Online appendix to accompany "Consumption, Reservation Wages, and Aggregate Labor Supply"

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In this online apendix, I show that the reservation wage can be defined uniquely conditional on consumption in many specification choices. In each case, the proof of Lemma 2 is almost the same. I focus on the single crossing between V^E and V^N across individuals with $b > 0$ (Lemma 1). I write c instead of c^E for notational brevity.

1 Single Earner's Problem

Proposition. *Consider a class of models in equation (1) in the main text. In each of the following cases, define the reservation wage conditional on consumption as in Definition 1.*

(I) (Separable Preferences, Both Margins) Suppose that the period utility function is

$$\tilde{u}(c,h) = u(c) + v(1-h) - F \cdot \mathbf{I}_{[h>0]},$$

where $u(\cdot)$ and $v(\cdot)$ is strictly increasing and concave, $F > 0$ is the fixed cost of working, $\mathbf{I}_{[h>0]}$ is 1 if $h > 0$ and 0 otherwise, and $0 \leq h \leq 1$. Then, (i)–(iii) in Proposition 1 hold.

(II) (Non-separable Preferences, Extensive Margin) Suppose that $h \in \{0, \bar{h}\}$ and the instantaneous utility function is non-separable between consumption and leisure:

$$\tilde{u}(c,h) = u(c, 1-h),$$

where $u(\cdot, 1-h)$ are strictly increasing and strictly concave in consumption, $u(c,\cdot)$ is strictly increasing in leisure. If $u_{cl} > 0$, then (i)–(iii) in Proposition 1 hold. If $u_{cl} < 0$, then (i) and (iii) in Proposition 1 hold.

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(III) (Non-separable Preferences, Both Margins) Suppose that the instantaneous utility function is

$$\tilde{u}(c,h) = u(c, 1-h) - F \cdot \mathbf{I}_{[h>0]},$$

where $u(\cdot,\cdot)$ is strictly increasing and strictly concave in consumption and leisure, and $0 \le h \le 1$. If $u_{cl} > 0$, then (i)–(iii) in Proposition 1 hold. If $u_{cl} < 0$ and consumption and leisure are normal goods, then (i)–(iii) in Proposition 1 hold up to the first order approximation of the instantaneous utility function.

Proof. **Part (I).** Suppose that we model both intensive and extensive margins of labor supply using the fixed cost of working in the utility.[1] Then, conditional on working, the fixed cost is sunk and the first order condition for hours (FOC) is satisfied: $v'(1-h) = \tilde{w}u'(c)$. Using this, one can write the implied hours as a function of consumption and wage $h(c,\tilde{w})$ which is an increasing (decreasing) function in wage (consumption).[2] They reflect substitution and wealth effects of the labor supply at the intensive margin. Note that given consumption, lower wages drive down the hours through the substitution effect. Thus, for some low enough wage, the hours will reach to zero. We simply write this as $\lim_{\tilde{w}\to 0} h(c,\tilde{w}) = 0$. The values within the period are given by

$$\begin{aligned} V^E(c,\tilde{w}) &= u(c) + v(1-h(c,\tilde{w})) - F, \\ V^N(c,\tilde{w}) &= u(c - \tilde{w}h(c,\tilde{w})). \end{aligned}$$

First, $\lim_{\tilde{w}\to 0}[V^N(c,\tilde{w}) - V^E(c,\tilde{w})] = F > 0$. Second, note that

$$\begin{aligned} \frac{\partial}{\partial \tilde{w}}(V^N - V^E) &= -u'(c - \tilde{w}h(c,\tilde{w}))[\tilde{w}h_w(c,\tilde{w}) + h(c,\tilde{w})] + \phi'(1-h(c,\tilde{w}))h_w(c,\tilde{w}) \\ &= \tilde{w}h_w(c,\tilde{w})[u'(c) - u'(c - \tilde{w}h(c,\tilde{w}))] - h(c,\tilde{w})u'(c - \tilde{w}h(c,\tilde{w})) \\ &< 0. \end{aligned}$$

In the second equality, I used $v'(1-h(c,\tilde{w})) = \tilde{w}u'(c)$. The inequality is from the fact

[1] It is well known that discontinuities in hours decision rule are required to generate the participation decisions. It could be generated either by consumption (or hours) costs for working (e.g. Cogan (1981)) or by nonlinear earnings convex in hours (e.g. Prescott et al. (2009) and Rogerson and Wallenius (2009)). The key is the penalty at low hours. Here I consider the fixed cost of working in terms of utility: if the agent considers low hours, the resulting consumption increase will not be enough to compensate the fixed cost of working and the lost leisure (see e.g. Blundell et al. (2011)). Note that the first order condition approach in Example 1 cannot be applied since the utility function is not continuous in hours, so we compare the utility levels when working and not working.

[2] To see this, write the FOC as $v'(1-h(c,\tilde{w})) = \tilde{w}u'(c)$. Applying the implicit function theorem gives

$$h_{\tilde{w}}(c,\tilde{w}) = -\frac{u'(c)}{v''(1-h(c,\tilde{w}))} > 0 \quad \text{and} \quad h_c(c,\tilde{w}) = -\frac{\tilde{w}u''(c)}{v''(1-h(c,\tilde{w}))} < 0.$$

that $u(\cdot)$ is strictly increasing and concave. Finally, $\lim_{\tilde{w}\to\bar{w}_c}[V^N(c,\tilde{w}) - V^E(c,\tilde{w})] < 0$, where $\bar{w}_c$ is the wage level given c such that $c - \bar{w}_c h(c,\bar{w}_c) = 0$ as in Assumption 1. With the same interpretation, there is a single crossing between V^E and V^N. Now to see the increasing property of the reservation wage curve, set $V^E(c, w^*(c)) = V^E(c, w^*(c))$. The implicit function theorem yields

$$\frac{dw^*(c)}{dc} = \frac{[u'(c - \tilde{w}h(c,\tilde{w})) - u'(c)] + \tilde{w}h_c(c,\tilde{w})[u'(c) - u'(c - \tilde{w}h(c,\tilde{w}))]}{\tilde{w}h_w(c,\tilde{w})[u'(c - \tilde{w}h(c,\tilde{w})) - u'(c)] + h(c,\tilde{w})u'(c - \tilde{w}h(c,\tilde{w}))} > 0,$$

which holds because $u(\cdot)$ is strictly increasing and concave, $h_w > 0$, and $h_c < 0$.

Part (II). With indivisible labor and non-separable preferences,

$$\begin{aligned} V^E(c,\tilde{w}) &= u(c, 1-\bar{h}), \\ V^N(c,\tilde{w}) &= u(c - \tilde{w}\bar{h}, 1). \end{aligned}$$

Note that $V^N(c,0) - V^E(c,0) = u(c,1) - u(c,1-\bar{h}) > 0$, $\frac{\partial}{\partial\tilde{w}}(V^N - V^E) = -\bar{h}u_c(c - \tilde{w}\bar{h}, 1) < 0$, and under Assumption 1, $\lim_{\tilde{w}\to\bar{w}_c}[V^N(c,\tilde{w}) - V^E(c,\tilde{w})] < 0$. Thus, again, there is a single crossing between V^E and V^N. Note that this does not depend on whether $u_{cl} > 0$ or $u_{cl} < 0$. Now to determine the reservation wage curve, set $u(c, 1-\bar{h}) = u(c - w^*\bar{h}, 1)$. Then,

$$\frac{dw^*(c)}{dc} = \frac{u_c(c - \tilde{w}\bar{h}, 1) - u_c(c, 1-\bar{h})}{u_c(c - \tilde{w}\bar{h}, 1)}.$$

If $u_{cl} > 0$ then it follows that

$$u_c(c - \tilde{w}h, 1) \overset{(i)}{>} u_c(c, 1) \overset{(ii)}{>} u_c(c, 1-h), \tag{1.1}$$

where (i) is from the strict concavity in consumption and (ii) is because $u_{cl} > 0$. Thus, the reservation wage is increasing in consumption if $u_{cl} > 0$. If $u_{ul} < 0$, the sign of the slope of the reservation wage curve cannot be determined. This is mainly because of the discreteness of the participation decision. Unlike this case, if we model both margins of labor supply so that the FOC is met with equality conditional on working, then the reservation wage curve is increasing in consumption even when $u_{cl} < 0$ (see below).

Part (III) (i) ($u_{cl} > 0$). Suppose that we model both margins of labor supply while the period utility function is non-separable between consumption and leisure. Conditional on working, the FOC is $u_l(c, 1-h(c,\tilde{w})) = \tilde{w}u_c(c, 1-h(c,\tilde{w}))$, where I write the implied hours as a function of consumption and wage $h(c,\tilde{w})$. Then,

$$h_{\tilde{w}}(c,\tilde{w}) = \frac{u_c}{\tilde{w}u_{cl} - u_{ll}} > 0 \quad \text{and} \quad h_c(c,\tilde{w}) = \frac{\tilde{w}u_{cc} - u_{cl}}{\tilde{w}u_{cl} - u_{ll}} < 0, \tag{1.2}$$

where the signs come from the strict concavity and $u_{cl} > 0$. Again, given consumption, lower wage dcreases hours by the substitution effect and $\lim_{\tilde{w}\to 0} h(c,\tilde{w}) = 0$. The values for the period are

$$\begin{aligned} V^E(c,\tilde{w}) &= u(c, 1 - h(c,\tilde{w})) - F, \\ V^N(c,\tilde{w}) &= u(c - \tilde{w}h(c,\tilde{w}), 1). \end{aligned}$$

First, $\lim_{\tilde{w}\to 0}(V^N - V^E) = F > 0$. Second, write $h = h(c,\tilde{w})$ for simplicity. Then,

$$\begin{aligned} \frac{\partial(V^N - V^E)}{\partial \tilde{w}} &= -u_c(c - \tilde{w}h, 1)(\tilde{w}h_{\tilde{w}}(c,\tilde{w}) + h) + u_l(c, 1-h)h_{\tilde{w}}(c,\tilde{w}) \\ &= -u_c(c - \tilde{w}h, 1)h - \tilde{w}h_{\tilde{w}}(c,\tilde{w})\big[u_c(c - \tilde{w}h, 1) - u_c(c, 1-h)\big], \end{aligned} \tag{1.3}$$

where I used the FOC $u_l(c, 1 - h(c,\tilde{w})) = \tilde{w}u_c(c, 1 - h(c,\tilde{w}))$. Since $u_c > 0$ and $h_{\tilde{w}} > 0$, we have the negative sign for the whole expression if the terms in the squared brackets are positive. To see this, note that

$$u_c(c - \tilde{w}h, 1) \overset{(i)}{>} u_c(c, 1) \overset{(ii)}{>} u_c(c, 1-h). \tag{1.4}$$

(i) is from the strict concavity in consumption and (ii) is because $u_{cl} > 0$. Thus, $\frac{\partial(V^N - V^E)}{\partial \tilde{w}} < 0$. Finally, we have $\lim_{\tilde{w}\to\bar{w}_c}[V^N(c,\tilde{w}) - V^E(c,\tilde{w})] < 0$, where $\bar{w}_c$ is the wage level such that $c - \bar{w}_c h(c,\bar{w}_c) = 0$ under Assumption 1. With the same interpretation, there is a single crossing between V^E and V^N. To see that the reservation wage curve is increasing, set $u(c, 1 - h(c,w^*)) - F = u(c - w^*h(c,w^*), 1)$. We have

$$\frac{dw^*(c)}{dc} = \frac{\big[u_c(c - \tilde{w}h, 1) - u_c(c, 1-h)\big] - \tilde{w}h_c(c,\tilde{w})\big[u_c(c - \tilde{w}h, 1) - u_c(c, 1-h)\big]}{\tilde{w}h_{\tilde{w}}(c,\tilde{w})\big[u_c(c - \tilde{w}h, 1) - u_c(c, 1-h)\big] + hu_c(c - \tilde{w}h, 1)}. \tag{1.5}$$

From (1.2), (1.4), and $u_c > 0$, it follows that $\frac{dw^*(c)}{dc} > 0$.

Part (III) (ii) ($u_{cl} < 0$, and consumption and leisure are normal goods). By normals goods, we mean that, holding the wage constant, the demand for consumption and leisure increase as the non-labor income increases. To see what this implies, note that conditional on working, the nested maximization problem can be written as

$$\max_{c,h}\ u(c, 1-h) - F \quad s.t. \quad c = \tilde{w}h + b,$$

where $b > 0$. Since u is strictly increasing and concave, the interior solution for hours satisfies the second order necessary condition (SOC): $\tilde{w}(\tilde{w}u_{cc} - u_{cl}) - (\tilde{w}u_{cl} - u_{ll}) < 0$. Thus, if leisure is a normal good, it holds that

$$\left.\frac{dh}{db}\right|_{d\tilde{w}=0} = \frac{\tilde{w}u_{cc} - u_{cl}}{-(\tilde{w}(\tilde{w}u_{cc} - u_{cl}) - (\tilde{w}u_{cl} - u_{ll}))} < 0,$$

which together with the SOC implies that (i) $\tilde{w}u_{cc} - u_{cl} < 0$. In a similar fashion, if consumption is a normal good, then we have (ii) $\tilde{w}u_{cl} - u_{ll} > 0$. Thus, in this case, the expressions in (1.2) hold. As the wage decreases, holding consumption constant, the substitution effect drives down the hours: $\lim_{\tilde{w}\to 0} h(c, \tilde{w}) = 0$. As the previous case, we have $\lim_{\tilde{w}\to 0}[V^N - V^E] > 0$. Next, consider the equation (1.3). To determine the sign of the terms in the squared brackets, note that

$$\begin{aligned} u_c(c - \tilde{w}h, 1) - u_c(c, 1-h) &\overset{(i)}{\simeq} u_{cc}(c, 1-h)(-\tilde{w}h) + u_{cl}(c, 1-h)h \\ &= -h\big[\tilde{w}u_{cc}(c, 1-h) - u_{cl}(c, 1-h)\big] \\ &\overset{(ii)}{>} 0, \end{aligned} \tag{1.6}$$

where $u(c - \tilde{w}h, 1)$ is first order approximated around $u(c, 1-h)$ for (i), and (ii) is from the fact that leisure is a normal good, $\tilde{w}u_{cc} - u_{cl} < 0$. Thus, $V^N - V^E$ is decreasing. Applying Assumption 1, we have the single crossing. Further, (1.5) hold by (1.6): the reservation wage curve is increasing. □

2 Two Earners' Problem

While adding another earner in the household complicates the problem, the proposition above turns out to be valid if labor is indivisible.

Proposition (Two Earners' Problem)**.** *Consider a class of models for two earners defined in the main text. Then,*

(I) (Separable Preferences, Extensive Margin) Suppose that the period utility function is

$$\tilde{u}(c, h) = u(c) + v_H(1 - h_H) + v_S(1 - h_S),$$

where $u(\cdot)$ is strictly increasing and concave, $v(\cdot)$ is strictly increasing, $v(1)$ is nor-

malized to zero, and $h_S, h_H \in \{0, \bar{h}\}$*. Then,*

(i) Conditional on the spouse being not employed, given c^{EN}*, there exists a unique wage level* $w^*_{H1}(c^{EN})$ *such that heads with* $\tilde{w}_H \geq w^*_{H1}(c^{EN})$ *work and other heads with* $\tilde{w}_H < w^*_{H1}(c^{EN})$ *do not work.*

(ii) Conditional on the spouse being employed, given c^{EE}*, there exists a unique wage level* $w^*_{H2}(c^{EE})$ *such that heads with* $\tilde{w}_H \geq w^*_{H2}(c^{EE})$ *work and other heads with* $\tilde{w}_H < w^*_{H2}(c^{EE})$ *do not work.*

(iii) If $c^{EN} = c^{EE}$ *between any two households, then* $w^*_{H1}(c^{EN}) = w^*_{H2}(c^{EE})$*. Thus, the reservation wage for heads can be written as* $w^*_H(c^{E\cdot})$*, where* $c^{E\cdot}$ *is the household consumption when heads work, irrespective of the spouse's working status or wage.*

(iv) By symmetry, (i)–(iii) hold for the reservation wage for spouses. Regardless of the head's working status and wage, the reservation wage for spouses $w^*_S(c^{\cdot E})$ *is unique.*

(v) $w^*_H(\cdot)$ *and* $w^*_S(\cdot)$ *are strictly increasing in consumption.*

(vi) $w^*_H(\cdot)$ *and* $w^*_S(\cdot)$ *are independent of any state variables.*

(II) (Non-separable Preferences, Extensive Margin) Suppose that the period utility function is

$$\tilde{u}(c, h) = u(c, 1 - h_H, 1 - h_S),$$

where $u(\cdot, 1 - h_H, 1 - h_S)$ *is strictly increasing and strictly concave in consumption,* $u(c, \cdot, \cdot)$ *is strictly increasing in each member's leisure, and* $h_H, h_S \in \{0, \bar{h}\}$*. If* $u_{cl} > 0$*, then (i), (ii), (v), and (vi) hold. Counterparts of (i), (ii), (v), and (vi) for the spouse reservation wage hold. If* $u_{cl} < 0$ *then (i), (ii), and (vi) and the counterparts of them for the spouse's reservation wage hold.*

Proof. **Part (I).** See Appendix B in the main text.

Part (II). In this case, the nested maximization problem is

$$\max\left\{\max_{c^{EN}, h_H}\left[V^{EN}(c^{EN}, \tilde{w}_H), V^{NN}(c^{EN}, \tilde{w}_H)\right], \max_{c^{EE}, h_H}\left[V^{EE}(c^{EE}, \tilde{w}_H), V^{NE}(c^{EE}, \tilde{w}_H)\right]\right\}, \tag{2.1}$$

where

$$
\begin{aligned}
V^{EN}(c^{EN}, \tilde{w}_H) &= u(c^{EN}, 1-\bar{h}, 1), \\
V^{NN}(c^{EN}, \tilde{w}_H) &= u(c^{EN} - \tilde{w}_H \bar{h}, 1, 1), \\
V^{EE}(c^{EE}, \tilde{w}_H) &= u(c^{EE}, 1-\bar{h}, 1-\bar{h}), \\
V^{NE}(c^{EE}, \tilde{w}_H) &= u(c^{EE} - \tilde{w}_H \bar{h}, 1, 1-\bar{h}).
\end{aligned}
$$

It is simple to check that the proof of Part (II) in Section 1 above applies to each of the maximization problem in (2.1). Thus, (i), (ii), and (vi) hold. All the remaining parts are also analogous to the discussion above. $\square$

References

Blundell, Richard, Antoine Bozio, and Guy Laroque (2011) "Extensive and Intensive Margins of Labor Supply: Working Hours in the US, UK and France," *Working Paper.*

Cogan, John F. (1981) "Fixed Costs and Labor Supply," *Econometrica*, Vol. 49, No. 4, pp. 945–963.

Prescott, Edward C., Richard Rogerson, and Johanna Wallenius (2009) "Lifetime Aggregate Labor Supply with Endogenous Workweek Length," *Review of Economic Dynamics*, Vol. 12, No. 1, pp. 23–36.

Rogerson, Richard and Johanna Wallenius (2009) "Micro and macro elasticities in a life cycle model with taxes," *Journal of Economic Theory*, Vol. 144, No. 6, pp. 2277–2292.