hold, and for a given stable transition matrix $\boldsymbol{\mathcal{T}}$ between policymaker projections and forecasts as in (A-55)*, the steady-state Kalman filter represents conditional expectations $\boldsymbol{\mathcal{S}}_{t|t} = E\left(\boldsymbol{\mathcal{S}}_t|\boldsymbol{Z}^t\right)$. For a given sequence of innovations in the measurement vector, $\tilde{\boldsymbol{Z}}_t$, the Kalman filter implies a stationary evolution of projections:*

$$\begin{aligned} \boldsymbol{\mathcal{S}}_{t+1|t+1} &= \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{S}}_{t|t} + \boldsymbol{\mathcal{K}}\tilde{\boldsymbol{Z}}_{t+1}\ , \\ &\textit{with} \quad \boldsymbol{\mathcal{K}} = \operatorname{Cov}\left(\tilde{\boldsymbol{\mathcal{S}}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\operatorname{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1} = \begin{bmatrix} 0' & \boldsymbol{K}_x' & \boldsymbol{K}_y' & \boldsymbol{K}_i' \end{bmatrix}'. \end{aligned} \tag{A-56}$$

Proof. In a linear, time-invariant stationary equilibrium, shocks are jointly normal and propagate linearly so that the sequences of $\boldsymbol{\mathcal{S}}_t$ and $\boldsymbol{Z}_t$ are joint normally distributed, and conditional expectations are identical to mean-squared-error optimal linear projections. By the law of iterated projections, we can decompose $E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^{t+1}\right) = E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^t\right) + E\left(\tilde{\boldsymbol{\mathcal{S}}}_{t+1}|\tilde{\boldsymbol{Z}}_{t+1}\right)$ and, based on (A-55), we have $E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^t\right) = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{S}}_{t|t}$. From Proposition 2 follows $E\left(\tilde{\boldsymbol{\mathcal{S}}}_{t+1}|\tilde{\boldsymbol{Z}}_{t+1}\right) = \boldsymbol{\mathcal{K}}\tilde{\boldsymbol{Z}}_{t+1}$. $\boldsymbol{K}_x$ and $\boldsymbol{K}_y$ are appropriate partitions of $\boldsymbol{K}$ as defined in (A-53) and $\boldsymbol{K}_i = \boldsymbol{\mathcal{G}}_{ix}\boldsymbol{K}_x$. The upper block of $\boldsymbol{\mathcal{K}}$, corresponding to the gain coefficients for the lagged policy instrument, is zero since $\boldsymbol{i}_{t-1} = \boldsymbol{i}_{t-1|t-1}$ $\Rightarrow \tilde{\boldsymbol{i}}_{t-1|t} = \boldsymbol{i}_{t-1|t} - \boldsymbol{i}_{t-1|t-1} = 0$. Projections are stationary since $\boldsymbol{\mathcal{T}}$ is stable. ■

So far, we have characterized $\boldsymbol{\mathcal{T}}$ merely as a stable matrix. We show now that in a stationary and linear imperfect information equilibrium that conforms to Definition 1, the transition matrix $\boldsymbol{\mathcal{T}}$ must be identical to its full-information counterpart $\bar{\boldsymbol{\mathcal{T}}}$ as described in equation (19). Specifically, we begin by defining a set of equilibrium conditions labelled a "projection condition" that imply $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$. We then proceed to establish that the projection condition needs to hold for any equilibrium that conforms to Definition 1.

DEFINITION 2 (Projection Condition) *The projection condition restricts the mapping between projected backward- and forward-looking variables to be identical to the full-information case:*

$$\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}\,, \qquad\qquad \text{and} \quad \boldsymbol{\mathcal{X}}_{t+1|t} = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{X}}_{t|t}\,. \tag{A-57}$$

where $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{P}}$ are the unique solution coefficients in the corresponding full-information case.

The projection condition is an equilibrium condition that imposes a linear mapping between projections of backward- and forward-looking variables.[18] Moreover, the projection condition is required to hold in any stationary equilibrium that is also linear and time-invariant.

PROPOSITION 4 (Projection Condition holds when full-information case is determinate) *If Assumption 1 (uniqueness of full-information equilibrium) holds, the projection condition must hold in any stationary linear and time-invariant imperfect information equilibrium that conforms to Definition 1. It follows that $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$.*

Proof. As noted before, conditioning down (12) and (13) onto $\boldsymbol{Z}^t$ yields a system of expectational linear difference equations that is akin to the full-information system shown in (17), except for the use of policymaker projections in lieu of full-information expectations: $\boldsymbol{\mathcal{J}} E_t \boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{A}} \boldsymbol{\mathcal{S}}_{t|t}$. Since $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$ are identical to the full-information case, we obtain the same complex generalized Schur decomposition as in the full-information case of equations (A-44) and (A-45), with the same partitioning into stable and unstable canonical variables (except for expressing these in terms of projections, $\boldsymbol{\mathcal{X}}^*_{t|t}$ and $\boldsymbol{\mathcal{Y}}^*_{t|t}$). As argued above, the transition equation for projections must be stable to ensure a stationary equilibrium. In terms of the canonical variables, this requires $\boldsymbol{\mathcal{Y}}^*_{t|t} = \boldsymbol{0}$ and following steps analogous to the full-information case as in (A-46) – (A-47) we obtain the projection condition. Given the definition of $\bar{\boldsymbol{\mathcal{T}}}$ in (19), it follows that $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$ when the projection condition holds. ■

Moreover, the projection condition imposes a *second-moment* restriction on the joint distribution of the innovations $\tilde{\boldsymbol{X}}_t$, $\tilde{\boldsymbol{Y}}_t$.[19] As a second-moment restriction, the projection condition only restricts co-movements of the innovations *on average*, but not for any particular realization of $\tilde{\boldsymbol{X}}_t$ and $\tilde{\boldsymbol{Y}}_t$. The projection condition (20) implies the following restriction between the Kalman gains of forward-

[18] In a setup similar to ours, Svensson and Woodford (2004) also embed a set of equations akin to the projection condition, based on an appeal to certainty equivalence. In contrast to our paper, they restrict the number of state variables in a way that disregards a role for belief shocks, and assume equilibrium uniqueness. Applications that build on Svensson and Woodford (2004) are, for example, Dotsey and Hornstein (2003), Aoki (2006), Nimark (2008), Carboni and Ellison (2011).

[19] In addition to $\boldsymbol{Y}_t$ and $\boldsymbol{X}_t$, $\boldsymbol{\mathcal{Y}}_t$ and $\boldsymbol{\mathcal{X}}_t$ also contain the current and lagged policy instrument, respectively. However, the projection condition does not impose a direct restriction on innovations in the policy instrument since $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$ and thus $\tilde{\boldsymbol{i}}_{t-1|t} = \boldsymbol{i}_{t-1|t} - \boldsymbol{i}_{t-1|t-1} = 0$.

and backward-looking variables:

$$\boldsymbol{Y}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{X}_{t|t} + \boldsymbol{\mathcal{G}}_{yi}\ \boldsymbol{i}_{t-1|t} \quad \Rightarrow \tilde{\boldsymbol{Y}}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\ \tilde{\boldsymbol{X}}_{t|t} \quad \Leftrightarrow \quad \left(\boldsymbol{K}_y - \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{K}_x\right)\tilde{\boldsymbol{Z}}_t = \boldsymbol{0}\,, \tag{A-58}$$

where $\boldsymbol{K}_y$ and $\boldsymbol{K}_x$ denote the corresponding partitions of the Kalman gain, $\boldsymbol{K}$, defined in (A-53). Since equation (A-58) must hold for every $\tilde{\boldsymbol{Z}}_t$, the projection condition therefore implies a restriction on the Kalman gains, which we summarize in the following proposition.

PROPOSITION 5 (Projection Condition for Kalman gains) *The projection condition* (20) *holds only if the Kalman gains satisfy* $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{K}_x$.

Proof. As noted in (A-58), a necessary condition for the projection to hold is $\left(\boldsymbol{K}_y - \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{K}_x\right)\tilde{\boldsymbol{Z}}_t = 0$ for all realizations of $\tilde{\boldsymbol{Z}}_t$. $\tilde{\boldsymbol{Z}}_t$ has a joint normal distribution and Assumption 3 implies that $\mathrm{Var}\left(\tilde{\boldsymbol{Z}}_t\right) = \boldsymbol{C}\,\mathrm{Var}\left(\boldsymbol{S}_t^*\right)\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'$ is strictly positive definite, so that the distribution of $\tilde{\boldsymbol{Z}}_t$ is non-degenerate. Thus, for (A-58) to hold we must have $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{K}_x$. Since

$$\boldsymbol{K}_y = \mathrm{Cov}\left(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t\right)\mathrm{Var}\left(\tilde{\boldsymbol{Z}}_t\right)^{-1},\ \text{and}\ \boldsymbol{K}_x = \mathrm{Cov}\left(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t\right)\mathrm{Var}\left(\tilde{\boldsymbol{Z}}_t\right)^{-1},$$

the projection condition equivalently requires $\mathrm{Cov}\left(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t\right) = \boldsymbol{\mathcal{G}}_{yx}\ \mathrm{Cov}\left(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t\right)$. ■

Intuitively, the Kalman gains are multivariate regression slopes and, as shown in Proposition 5, the projection condition imposes a linear restriction on covariances between $\tilde{\boldsymbol{Y}}_t$ and $\tilde{\boldsymbol{X}}_t$. It is this feature of our framework that moves the solution of the underlying linear rational expectations model out of certainty equivalence and poses an intricate fixed-point problem with a highly non-linear solution: The Kalman gains depend on the second moments of the solution, which in turn depends on the Kalman gains via the projection condition.

We are now in a position to derive a representation of the system of expectational difference equations (12) - (14), where the expectations of the fully informed agent are captured by the concept of endogenous forecast errors known from Definition 2 and the expectation formation of the less-informed agent is represented by the Kalman filter. As summarized in the following theorem, equilibrium dynamics then follow a state vector that tracks both projections and actual values of backward- and forward-looking variables as well as the policy instrument.[20]

THEOREM 1 (Difference System Under Imperfect Information) *Consider the system of difference equations* (12) *and* (13) *with a measurement vector that is linear in* $\boldsymbol{S}_t$ *as defined in* (15)*.*

[20] In fact, the joint vector of $\boldsymbol{\mathcal{S}}_t$ and $\boldsymbol{\mathcal{S}}_{t|t}$ does not need to tracked in its entirety. First, $\boldsymbol{\mathcal{S}}_t$ already includes the policy instrument $\boldsymbol{i}_t$, which lies in the space of central bank projections. The state of the economy can thus be described by $\boldsymbol{\mathcal{S}}_{t|t}$ and $\boldsymbol{S}_t$, which differs from $\boldsymbol{\mathcal{S}}_t$ in omitting $\boldsymbol{i}_t$. Second, the state of the economy is equivalently described by $\boldsymbol{S}_t^* = \boldsymbol{S}_t - \boldsymbol{S}_{t|t}$ and $\boldsymbol{\mathcal{S}}_{t|t}$. Third, when the projection condition (20) is satisfied, we need only track $\boldsymbol{\mathcal{X}}_{t|t}$ rather than $\boldsymbol{\mathcal{S}}_{t|t}' = \begin{bmatrix}\boldsymbol{\mathcal{X}}_{t|t}' & \boldsymbol{\mathcal{Y}}_{t|t}'\end{bmatrix}$, since $\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}$.

Let Assumptions 1, 2, 3, and 4 hold and consider stationary, linear, time-invariant equilibria that satisfy the projection condition, as stated in Definitions 1 and 2. Equilibrium dynamics are then characterized by the evolution of the following vector system:

$$\overline{\boldsymbol{\mathcal{S}}}_{t+1} \equiv \begin{bmatrix} \boldsymbol{S}^*_{t+1} \\ \boldsymbol{\mathcal{X}}_{t+1|t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} (\boldsymbol{A}-\boldsymbol{K}\boldsymbol{C}) & 0 \\ \boldsymbol{\mathcal{K}}_x\boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{P}} \end{bmatrix}}_{\overline{\boldsymbol{\mathcal{A}}}} \overline{\boldsymbol{\mathcal{S}}}_t + \begin{bmatrix} (\boldsymbol{I}-\boldsymbol{K}\boldsymbol{H}) \\ \boldsymbol{\mathcal{K}}_x\boldsymbol{H} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{\eta}_{t+1} \end{bmatrix},$$

where $\boldsymbol{\mathcal{K}}'_x = \begin{bmatrix} 0 & \boldsymbol{K}'_x \end{bmatrix}$, $\boldsymbol{\mathcal{C}}$ *as defined in* (A-52)*, and* $\boldsymbol{\mathcal{P}}$ *known from the unique full-information solution.*

Proof. $\boldsymbol{S}_t$ can be decomposed into $\boldsymbol{S}_t = \boldsymbol{S}^*_t + \boldsymbol{S}_{t|t}$. We need to show that $\boldsymbol{S}_{t|t}$ can be constructed from $\overline{\boldsymbol{\mathcal{S}}}_t$, and that the policy instrument $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$ can be constructed from the proposed state vector $\overline{\boldsymbol{\mathcal{S}}}_t$. Recalling the definitions of $\boldsymbol{S}_t$, $\boldsymbol{\mathcal{X}}_t$ and $\boldsymbol{\mathcal{Y}}_t$ in (14), and (16), the projection condition then implies that:

$$\boldsymbol{S}_{t|t} = \begin{bmatrix} \boldsymbol{X}_{t|t} \\ \boldsymbol{Y}_{t|t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \boldsymbol{I} \\ \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \end{bmatrix}}_{\boldsymbol{\mathcal{G}}_S} \boldsymbol{\mathcal{X}}_{t|t}, \qquad \boldsymbol{i}_t = \underbrace{\begin{bmatrix} \boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix}}_{\boldsymbol{\mathcal{G}}_i} \boldsymbol{\mathcal{X}}_{t|t}, \qquad \text{with} \quad \boldsymbol{\mathcal{X}}_{t|t} = \begin{bmatrix} \boldsymbol{i}_{t-1} \\ \boldsymbol{X}_{t|t} \end{bmatrix},$$

where block matrices are partitioned along the lines of $\boldsymbol{\mathcal{X}}_{t|t}$ above. The various coefficient matrices "$\boldsymbol{\mathcal{G}}_{..}$" are known from the full-information solution given in (18). The dynamics of $\boldsymbol{S}^*_{t+1}$, as captured by the top rows of $\overline{\boldsymbol{\mathcal{S}}}$, follow from the innovation state space (A-50) - (A-51) and the steady-state Kalman filter described in Appendix B.4. The dynamics of $\boldsymbol{\mathcal{X}}_{t+1|t+1}$, as captured by the bottom rows of $\overline{\boldsymbol{\mathcal{S}}}$, follow from (A-56) together with the projection condition (20) and the dynamics of $\tilde{\boldsymbol{Z}}_t$ given in (A-51). ■

Equation (21) forms the basis for solving a rational expectations model with heterogeneous information sets. Based on this representation we can derive key insights into the nature of the equilibria in this setting. The state vector $\overline{\boldsymbol{\mathcal{S}}}_t$ follows a first-order linear difference system given in (21). The stability of the system depends on the eigenvalues of its transition matrix $\overline{\boldsymbol{\mathcal{A}}}$. The transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ depends on the Kalman gain $\boldsymbol{K}$, which depends on the yet to be determined shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_t$. Nevertheless, existence of a steady-state Kalman filter allows us to conclude that $\overline{\boldsymbol{\mathcal{A}}}$, is stable.

COROLLARY 1 (Stable Transition Matrix) *Provided that a steady-state Kalman filter exists, the transition matrix* $\overline{\boldsymbol{\mathcal{A}}}$ *in* (21) *is stable. The eigenvalues of* $\overline{\boldsymbol{\mathcal{A}}}$ *are given by the eigenvalues of* $\boldsymbol{\mathcal{P}}$*, which is stable and known from the full-information solution* (18)*, and* $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$*, whose stability is assured by the existence of the steady-state Kalman filter.*

Proof. The stability of $\mathcal{P}$ follows from Assumption 1 and the resulting solution of the full-information case. The stability of $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ follows from Theorem 2 in Appendix B.4. ■

One key implication of Proposition 1 is that the usual root-counting arguments, as in Blanchard and Kahn (1980) or generalized in Sims (2002), do not pin down the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_{t+1}$ in (21). This follows from the fact that $\overline{\boldsymbol{\mathcal{A}}}$ is a stable matrix for any choice of $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ consistent with the existence of a steady-state Kalman filter. Moreover, the projection condition does typically not place sufficiently many restrictions on $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ to uniquely identify the shock loadings. This observation is the key result of our paper, namely that equilibria are indeterminate.

PROPOSITION 6 (Indeterminacy) *With $\overline{\boldsymbol{\mathcal{A}}}$ stable, the endogenous forecast errors are only restricted by the projection condition given in Definition 2. The shock loadings of the endogenous forecast errors, $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, have $N_y \times (N_\varepsilon + N_y)$ unknown conditions. The projection condition stated in* (A-58) *imposes only $N_y \times N_z$ restrictions. However, a necessary condition for Assumption 3 to hold is $N_z \leq N_\varepsilon + N_y$. As a result, the projection condition cannot uniquely identify the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$.*

To summarize, in this section we derive two key insights for our linear rational expectations framework with heterogeneous information sets. First, we show that under Gaussian shocks a Kalman filter exists, which is the optimal filter in this environment. The filter is available to the less-informed agent to gain information about realizations of variables that are available to the fully informed agent. Second, we show that the Kalman filter represents the conditional expectations of the less informed agent. This insight allows us to construct a solution approach for this framework which deviates from standard methods in that certainty equivalence no longer holds. Specifically, expectation formation of the less informed agents now depends on second moments embedded in the steady-state Kalman filter. Moreover, it is the intersection of the different expectation formation processes under rational expectations that lies at the heart of indeterminacy in this framework.

B.3 Determination of the Endogenous Forecast Errors

In the full-information literature, restrictions for the endogenous forecast errors $\boldsymbol{\eta}_t$ emanate from explosive roots in the dynamic system, as described by Sims (2002) and Lubik and Schorfheide (2003). In our imperfect information case, further restrictions result from the projection condition stated in equation 20. As noted in Corollary 1 above, the transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ is always stable in a time-invariant equilibrium with a steady-state Kalman filter. Restrictions on $\boldsymbol{\eta}_t$ can therefore only result from the projection condition. At the same time, the projection condition does generally not provide sufficiently many restrictions to pin down $\boldsymbol{\eta}_t$ uniquely, as discussed in Proposition 6.

Determination of shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$ for the endogenous forecast errors that are consistent with the projection condition poses an intricate fixed problem between shock loadings and Kalman gains. As noted already by Sargent (1991), the Kalman gains are endogenous equilibrium objects when the observable signals reflect information contained in endogenous variables. In contrast, as discussed in an earlier working paper version Lubik, Matthes, and Mertens (2019), the Kalman filtering problem can be solved independently of the equilibrium dynamics of the system when the signal vector consists only of exogenous variables.[21]

Equation (15) describes the central bank's measurement vector generically as a linear combination of backward- and forward-looking variables, $\boldsymbol{Z}_t = \boldsymbol{H}_x\boldsymbol{X}_t + \boldsymbol{H}_y\boldsymbol{Y}_t$. To simplify some of the algebra, we limit ourselves to signal vectors that have the same length as the vector of forward-looking variables ($\boldsymbol{Y}_t$) and that have no rank-deficient loading on $\boldsymbol{Y}_t$, so that $\boldsymbol{H}_y$ is square and invertible, and can be normalized to the identity matrix:[22]

$$\boldsymbol{Z}_t = \boldsymbol{H}_x\boldsymbol{X}_t + \boldsymbol{Y}_t \qquad \text{and thus} \qquad \boldsymbol{H} = \begin{bmatrix}\boldsymbol{H}_x & \boldsymbol{I}\end{bmatrix}. \tag{A-59}$$

This setup also includes the case where each forward-looking variable is observed with error, as in $\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t$, where $\boldsymbol{\nu}_t$ is an exogenous measurement error to be included among the set of backward-looking variables in $\boldsymbol{X}_t$.

We pursue a numerical the solution for the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ that are consistent with the projection condition. Our procedure combines conventional techniques for solving linear RE models with standard algorithms for solving the Kalman filter's Riccati equation, while ensuring consistency with the projection condition. The algorithm searches for shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ that satisfy the projection condition for a Kalman filter that is consistent with equilibrium outcomes of all variables.

From the measurement equation (A-59), we can deduce that $\boldsymbol{Y}_t^* = -\boldsymbol{H}_x\boldsymbol{X}_t^*$, which allows to simplify the innovation system given by (A-50) and (A-51) as follows: $\tilde{\boldsymbol{X}}_{t+1} = \tilde{\boldsymbol{A}}\boldsymbol{X}_t^* + \tilde{\boldsymbol{B}}\boldsymbol{w}_{t+1}$, and $\tilde{\boldsymbol{Z}}_{t+1} = \tilde{\boldsymbol{C}}\boldsymbol{X}_t^* + \tilde{\boldsymbol{D}}\boldsymbol{w}_{t+1}$ with $\tilde{\boldsymbol{A}} = \boldsymbol{A}_{xx} - \boldsymbol{A}_{xy}\boldsymbol{H}_x$, $\tilde{\boldsymbol{B}} = \begin{bmatrix}\boldsymbol{B}_{x\varepsilon} & 0\end{bmatrix}$, $\tilde{\boldsymbol{C}} = \boldsymbol{H}_x\ (\boldsymbol{A}_{xx} - \boldsymbol{A}_{xy}\boldsymbol{H}_x)+$

[21]The case of a purely exogenous signal arises when $\boldsymbol{H}_y = \boldsymbol{0}$ in (15) and $\boldsymbol{X}_t$ exogenous (i.e. without dependence on lagged forward-looking variables). The working paper version of this article contains further discussion and a general analytical characterization of the solution for this case. In Lubik, Matthes, and Mertens (2019), we derive two general results for the exogenous-signal case: First, the projection condition does not restrict the belief shock loadings of the endogenous forecast errors, $\boldsymbol{\Gamma}_b$, when the signal is exogenous. Second, we derive an analytical expression for the restrictions on the loadings of the endogenous forecast errors on fundamental shocks (including the measurement errors) that result from the projection condition. However, the exogenous-signal case is arguably less realistic since variables relevant to the policymaker are typical endogenous.

[22]Consider the case of a signal $\hat{\boldsymbol{Z}}_t = \hat{\boldsymbol{H}}_x\boldsymbol{X}_t + \hat{\boldsymbol{H}}_y\boldsymbol{Y}_t$, where $\hat{\boldsymbol{H}}_y$ is square and invertible. The information content provided by $\hat{\boldsymbol{Z}}_t$ is equivalent to what is spanned by $\boldsymbol{Z}_t = \hat{\boldsymbol{H}}_y^{-1}\hat{\boldsymbol{Z}}_t$ with $\boldsymbol{H}_x = \hat{\boldsymbol{H}}_y^{-1}\hat{\boldsymbol{H}}_x$.

$\boldsymbol{A}_{yx} - \boldsymbol{A}_{yy}\,\boldsymbol{H}_x$,[23]

$$\text{and} \quad \tilde{\boldsymbol{D}} = \begin{bmatrix} (\boldsymbol{H}_x \boldsymbol{B}_{x\varepsilon} + \boldsymbol{\Gamma}_\varepsilon) & \boldsymbol{\Gamma}_b \end{bmatrix} . \tag{A-60}$$

For a given $\tilde{\boldsymbol{D}}$, which embodies a guess of $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, the Kalman-filtering solution to this system generates a Kalman gain $\boldsymbol{K}_x$, which can be used to form projections $\tilde{\boldsymbol{X}}_{t|t} = \boldsymbol{K}_x \tilde{\boldsymbol{Z}}_t$. The algorithm checks whether this guess for $\tilde{\boldsymbol{D}}$ also satisfies the projection condition, which requires $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{K}_x$. Together with the projection condition, the measurement equation (A-59) implies $\boldsymbol{I} = \boldsymbol{H}_x\boldsymbol{K}_x + \boldsymbol{K}_y = (\boldsymbol{H}_x + \boldsymbol{\mathcal{G}}_{yx})\,\boldsymbol{K}_x$. We employ a numerical solver that searches for a $\tilde{\boldsymbol{D}}$ that generates a Kalman gain $\boldsymbol{K}_x$ such that $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$, where $\boldsymbol{L} = \boldsymbol{H}_x + \boldsymbol{\mathcal{G}}_{yx}$. Given a solution for $\tilde{\boldsymbol{D}}$ that satisfies the projection condition $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$, we can then back out $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ based on (A-60).

B.4 The Steady State Kalman Filter

This section describes details of the steady-state Kalman filter for the innovations state space (A-50) and (A-51) when Assumption 3 holds. Existence of a steady-state Kalman filter relies on finding an ergodic distribution for $\boldsymbol{S}^*_t$ (and thus $\tilde{\boldsymbol{S}}_t$) with constant second moments $\boldsymbol{\Sigma} \equiv \mathrm{Var}\,(\boldsymbol{S}^*_t)$. When a steady-state filter exists, a constant Kalman gain, $\boldsymbol{K}$ relates projected innovations of $\tilde{\boldsymbol{S}}_t$ to innovations in the signal, $\tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t$ with:[24]

$$\boldsymbol{K} = \mathrm{Cov}\left(\tilde{\boldsymbol{S}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\mathrm{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1} = \left(\boldsymbol{A}\Sigma\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)\left(\boldsymbol{C}\Sigma\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'\right)^{-1} .$$

The dynamics of $\boldsymbol{S}^*_t$ are then characterized by

$$\boldsymbol{S}^*_{t+1} = (\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})\;\boldsymbol{S}^*_t + (\boldsymbol{B} - \boldsymbol{K}\boldsymbol{D})\,\boldsymbol{w}_{t+1} \tag{A-61}$$

Existence of a steady-state filter depends on finding a symmetric, positive (semi) definite solution $\boldsymbol{\Sigma}$ to the following Riccati equation:

$$\begin{aligned} \Sigma &= (\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})\Sigma(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})' + \boldsymbol{B}\boldsymbol{B}' \\ &= \boldsymbol{A}\Sigma\boldsymbol{A}' + \boldsymbol{B}\boldsymbol{B}' - \left(\boldsymbol{A}\Sigma\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)\left(\boldsymbol{C}\Sigma\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'\right)^{-1}\left(\boldsymbol{A}\Sigma\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)' . \end{aligned} \tag{A-62}$$

Intuitively, the Kalman filter seeks to construct mean-squared error optimal projections $\boldsymbol{S}_{t|t}$ that minimize Σ. A necessary condition for the existence of a solution to this minimization problem is the ability to find at least some gain $\hat{\boldsymbol{K}}$ for which $\boldsymbol{A} - \hat{\boldsymbol{K}}\boldsymbol{C}$ is stable; otherwise, $\boldsymbol{S}^*$ would

[23] $\boldsymbol{A}_{xx}$, $\boldsymbol{A}_{xy}$, etc. denote suitable sub-matrices of $\boldsymbol{A}$.

[24] See also (A-53) in the paper.

have unstable dynamics as can be seen from (A-61). Thus, existence of the second moment for the residuals, $\text{Var}\left(\boldsymbol{S}_t^*\right) = \boldsymbol{\Sigma} \geq \boldsymbol{0}$, is synonymous with a stable transition matrix $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$.

Formal conditions for the existence of a time-invariant Kalman filter have been stated, among others, by Anderson and Moore (1979), Anderson, McGrattan, Hansen, and Sargent (1996), Kailath, Sayed, and Hassibi (2000), and Hansen and Sargent (2007). Necessary and sufficient conditions for the existence of a unique and stabilizing solution that is also positive semi-definite depend on the "detectability" and "unit-circle controllability" of certain matrices in our state space. We restate those concepts next.

DEFINITION 3 (Detectability) *A pair of matrices $(\boldsymbol{A}, \boldsymbol{C})$ is detectable when no right eigenvector of $\boldsymbol{A}$ that is associated with an unstable eigenvalue is orthogonal to the row space of $\boldsymbol{C}$. That is, there is no non-zero column vector $\boldsymbol{v}$ such that $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{v}\lambda$ and $|\lambda| \geq 1$ with $\boldsymbol{C}\boldsymbol{v} = \boldsymbol{0}$.*

Detectability alone is already sufficient for the existence of *some* solution to the Riccati equation such that $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ is stable; see Kailath, Sayed, and Hassibi (2000, Table E.1). Evidently, detectability is assured when $\boldsymbol{A}$ is a stable matrix, regardless of $\boldsymbol{C}$. To gain further intuition for the role of detectability, consider transforming $\boldsymbol{S}_t$ into "canonical variables" by premultiplying $\boldsymbol{S}_t$ with the matrix of eigenvectors of $\boldsymbol{A}$ — this transformation into canonical variables is at the heart of procedures for solving rational expectations models known from Blanchard and Kahn (1980), King and Watson (1998), Klein (2000), Sims (2002). Detectability then requires the signal equation (A-51) to provide some signal (i.e. to have non-zero loadings) for any unstable canonical variables.[25]

To establish existence of a solution to the Riccati equation that is unique and positive semi-definite, we follow Kailath, Sayed, and Hassibi (2000) and require unit-circle controllability, defined as follows.

DEFINITION 4 (Unit-circle controllability) *The pair $(\boldsymbol{A}, \boldsymbol{B})$ is unit-circle controllable when no left-eigenvector of $\boldsymbol{A}$ associated with an eigenvalue on the unit circle is orthogonal to the column space of $\boldsymbol{B}$. That is, there is no non-zero row vector $\boldsymbol{v}$ such that $\boldsymbol{v}\boldsymbol{A} = \boldsymbol{v}\lambda$ with $|\lambda| = 1$ and $\boldsymbol{v}\boldsymbol{B} = \boldsymbol{0}$.*

In our state space, with $\boldsymbol{B}\boldsymbol{D}' \neq \boldsymbol{0}$, shocks to state and measurement equation are generally

[25]Specifically, let $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ with $\boldsymbol{\Lambda}$ diagonal be the eigenvalue-eigenvector factorization of $\boldsymbol{A}$ so that the columns of $\boldsymbol{V}$ correspond to the right eigenvectors of $\boldsymbol{A}$. Define canonical variables $\boldsymbol{S}_t^C \equiv \boldsymbol{V}^{-1}\boldsymbol{S}_t$. The signal equation can then be stated as $\boldsymbol{Z}_t = \boldsymbol{C}\boldsymbol{V}\boldsymbol{S}_t^C$ and detectability requires the signal equation to have non-zero loadings on at least every canonical variable associated with an unstable eigenvalue in $\boldsymbol{\Lambda}$.

correlated. Unit-circle controllability is thus applied to the following transformations of $\boldsymbol{A}$, $\boldsymbol{B}$:[26]

$$\boldsymbol{A}^C \equiv \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{C} \qquad\qquad \boldsymbol{B}^C \equiv \boldsymbol{B}\left(\boldsymbol{I} - \boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{D}\right)$$

Based on these definitions, the following theorem restates results from Kailath, Sayed, and Hassibi (2000) in our notation:

THEOREM 2 (Stabilizing Solution to Riccati Equation) *Provided Assumption 3 holds, a stabilizing and positive semi-definite solution to the Riccati equation* (A-62) *exists when* $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ *is unit-circle controllable and* $(\boldsymbol{A}, \boldsymbol{C})$ *is detectable. The steady-state Kalman gain is such that* $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ *is a stable matrix; moreover, the stabilizing solution is unique.*[27]

Proof. See Theorem E.5.1 of Kailath, Sayed, and Hassibi (2000); related results are also presented in Anderson, et al. (1996), or Chapter 4 of Anderson and Moore (1979). ■

In our context, with $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ and $\boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}$, the conditions for detectability and unit-circle controllability can also be restated as follows.

PROPOSITION 7 (Detectability of $(\boldsymbol{A}, \boldsymbol{H})$) *With* $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$*, detectability of* $(\boldsymbol{A}, \boldsymbol{C})$ *is equivalent to detectability of* $(\boldsymbol{A}, \boldsymbol{H})$

Proof. When $(\boldsymbol{A}, \boldsymbol{C})$ are detectable, we have $\boldsymbol{C}\boldsymbol{v} \neq \boldsymbol{0}$ for any right-eigenvector of $\boldsymbol{A}$ associated with an eigenvalue λ on or outside the unit circle, $|\lambda| \geq 1$. With $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ we then also have $\boldsymbol{C}\boldsymbol{v} = \boldsymbol{H}\boldsymbol{A}\boldsymbol{v} = \boldsymbol{H}\boldsymbol{v}\lambda \neq 0 \Leftrightarrow \boldsymbol{H}\boldsymbol{v} \neq 0$ ■

Furthermore, with $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ and $\boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}$, the above expressions for $\boldsymbol{A}^C$ and $\boldsymbol{B}^C$ can be transformed as follows:

$$\boldsymbol{A}^C = (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{A} \quad \text{and} \quad \boldsymbol{B}^C = (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{B} \quad \text{with} \quad \boldsymbol{P}^C \equiv \boldsymbol{B}\boldsymbol{H}'\left(\boldsymbol{H}\boldsymbol{B}\boldsymbol{B}'\boldsymbol{H}'\right)^{-1}\boldsymbol{H}. \quad \text{(A-63)}$$

$\boldsymbol{P}^C$ is a non-symmetric, idempotent projection matrix with $\boldsymbol{H}\boldsymbol{P}^C = \boldsymbol{H}$.[28]

PROPOSITION 8 (Unit-circle controllability of $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$) *With* $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ *and* $\boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}$*, unit-circle controllability of* $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ *is equivalent to unit-circle controllability of* $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$ *with* $\boldsymbol{P}^C$ *defined in* (A-63)*.*

[26] Notice that $\boldsymbol{B}^C = \boldsymbol{B}\boldsymbol{\mathcal{M}}^D$ where $\boldsymbol{\mathcal{M}}^D = \boldsymbol{I} - \boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{D}$ is a projection matrix, which is symmetric and idempotent, $\boldsymbol{\mathcal{M}}^D = \boldsymbol{\mathcal{M}}^D\boldsymbol{\mathcal{M}}^D$, and orthogonal to the row space of $\boldsymbol{D}$.To appreciate the role of $\mathcal{M}^D$, consider the following thought experiment: $\boldsymbol{\mathcal{M}}^D$ construct the residual in projecting the shocks of the system *off* the shocks in the signal equation, $\boldsymbol{w}_t - E(\boldsymbol{w}_t|\boldsymbol{D}\boldsymbol{w}_t) = \boldsymbol{\mathcal{M}}^D\boldsymbol{w}_t$.

[27] There may be other, non-stabilizing positive semi-definite solutions.

[28] An idempotent matrix is equal to its own square, that is $\boldsymbol{P}^C = \boldsymbol{P}^C\boldsymbol{P}^C$, and the eigenvalues of an idempotent matrix are either zero or one and we have $|\boldsymbol{P}^C| = \boldsymbol{0}$.

Proof. Suppose $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ are unit-circle controllable. Let $\tilde{\boldsymbol{v}} \equiv \boldsymbol{v}(\boldsymbol{I} - \boldsymbol{P}^C)$ and note that left-eigenvectors of $\boldsymbol{A}^C$ associated with eigenvalues on the unit circle cannot be orthogonal to $\boldsymbol{P}^C$ (otherwise we would have $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{0}$). Accordingly, $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{v}\lambda$ with $|\lambda| = 1$, $\boldsymbol{v}\boldsymbol{B}^C \neq \boldsymbol{0}$ and $\boldsymbol{v} \neq \boldsymbol{0}$ is equivalent to $\tilde{\boldsymbol{v}}\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C) = \tilde{\boldsymbol{v}}\lambda$ with $|\lambda| = 1$, $\tilde{\boldsymbol{v}} \neq \boldsymbol{0}$ $\tilde{\boldsymbol{v}}\boldsymbol{B} \neq \boldsymbol{0}$. The converse reasoning applies as well. ■

As discussed in the main text, an upshot of Proposition 8 is that a sufficient condition for unit-circle controllability of $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ is for $\boldsymbol{B}$ to have full rank. Finally, for convenience, we define the the joint concept of detectability and unit-circle controllability for the triplet $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H})$.

DEFINITION 5 (Joint detectability and unit-circle controllability) *The triplet $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H}$ is detectable and unit-circle controllable when ($\boldsymbol{A}, \boldsymbol{H}$) is detectable and $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$ is unit-circle controllable, where $\boldsymbol{P}^C$ is defined in* (A-63).

References

Anderson, Brian D. O., and John B. Moore (1979), *Optimal Filtering*, Information and System Sciences Series, Englewood Cliffs, New Jersey: Prentice-Hall Inc.

Anderson, Evan W., Ellen R. McGrattan, Lars Peter Hansen, and Thomas J. Sargent (1996), "Mechanics of forming and estimating dynamic linear economies," in *Handbook of Computational Economics* eds. by Hans M. Amman, David A. Kendrick, and John Rust, 1: Elsevier, chap. 4, 171–252.

Aoki, Kosuke (2006), "Optimal commitment policy under noisy information," *Journal of Economic Dynamics and Control*, 30, 81–109.

Baxter, Brad, Liam Graham, and Stephen Wright (2011), "Invertible and non-invertible information sets in linear rational expectations models," *Journal of Economic Dynamics and Control*, 35, 295–311.

Blanchard, Oliver Jean, and Charles M. Kahn (1980), "The solution of linear difference models under rational expectations," *Econometrica*, 48, 1305–1312.

Bullard, James, and Kaushik Mitra (2002), "Learning about monetary policy rules," *Journal of Monetary Economics*, 49, 1105–1129.

Carboni, Giacomo, and Martin Ellison (2011), "Inflation and output volatility under asymmetric incomplete information," *Journal of Economic Dynamics and Control*, 35, 40–51.

Dotsey, Michael, and Andreas Hornstein (2003), "Should a monetary policymaker look at money?," *Journal of Monetary Economics*, 50, 547–579.

Farmer, Roger, Vadim Khramov, and Giovanni Nicolò (2015), "Solving and estimating indeterminate DSGE models," *Journal of Economic Dynamics and Control*, 54, 17–36.

Fernández-Villaverde, Jesús, Juan F. Rubio-Ramírez, Thomas J. Sargent, and Mark W. Watson (2007), "ABCs (and Ds) of understanding VARs," *American Economic Review*, 97, 1021–1026.

Hansen, Lars P., and Thomas J. Sargent (2007), *Robustness*: Princeton University Press.

Kailath, Thomas, Ali H. Sayed, and Babak Hassibi (2000), *Linear Estimation*, Prentice Hall Information and System Sciences Series: Pearson Publishing.

King, Robert G., and Mark W. Watson (1998), "The solution of singular linear difference systems under rational expectations," *Internatinal Economic Review*, 39, 1015–1026.

Klein, Paul (2000), "Using the generalized Schur form to solve a multivariate linear rational expectations model," *Journal of Economic Dynamics and Control*, 24, 1405–1423.

Lubik, Thomas A., Christian Matthes, and Elmar Mertens (2019), "Indeterminacy and imperfect information," Working Paper 19-17, Federal Reserve Bank of Richmond.

Lubik, Thomas A., and Frank Schorfheide (2003), "Computing sunspot equilibria in linear rational expectations models," *Journal of Economic Dynamics and Control*, 28, 273–285.

Nimark, Kristoffer (2008), "Monetary policy with signal extraction from the bond market," *Journal of Monetary Economics*, 55, 1389–1400.

Sims, Christopher A (2002), "Solving linear rational expectations models," *Computational Economics*, 20, 1–20.

Svensson, Lars E. O., and Michael Woodford (2004), "Indicator variables for optimal policy under asymmetric information," *Journal of Economic Dynamics and Control*, 28, 661–690.

Woodford, Michael (2003), "Optimal interest-rate smoothing," *Review of Economic Studies*, 70, 861–886.