A Price Equilibrium

On the one hand, at time t, a monopolist entering the market for a particular good q cannot set a price higher than the competitive price without giving way to a competitive fringe of firms. On the other hand, could he seriously consider increasing profits by lowering the price unilaterally below $p(t) = \alpha w(t)/A(t)$ (i.e. while all other firms keep their price unchanged)? The answer is no, as long as the marginal profit satisfies the following condition:

$$\begin{aligned}\frac{\partial \pi(q,t)}{\partial \widehat{p}(q,t)} &= \frac{\partial \widehat{x}(q,t)}{\partial \widehat{p}(q,t)}\left[\widehat{p}(q,t) - \frac{w(t)}{A(t)}\right] + \widehat{x}(q,t) > 0 \\ &\Leftrightarrow -\frac{\partial \widehat{x}(q,t)}{\partial \widehat{p}(q,t)}\frac{\widehat{p}(q,t)}{\widehat{x}(q,t)}\left[\frac{\widehat{p}(q,t) - w(t)/A(t)}{\widehat{p}(q,t)}\right] < 1,\end{aligned} \tag{A.1}$$

with $\widehat{p}(q,t) \leq p(t)$, and where $\widehat{x}(q,t) = [1 - F(\gamma(\widehat{q},t))]L$ is the effective demand for good q produced at $\widehat{p}(q,t)$. In other words, the price elasticity of demand multiplied by the price-cost margin should not exceed unity.

Let us define $\widehat{q} \leq q$ such that

$$\frac{1}{q}\frac{1}{\widehat{p}(q,t)} = \frac{1}{\widehat{q}}\frac{1}{p(t)} \Rightarrow \widehat{q} = \frac{\widehat{p}(q,t)}{p(t)}q.$$

Customers for the variety of good q include all those that are rich enough to buy $\widehat{q}$ (i.e. households of type $\gamma \geq \gamma(\widehat{q},t)$ where $\gamma(\widehat{q},t) = \widehat{q}(t)/y(t)L = (\widehat{p}(q,t)q)\,/\,(p(t)y(t)L)$). Let $f(\gamma)$ be the density of γ -type households. We also define $\beta(\gamma) \equiv \gamma f(\gamma)/\left[1 - F(\gamma)\right]$. The price elasticity of demand for good q can be written as follows

$$\begin{aligned}-\frac{\partial \widehat{x}(q,t)}{\partial \widehat{p}(q,t)}\frac{\widehat{p}(q,t)}{\widehat{x}(q,t)} &= \frac{f(\gamma(\widehat{q},t))q}{p(t)y(t)}\frac{\widehat{p}(q,t)}{[1 - F(\gamma(\widehat{q},t))]L} \\ &= \frac{f(\gamma(\widehat{q},t))\gamma(\widehat{q},t)}{[1 - F(\gamma(\widehat{q},t))])L}\frac{p(t)y(t)L}{p(t)y(t)} = \frac{\gamma(\widehat{q},t)f(\gamma(\widehat{q},t))}{1 - F(\gamma(\widehat{q},t))} = \beta(\gamma(\widehat{q},t)).\end{aligned} \tag{A.2}$$

We can show that using (A.1) and (A.2), we have:

$$\begin{aligned}-\frac{\partial \widehat{x}(q,t)}{\partial \widehat{p}(q,t)}\frac{\widehat{p}(q,t)}{\widehat{x}(q,t)}\left[\frac{\widehat{p}(q,t) - w(t)/A(t)}{\widehat{p}(q,t)}\right] &= \beta(\gamma(\widehat{q},t))\left[\frac{\widehat{p}(q,t) - w(t)/A(t)}{\widehat{p}(q,t)}\right] \\ &< \beta(\gamma(\widehat{q},t))\left[\frac{p(q,t) - w(t)/A(t)}{p(q,t)}\right] = \frac{\alpha - 1}{\alpha}\beta(\gamma(\widehat{q},t)).\end{aligned}$$

Therefore, the following inequality provides a sufficient condition for ruling out price-cutting equilibria:

$$\frac{\alpha-1}{\alpha}\beta\left(\gamma(\widehat{q},t)\right)<1 \Rightarrow -\frac{\partial\widehat{x}(q,t)}{\partial\widehat{p}(q,t)}\frac{\widehat{p}(q,t)}{\widehat{x}(q,t)}\left[\frac{\widehat{p}(q,t)-w(t)/A(t)}{\widehat{p}(q,t)}\right]<1. \quad \text{(A.3)}$$

Indeed, as long as (A.1) is satisfied, when a firm with access to IRS technology in sector q aims to cut the price below $\alpha w(t)/A(t)$, it is unable to expand its customer base to such an extent as to compensate for the loss in profit per customer, thus discouraging price-cutting. In our framework, such a condition results in (A.3). The income distribution should not degenerate around any γ -type.

B Proof of Proposition 3: Scale Economies, Learning Potential, and Growth

Using the break-even condition (7), it is easy to verify that

$$\tfrac{d\gamma^*}{d\alpha}=\tfrac{1-F(\gamma^*)}{(\alpha-1)f(\gamma^*)}>0, \quad \tfrac{d\gamma^*}{dL}=\tfrac{1-F(\gamma^*)}{f(\gamma^*)L}>0, \quad \tfrac{d\gamma^*}{dC}=-\tfrac{1}{(\alpha-1)f(\gamma^*)L}<0,$$

which implies that

$$\begin{aligned}\frac{dE^c(.)}{d\alpha} &= \frac{1}{\gamma^* L}\left[1-\frac{\alpha}{\alpha-1}\frac{1-F(\gamma^*)}{f(\gamma^*)\gamma^*}+\left[\frac{1-F(\gamma^*)}{f(\gamma^*)\gamma^*}-1\right]T-\left[1-F(\gamma^*)\right]\gamma^*\right]<0\\ &\Leftrightarrow \frac{\alpha-1}{\alpha}\frac{f(\gamma^*)\gamma^*}{1-F(\gamma^*)}<1 \text{ (see the price equilibrium section above);}\end{aligned}$$

$$\begin{aligned}\frac{dE^c(.)}{dL} &= -\left[\frac{E^c(.)}{\gamma^*}+(\alpha-1)f(\gamma^*)\right]\frac{(1-F(\gamma^*))}{f(\gamma^*)L}-\frac{\alpha}{\gamma^*(L)^2}<0;\\ \frac{dE^c(.)}{dC} &= \left[\frac{E^c(.)}{\gamma^*}+(\alpha-1)f(\gamma^*)\right]\frac{1}{(\alpha-1)f(\gamma^*)L}>0.\end{aligned}$$

C Proof of Proposition 4: Very Unequal Versus Almost Perfectly Egalitarian Income Distribution and Growth

First, for a very unequal distribution of income, the Lorenz curve is such that its slope tends to be zero for $\underline{\gamma}\le\gamma\le\gamma^*$. Thus, we have $\underline{\gamma}\approx\widetilde{\gamma}\approx\gamma^*\approx 0$ which implies both $T\approx\widetilde{T}\approx 0$ and $M\approx 1$. Replacing in (13) immediately leads to $g=0$. Second, for an almost perfectly egalitarian distribution of income, the Lorenz curve approaches the 45° line, and we have $\underline{\gamma}\approx\gamma^*\approx 1/L$ which implies that $T \approx 1 - N^*/L$, $\widetilde{T} \approx 0$, and $M \approx \alpha/(1 + C/L)$. Again, replacing in (13) we obtain:

$$g = \lambda \widetilde{E} L \frac{1}{1 + C/L} \left[1 - \widetilde{\gamma} L\right].$$

Using (19) and $F(\gamma) \approx 0$ for $\gamma < \gamma^*$, we get

$$\frac{g}{\lambda} = L \int_{\widetilde{\gamma}}^{\gamma^*} \frac{1}{\gamma} d\gamma = L \ln \left(\frac{1}{\widetilde{\gamma} L} \right),$$

which implies that $\widetilde{\gamma} L = \exp(-g/\lambda L)$. Eventually, if we substitute this result into the above expression, the steady-state growth rate becomes

$$\begin{aligned} g = \Lambda(g) &= \lambda \widetilde{E} L \frac{1}{1 + C/L} \left[1 - \exp(-g/\lambda L)\right] \\ \Rightarrow \left. \frac{d\Lambda(g)}{dg} \right|_{g=0} &= \frac{\widetilde{E}}{1 + C/L} > 1 \Leftrightarrow \widetilde{E} > E^c\left(.\right) = 1 + \frac{C}{L}. \end{aligned}$$

D Proof of Proposition 5: Inequality, Transformation of Income, and Growth

The proof proceeds in two parts. The first part consists of demonstrating that an income transformation that increases the purchasing power of either the pre-tax middle class or the pre-tax lower class to the detriment of the initial upper class will produce an upward shift of $\Lambda(g)$, which eventually leads to a new equilibrium with a higher steady-state rate of growth. In the second part, we show that an income transformation that benefits the pre-tax lower class at the expense of the initial middle class lowers economic growth.

1.a/ Let us assume the following income transformation so that the post-transfer income distribution denoted by $F_r\left(\gamma\right)$ Lorenz dominates the pre-tax distribution of income $F\left(\gamma\right)$ (see Figure D.1):

$$\gamma_r \equiv \psi\left(\gamma, \tau\right) = \begin{cases} \gamma & \forall \gamma \in \left[\underline{\gamma}, \widetilde{\gamma}\right[\\ \gamma + \varphi\left(\gamma, \tau\right) & \forall \gamma \in \left[\widetilde{\gamma}, \gamma^*\right] \\ \gamma - \rho\left(\gamma, \tau\right) & \forall \gamma \in \left]\gamma^*, \infty\right[\end{cases},$$

where $\psi(.)$ is positive, monotone-increasing, and continuous with γ (i.e. $\partial\varphi\left(\gamma, \tau\right)/\partial\gamma > -1$, $\partial\rho\left(\gamma, \tau\right)/\partial\gamma < 1$), and $\tau > 0$ reflecting the intensity of income redistribution with $\partial\varphi\left(\gamma, \tau\right)/\partial\tau \geq 0$ and $\partial\rho\left(\gamma, \tau\right)/\partial\tau \geq 0$. In addition, we have $\widetilde{\gamma}_r = \widetilde{\gamma}$ and $\gamma_r^* = \gamma^*$, which implies that $\varphi\left(\widetilde{\gamma}, \tau\right) = \varphi\left(\gamma^*, \tau\right) = \rho\left(\gamma^*, \tau\right) = 0$, and $\int_{\widetilde{\gamma}}^{\gamma^*} \varphi\left(\gamma, \tau\right) f\left(\gamma\right) d\gamma = \int_{\gamma^*}^{\infty} \rho\left(\gamma, \tau\right) f\left(\gamma\right) d\gamma$.

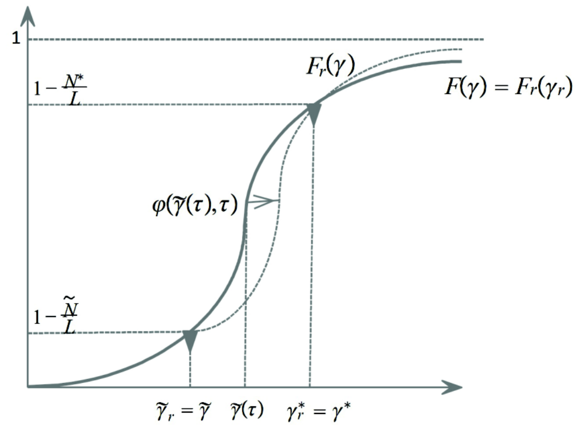

Fig. D.1. Income transformation from the upper to the middle class.

First, such an income transformation satisfies both the single-crossing condition and the principle of rank-preserving transfers. Second, note that it implies that $F_r(\gamma_r) = F(\gamma)$. Then, using $d\gamma_r = \frac{\partial\psi(\gamma,\tau)}{\partial\gamma}d\gamma$, we get

$$dF_r(\gamma_r) = dF(\gamma) \Rightarrow f_r(\gamma_r) = f(\gamma)\left(\frac{\partial\psi(\gamma,\tau)}{\partial\gamma}\right)^{-1}.$$

The middle-class share of post-tax income becomes:

$$T_r(\gamma_r^*,\tau) = \int_{\underline{\gamma}_r}^{\gamma_r^*} \gamma_r f_r(\gamma_r)\, d\gamma_r = \int_{\underline{\gamma}}^{\gamma^*} \psi(\gamma,\tau) f(\gamma)\, d\gamma = T(\gamma^*) + \int_{\widetilde{\gamma}}^{\gamma^*} \varphi(\gamma,\tau) f(\gamma)\, d\gamma.$$

Let $\gamma_r^*(\tau)$ be the implicit solution to $1 - F_r(\gamma_r^*(\tau)) = C/((\alpha-1)L)$, where $\gamma_r^*(\tau)$ is the post-transfer income of household of type $\gamma^*(\tau)$, which is equal to $\psi^{-1}(\gamma_r^*(\tau))$ and defined as the upper cut-off for the middle class after transformation. Given that $1 - F(\gamma^*) = C/((\alpha-1)L)$ and $\gamma_r^*(\tau) = \psi(\gamma^*(\tau)) = \gamma^*(\tau)$, we get $\gamma_r^*(\tau) = \gamma^*(\tau) = \gamma^*$.

We now analyze the effect of an income transformation on $\Lambda(g)$ (see (21)). The system of equations (20) can be rewritten in terms of both the transformed multiplier $M_r(\tau)$ and the proportion of total income spent in the intermediate range of sectors where learning takes place, which is denoted by $\mathbb{N}(\widetilde{\gamma}(\tau),\tau)$:

$$g = \Lambda(\widetilde{\gamma}(\tau),\tau) = \lambda\widetilde{E}L\frac{M_r(\tau)}{\alpha}\mathbb{N}(\widetilde{\gamma}(\tau),\tau) \tag{D.1}$$

$$\begin{aligned} g/\lambda &= \Gamma(\widetilde{\gamma}(\tau),\tau) \qquad\qquad (D.2)\\ &= N^*\ln(\gamma^*) - [1 - F(\widetilde{\gamma}(\tau))]L\ln(\widetilde{\gamma}(\tau) + \varphi(\widetilde{\gamma}(\tau),\tau)) \\ &\quad + \int_{\widetilde{\gamma}(\tau)}^{\gamma^*} \ln(\gamma + \varphi(\gamma,\tau)) f(\gamma) L d\gamma \end{aligned}$$

where $\widetilde{\gamma}(\tau) = \widetilde{\gamma}(g,\tau)$ is the implicit solution to (D.2) so that the household of

$\widetilde{\gamma}(\tau)$ -type is the new lower cut-off for the middle class after transformation, $\mathbb{N}(\widetilde{\gamma}(\tau),\tau) \equiv \int_{\widetilde{\gamma}_r(\tau)}^{\gamma^*} [1-F_r(\gamma_r)]\, L d\gamma_r$, and $\widetilde{\gamma}_r(\tau) = \widetilde{\gamma}(\tau) + \varphi(\widetilde{\gamma}(\tau),\tau)$ is the post-transfer income of household of type $\widetilde{\gamma}(\tau)$. The righ-hand side of (D.2) is the result of the integration by part of $\int_{\widetilde{\gamma}_r(\tau)}^{\gamma^*} \frac{1-F_r(\gamma_r)}{\gamma_r} d\gamma_r$.

Total differentiation of (D.1) leads to

$$dg = \frac{\lambda \widetilde{E} L}{\alpha} \left[\begin{array}{c} \frac{\partial M_r(\tau)}{\partial \tau} \mathbb{N}(\widetilde{\gamma}(\tau),\tau) + M_r(\tau) \frac{\partial \mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial \tau} d\tau \\ + M_r(\tau) \left[\frac{\partial \mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial \widetilde{\gamma}(\tau)} \frac{\partial \widetilde{\gamma}(g,\tau)}{\partial g} dg + \frac{\partial \mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial \widetilde{\gamma}(\tau)} \frac{\partial \widetilde{\gamma}(g,\tau)}{\partial \tau} d\tau \right] \end{array} \right].$$

Therefore, we have

$$\frac{dg/d\tau}{g} = \left[1 - \lambda \widetilde{E} L \frac{M_r(\tau)}{\alpha} \frac{\partial \mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial \widetilde{\gamma}(\tau)} \frac{\partial \widetilde{\gamma}(g,\tau)}{\partial g}\right]^{-1} \left[\frac{\frac{\partial M_r(\tau)}{\partial \tau}}{M_r(\tau)} + \frac{\frac{\partial \mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial \tau} + \frac{\partial \mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial \widetilde{\gamma}(\tau)} \frac{\partial \widetilde{\gamma}(g,\tau)}{\partial \tau}}{\mathbb{N}(\widetilde{\gamma}(\tau),\tau)} \right]. \tag{D.3}$$

First, note that

$$\lambda \widetilde{E} L \frac{M_r(\tau)}{\alpha} \frac{\partial \mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial \widetilde{\gamma}(\tau)} \frac{\partial \widetilde{\gamma}(g,\tau)}{\partial g} = \lambda \widetilde{E} L \frac{M_r(\tau)}{\alpha} \left(\widetilde{\gamma}(\tau) + \varphi(\widetilde{\gamma}(\tau),\tau)\right),$$

where $\partial \widetilde{\gamma}(g,\tau)/\partial g$ is the implicit derivative of $\widetilde{\gamma}(g,\tau)$ using (D.2):

$$\frac{\partial \widetilde{\gamma}(g,\tau)}{\partial g} = \frac{1}{\lambda} \left[\frac{\partial \Gamma(\widetilde{\gamma}(\tau),\tau)}{\partial \widetilde{\gamma}(\tau)}\right]^{-1}.$$

Using properties of $\Lambda(g)$ as discussed in Section 5.1, we infer that for g such that $\Lambda(g,\tau) = g$, we have $\partial \Lambda(g,\tau)/\partial g < 1$, where

$$\frac{\partial \Lambda(g,\tau)}{\partial g} = \lambda \widetilde{E} L \frac{M_r(\tau)}{\alpha} \left(\widetilde{\gamma}(\tau) + \varphi(\widetilde{\gamma}(\tau),\tau)\right),$$

which implies that

$$\left[1 - \frac{\lambda \widetilde{E} L}{\alpha} M_r(\tau) \frac{\partial \mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial \widetilde{\gamma}(\tau)} \frac{\partial \widetilde{\gamma}(g,\tau)}{\partial g}\right]^{-1} > 0.$$

Second, the transformed multiplier is given by

$$M_r(\tau) = \frac{1}{1-\frac{\alpha-1}{\alpha}\left[T\left(\gamma^*\right)+\int_{\widetilde{\gamma}(\tau)}^{\gamma^*}\varphi\left(\gamma,\tau\right)f\left(\gamma\right)d\gamma\right]}$$
$$\Rightarrow \frac{\partial M_r\left(\tau\right)/\partial\tau}{M_r(\tau)} = M_r\frac{\alpha-1}{\alpha}\int_{\widetilde{\gamma}(\tau)}^{\gamma^*}\frac{d\varphi\left(\gamma,\tau\right)}{d\tau}f\left(\gamma\right)d\gamma > 0.$$

Finally, the share of total income spent in the "transformed" range of sectors where learning takes place can be written as

$$\begin{aligned}\mathbb{N}\left(\widetilde{\gamma}\left(\tau\right),\tau\right) = {} & \gamma^* N^* - \left[\widetilde{\gamma}\left(\tau\right)+\varphi\left(\widetilde{\gamma}\left(\tau\right),\tau\right)\right]\left[1-F\left(\widetilde{\gamma}\left(\tau\right)\right)\right]L \\ & +T\left(\gamma^*\right)-T\left(\widetilde{\gamma}\left(\tau\right)\right)+\int_{\widetilde{\gamma}(\tau)}^{\gamma^*}\varphi\left(\gamma,\tau\right)f\left(\gamma\right)Ld\gamma.\end{aligned}$$

Given that

$$\frac{\partial\widetilde{\gamma}\left(g,\tau\right)}{\partial\tau} = -\frac{\partial\Gamma\left(\widetilde{\gamma}\left(\tau\right),\tau\right)/\partial\tau}{\partial\Gamma\left(\widetilde{\gamma}\left(\tau\right),\tau\right)/\partial\widetilde{\gamma}\left(\tau\right)},$$

we get

$$\begin{aligned}& \frac{\partial\mathbb{N}\left(\widetilde{\gamma}\left(\tau\right),\tau\right)}{\partial\tau}+\frac{\partial\mathbb{N}\left(\widetilde{\gamma}\left(\tau\right),\tau\right)}{\partial\widetilde{\gamma}\left(\tau\right)}\frac{\partial\widetilde{\gamma}\left(g,\tau\right)}{\partial\tau} \\ & = \int_{\widetilde{\gamma}(\tau)}^{\gamma^*}\left[1-\frac{\widetilde{\gamma}\left(\tau\right)+\varphi\left(\widetilde{\gamma}\left(\tau\right),\tau\right)}{\gamma+\varphi\left(\gamma,\tau\right)}\right]\frac{\partial\varphi\left(\gamma,\tau\right)}{\partial\tau}f\left(\gamma\right)Ld\gamma > 0 \\ & \Leftrightarrow \gamma+\varphi\left(\gamma,\tau\right) > \widetilde{\gamma}\left(\tau\right)+\varphi\left(\widetilde{\gamma}\left(\tau\right),\tau\right),\ \text{for } \widetilde{\gamma}\left(\tau\right)<\gamma<\gamma^*.\end{aligned}$$

1.b/ We use the same approach to prove that an income transformation, which increases the purchasing power of the pre-tax lower class at the expense of the initial upper class will lead to a rise in the steady-state growth rate. Consider the following income transformation:

$$\gamma_r \equiv \psi\left(\gamma,\tau\right) = \begin{cases}\gamma+\varphi\left(\gamma,\tau\right) & \forall\gamma\in\left[\underline{\gamma},\widetilde{\gamma}\right[\\ \gamma & \forall\gamma\in\left[\widetilde{\gamma},\gamma^*\right] \\ \gamma-\rho\left(\gamma,\tau\right) & \forall\gamma\in\left]\gamma^*,\infty\right[\end{cases},$$

where $\psi(.)$ is again positive, monotone-increasing and continuous with γ, both $\widetilde{\gamma}_r = \widetilde{\gamma}$ and $\gamma_r^* = \gamma^*$ imply that $\varphi\left(\widetilde{\gamma},\tau\right) = \rho\left(\gamma^*,\tau\right) = 0$, and $\int_{\underline{\gamma}}^{\widetilde{\gamma}}\varphi\left(\gamma,\tau\right)f\left(\gamma\right)d\gamma = \int_{\gamma^*}^{\infty}\rho\left(\gamma,\tau\right)f\left(\gamma\right)d\gamma$. Such an income transformation also satisfies the single-crossing condition and the principle of rank-preserving income transfers, and implies that $F_r(\gamma_r) = F(\gamma)$. The middle-class share of post-tax income becomes

$$T_r(\gamma^*,\tau)=\int_{\underline{\gamma}}^{\gamma^*}\psi(\gamma,\tau)\,f(\gamma)\,d\gamma=T(\gamma^*)+\int_{\underline{\gamma}}^{\widetilde{\gamma}}\varphi(\gamma,\tau)\,f(\gamma)\,d\gamma.$$

In this case, (D.2) is given by

$$\begin{aligned}\Gamma(\widetilde{\gamma}(\tau),\tau)=&N^*\ln(\gamma^*)-[1-F(\widetilde{\gamma}(\tau))]\,L\ln(\widetilde{\gamma}(\tau)+\varphi(\widetilde{\gamma}(\tau),\tau))\\&+\int_{\widetilde{\gamma}(\tau)}^{\widetilde{\gamma}}\ln(\gamma+\varphi(\gamma,\tau))\,f(\gamma)\,Ld\gamma+\int_{\widetilde{\gamma}}^{\gamma^*}\ln(\gamma)\,f(\gamma)\,Ld\gamma.\end{aligned}$$

We show that:

$$\tfrac{\partial M_r(\tau)}{\partial\tau}>0 \text{ and } \tfrac{\partial\mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial\tau}+\tfrac{\partial\mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial\widetilde{\gamma}(\tau)}\tfrac{\partial\widetilde{\gamma}(g,\tau)}{\partial\tau}>0,$$

which given (D.3) implies that $dg/g>0$. On the one hand, we have

$$\begin{aligned}M_r(\tau)&=\left[1-\frac{\alpha-1}{\alpha}\left[T(\gamma^*)+\int_{\underline{\gamma}}^{\widetilde{\gamma}}\varphi(\gamma,\tau)\,f(\gamma)\,d\gamma\right]\right]^{-1}\\&\Rightarrow\frac{\partial M_r(\tau)/\partial\tau}{M_r(\tau)}=M_r(\tau)\,\frac{\alpha-1}{\alpha}\int_{\underline{\gamma}}^{\widetilde{\gamma}}\frac{\partial\varphi(\gamma,\tau)}{\partial\tau}f(\gamma)\,Ld\gamma>0.\end{aligned}$$

On the other hand, the post-transfer proportion of total income spent in the range of sectors where learning now takes place is given by[1]

$$\begin{aligned}\mathbb{N}(\widetilde{\gamma}(\tau),\tau)=&\gamma^*N^*-[1-F(\widetilde{\gamma}(\tau))]\,L(\widetilde{\gamma}(\tau)+\varphi(\widetilde{\gamma}(\tau),\tau))\\&+T(\gamma^*)-T(\widetilde{\gamma}(\tau))+\int_{\widetilde{\gamma}(\tau)}^{\widetilde{\gamma}}\varphi(\gamma,\tau)\,f(\gamma)\,Ld\gamma\end{aligned}$$

$$\begin{aligned}&\Rightarrow\frac{\partial\mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial\tau}+\frac{\partial\mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial\widetilde{\gamma}(\tau)}\frac{\partial\widetilde{\gamma}(g,\tau)}{\partial\tau}\\&=\int_{\widetilde{\gamma}(\tau)}^{\widetilde{\gamma}}\left(1-\frac{\widetilde{\gamma}(\tau)+\varphi(\widetilde{\gamma}(\tau),\tau)}{\gamma+\varphi(\gamma,\tau)}\right)\frac{\partial\varphi(\gamma(\tau),\tau)}{\partial\tau}f(\gamma)\,Ld\gamma>0\\&\Leftrightarrow\gamma+\varphi(\gamma,\tau)>\widetilde{\gamma}(\tau)+\varphi(\widetilde{\gamma}(\tau),\tau),\text{ for }\widetilde{\gamma}(\tau)<\gamma<\widetilde{\gamma}.\end{aligned}$$

[1] We make use of the inequality $\widetilde{\gamma}(\tau)<\widetilde{\gamma}$, which can be shown using

$$\frac{d\widetilde{\gamma}(\tau)}{d\tau}=\frac{\partial\widetilde{\gamma}(g,\tau)}{\partial g}\frac{dg}{d\tau}+\frac{\partial\widetilde{\gamma}(g,\tau)}{\partial\tau}<0.$$

2/ Let us assume

$$\gamma_r \equiv \psi(\gamma,\tau) = \begin{cases} \gamma + \varphi(\gamma,\tau) & \forall \gamma \in \left[\underline{\gamma}, \widetilde{\gamma}\right[\\ \gamma - \rho(\gamma,\tau) & \forall \gamma \in [\widetilde{\gamma}, \gamma^*] \\ \gamma & \forall \gamma \in]\gamma^*, \infty[\end{cases},$$

where $\psi(.)$ has the same properties as above, and $\int_{\underline{\gamma}}^{\widetilde{\gamma}} \varphi(\gamma,\tau) f(\gamma) d\gamma = \int_{\widetilde{\gamma}}^{\gamma^*} \rho(\gamma,\tau) f(\gamma) d\gamma$. The middle-class share of post-tax income is equal to

$$T_r(\gamma^*,\tau) = T(\gamma^*) + \int_{\underline{\gamma}}^{\widetilde{\gamma}} \varphi(\gamma,\tau) f(\gamma) d\gamma - \int_{\widetilde{\gamma}}^{\gamma^*} \rho(\gamma,\tau) f(\gamma) d\gamma = T(\gamma^*).$$

Note that (D.2) is given by

$$\begin{aligned} \Gamma(\widetilde{\gamma}(\tau),\tau) = {} & N^* \ln(\gamma^*) - [1 - F(\widetilde{\gamma}(\tau))] L \ln(\widetilde{\gamma}(\tau) + \varphi(\widetilde{\gamma}(\tau),\tau)) \\ & + \int_{\widetilde{\gamma}(\tau)}^{\widetilde{\gamma}} \ln(\gamma + \varphi(\gamma,\tau)) f(\gamma) L d\gamma + \int_{\widetilde{\gamma}}^{\gamma^*} \ln(\gamma - \rho(\gamma,\tau)) f(\gamma) L d\gamma. \end{aligned}$$

Here, we show that:

$$\frac{\partial M_r(\tau)}{\partial \tau} = 0 \text{ and } \frac{\partial \mathbb{N}\left(\widetilde{\gamma}(\tau),\tau\right)}{\partial \tau} + \frac{\partial \mathbb{N}\left(\widetilde{\gamma}(\tau),\tau\right)}{\partial \widetilde{\gamma}(\tau)} \frac{\partial \widetilde{\gamma}(g,\tau)}{\partial \tau} > 0,$$

which implies that $dg/g > 0$. We focus on the post-transfer share of total income spent in the intermediate range of sectors where learning takes place:

$$\begin{aligned} \mathbb{N}(\widetilde{\gamma}(\tau),\tau) = {} & N^*\gamma^* - [1 - F(\widetilde{\gamma}(\tau))] L (\widetilde{\gamma}(\tau) + \varphi(\widetilde{\gamma}(\tau),\tau)) + T(\gamma^*) - T(\widetilde{\gamma}(\tau)) \\ & + \int_{\widetilde{\gamma}(\tau)}^{\widetilde{\gamma}} \varphi(\gamma,\tau) f(\gamma) L d\gamma - \int_{\widetilde{\gamma}}^{\gamma^*} \rho(\gamma,\tau) f(\gamma) L d\gamma \end{aligned}$$

$$\begin{aligned} & \Rightarrow \frac{\partial \mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial \tau} + \frac{\partial \mathbb{N}(\widetilde{\gamma}(\tau),\tau)}{\partial \widetilde{\gamma}(\tau)} \frac{\partial \widetilde{\gamma}(g,\tau)}{\partial \tau} \\ & = \int_{\widetilde{\gamma}(\tau)}^{\widetilde{\gamma}} \left(1 - \frac{\widetilde{\gamma}_r(\tau)}{\gamma + \varphi(\gamma,\tau)}\right) \frac{\partial \varphi(\gamma(\tau),\tau)}{\partial \tau} f(\gamma) L d\gamma \\ & - \int_{\widetilde{\gamma}}^{\gamma^*} \left(1 - \frac{\widetilde{\gamma}_r(\tau)}{\gamma - \rho(\gamma,\tau)}\right) \frac{\partial \rho(\gamma(\tau),\tau)}{\partial \tau} f(\gamma) L d\gamma < 0, \end{aligned}$$

recalling that $\widetilde{\gamma}_r(\tau) = \widetilde{\gamma}(\tau) + \varphi(\widetilde{\gamma}(\tau),\tau)$.

Note first that the following two inequalities hold:

$$\int_{\widetilde{\gamma}(\tau)}^{\widetilde{\gamma}} \left(1 - \frac{\widetilde{\gamma}_r(\tau)}{\gamma + \varphi(\gamma,\tau)}\right) \frac{\partial \varphi(\gamma(\tau),\tau)}{\partial \tau} f(\gamma) d\gamma < \int_{\widetilde{\gamma}(\tau)}^{\widetilde{\gamma}} \left(1 - \frac{\widetilde{\gamma}_r(\tau)}{\widetilde{\gamma}}\right) \frac{\partial \varphi(\gamma(\tau),\tau)}{\partial \tau} f(\gamma) d\gamma,$$

$$\int_{\widetilde{\gamma}}^{\gamma^*}\left(1-\frac{\widetilde{\gamma}_r\left(\tau\right)}{\widetilde{\gamma}}\right)\frac{\partial\rho\left(\gamma\left(\tau\right),\tau\right)}{\partial\tau}f\left(\gamma\right)d\gamma<\int_{\widetilde{\gamma}}^{\gamma^*}\left(1-\frac{\widetilde{\gamma}_r\left(\tau\right)}{\gamma-\rho\left(\gamma,\tau\right)}\right)\frac{\partial\rho\left(\gamma\left(\tau\right),\tau\right)}{\partial\tau}f\left(\gamma\right)d\gamma.$$

Second, the above budget constraint leads to:

$$\int_{\widetilde{\gamma}(\tau)}^{\widetilde{\gamma}}\left(1-\frac{\widetilde{\gamma}_r\left(\tau\right)}{\widetilde{\gamma}}\right)\frac{\partial\varphi\left(\gamma\left(\tau\right),\tau\right)}{\partial\tau}f\left(\gamma\right)d\gamma<\int_{\widetilde{\gamma}}^{\gamma^*}\left(1-\frac{\widetilde{\gamma}_r\left(\tau\right)}{\widetilde{\gamma}}\right)\frac{\partial\rho\left(\gamma\left(\tau\right),\tau\right)}{\partial\tau}f\left(\gamma\right)d\gamma.$$

Finally, the following inequality is obtained using the principles of transitivity:

$$\int_{\widetilde{\gamma}(\tau)}^{\widetilde{\gamma}}\left(1-\frac{\widetilde{\gamma}_r\left(\tau\right)}{\gamma+\varphi\left(\gamma,\tau\right)}\right)\frac{\partial\varphi\left(\gamma\left(\tau\right),\tau\right)}{\partial\tau}f\left(\gamma\right)d\gamma<\int_{\widetilde{\gamma}}^{\gamma^*}\left(1-\frac{\widetilde{\gamma}_r\left(\tau\right)}{\gamma-\rho\left(\gamma,\tau\right)}\right)\frac{\partial\rho\left(\gamma\left(\tau\right),\tau\right)}{\partial\tau}f\left(\gamma\right)d\gamma.$$