Intersectoral Linkages, Diverse Information, and Aggregate Dynamics: Online Appendix

Manoj Atolia*
Florida State University

Ryan Chahrour†
Boston College

October 17, 2019

Abstract

We present technical details and proofs of theorems for Intersectoral Linkages, Diverse Information, and Aggregate Dynamics published in the *Review of Economic Dynamics*.

*Department of Economics, Florida State University, Tallahassee, FL 32306, U.S.A. Telephone: 850-644-7088. Email: matolia@fsu.edu.

†Department of Economics, Boston College, Chestnut Hill, MA 02467, U.S.A. Telephone: 617-552-0902. Email: ryan.chahrour@bc.edu.

A Steady-State Production Parameters

With a nested-CES production structure, the mapping between long-run sector shares and the parameters of production is non trivial. In this appendix, we describe in detail the steps required to infer these parameters. Recall that we take $p = 1$ to be the numeraire in the economy. In steady state, the following sector-specific equations must hold for each sector j:

$$\lambda_i = C_i^{-\frac{1}{\tau}}(1-L_i)^{\varphi(1-\frac{1}{\tau})} \tag{1}$$

$$\lambda_i w_i = \varphi C_i^{1-\frac{1}{\tau}}(1-L_i)^{\varphi(1-\frac{1}{\tau})-1} \tag{2}$$

$$w_i = P_i F_{l,i} \tag{3}$$

$$P_j = P_i F_{x_{ij},i} \quad \forall j \text{ s.t. } a_{ij} > 0 \tag{4}$$

$$1 = \beta(P_j F_{k,i} + 1 - \delta) \tag{5}$$

$$Z_i = a_i P_i^{-\zeta} y \tag{6}$$

$$Q_i = Z_i + \sum_j X_{ji} \tag{7}$$

$$P_i Z_i = C_i + I_i \tag{8}$$

$$Q_i = F(K_i, L_i, \{X_{ij}\}) \tag{9}$$

$$I_i = \delta K_i \tag{10}$$

where

$$F(K_i, L_i, \{X_{ij}\}) = \left[\left\{\sum_j a_{ij} X_{ij}^{1-\frac{1}{\xi}}\right\}^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\xi}}} + \left\{a_{il} L_i^{1-\frac{1}{\kappa}} + a_{ik} K_i^{1-\frac{1}{\kappa}}\right\}^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\kappa}}}\right]^{\frac{1}{1-\frac{1}{\sigma}}} \tag{11}$$

and

$$F_{l,i} = Q_i^{\frac{1}{\sigma}} \left\{a_{il} L_i^{1-\frac{1}{\kappa}} + a_{ik} K_i^{1-\frac{1}{\kappa}}\right\}^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\kappa}}-1} a_{il} L_i^{-\frac{1}{\kappa}} \tag{12}$$

$$F_{k,i} = Q_i^{\frac{1}{\sigma}} \left\{a_{il} L_i^{1-\frac{1}{\kappa}} + a_{ik} K_i^{1-\frac{1}{\kappa}}\right\}^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\kappa}}-1} a_{ik} K_i^{-\frac{1}{\kappa}} \tag{13}$$

$$F_{x_{ij},i} = Q_i^{\frac{1}{\sigma}} \left\{\sum_j a_{ij} X_{ij}^{1-\frac{1}{\xi}}\right\}^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\xi}}-1} a_{ij} X_{ij}^{-\frac{1}{\xi}}. \tag{14}$$

Moreover, the following aggregate conditions must also hold:

$$Y = \left\{\sum_{i=1}^{N} a_i^{\frac{1}{\zeta}} Z_i^{1-\frac{1}{\zeta}}\right\}^{\frac{1}{1-\frac{1}{\zeta}}} \tag{15}$$

We proceed by fixing the share of good i in final production, the capital share of *value added output* in sector i, and the share of sector i's revenue dedicated to purchasing inputs from sector j. Additionally, we normalize the steady-state prices of all intermediate goods to $P_i = 1$. Call these shares ψ_{iy}, ψ_{ik}, and ψ_{ij}, respectively. Note that $\sum_{i=1}^{N} \psi_{iy}$ must equal one. These values, along with the normalization of aggregate output, $Y = 1$, fix the production parameters $a_i, a_{ij}, a_{ik}, a_{il}$. Since we have little a priori guidance on the value of φ, we calibrate φ to match a value for the steady-state Frisch elasticity.

From equation (6) and the normalization $Y = P_i = 1$, it immediately follows that

$$Z_i = a_i = \psi_{iy}. \tag{16}$$

Substituting the shares of revenue devoted to intermediate inputs into the market-clearing condition in (7), we have that

$$Q_i = Z_i + \sum_j \psi_{ji} \frac{P_j Q_j}{P_i}. \tag{17}$$

Combining the N equations yields a matrix expression for the values of $P_i Y_i$,

$$\mathbf{PQ} = (I - IO')^{-1} \mathbf{PZ} \tag{18}$$

where boldface letters represent the vector of sector values (e.g., $\mathbf{P} = [P_1, P_2, ..., P_n]'$), and IO is matrix of intermediate shares defined in the text. Having solved for the vector $\mathbf{PQ}$, we can directly back out the values of sectoral production, Q_i. It follows from the definition of $\psi_{ij} \equiv \frac{P_j X_{ij}}{P_i Q_i}$ that

$$X_{ij} = P_i Q_i \frac{\psi_{ij}}{P_j}. \tag{19}$$

Multiply the intermediate inputs' first-order condition in equation (4) by X_{ij}, and sum sectors i for which $a_{ij} > 0$ to get

$$\sum_j P_j X_{ij} = P_i Q_i^{\frac{1}{\sigma}} \left\{ \sum_j a_{ij} X_{ij}^{1-\frac{1}{\xi}} \right\}^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\xi}} - 1} \sum_j a_{ij} X_{ij}^{1-\frac{1}{\xi}} \tag{20}$$

$$= P_i Q_i^{\frac{1}{\sigma}} \left\{ \sum_j a_{ij} X_{ij}^{1-\frac{1}{\xi}} \right\}^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\xi}}}, \tag{21}$$

which can easily be solved for $\Omega_{1,i} \equiv \sum_j a_{ij} X_{ij}^{1-\frac{1}{\xi}}$. Plugging this value back into equation (4) yields a solution for a_{ij}:

$$a_{ij} = \frac{P_j}{P_i} X_{ij}^{\frac{1}{\xi}} Q_i^{-\frac{1}{\sigma}} \Omega_{1,i}^{1-\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\xi}}}. \tag{22}$$

Using a similar procedure, we can now solve a_{ik} and a_{il}. First, use the production function to solve for $\Omega_{2,i} \equiv a_{il} L_i^{1-\frac{1}{\kappa}} + a_{ik} K_i^{1-\frac{1}{\kappa}}$:

$$\Omega_{2,i} = \left(Q_i^{1-\frac{1}{\sigma}} - \Omega_{1,i}^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\xi}}} \right)^{\frac{1-\frac{1}{\kappa}}{1-\frac{1}{\sigma}}}. \tag{23}$$

To back out K_i, note that

$$\begin{aligned} \psi_{ik} &\equiv \frac{P_i F_{k,i} K_i}{P_i Q_i - \sum_j P_j X_{ij}} \\ &= \frac{F_{k,i} K_i / Q_i}{1 - \sum_j \psi_{ij}}. \end{aligned} \tag{24}$$

Rearranging equation (5) gives the following expression for capital in sector i:

$$K_i = \frac{P_i \psi_{ik} Q_i}{\beta^{-1} - 1 + \delta} \left(1 - \sum_j \psi_{ij} \right). \tag{25}$$

Sectoral investment is now simply $I_i = \delta K_i$. To solve for a_{ik}, use the above result and the expression for $F_{k,i}$ to find

$$a_{ik} = \psi_{ik} \left(1 - \sum_j \psi_{ij} \right) q_i^{1-\frac{1}{\sigma}} \Omega_{2,i}^{1-\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\kappa}}} K_i^{\frac{1}{\kappa}-1}. \tag{26}$$

From this, we can also easily determine

$$a_{il} L_i^{1-\frac{1}{\kappa}} = \Omega_{2,i} - a_{ik} K_i^{1-\frac{1}{\kappa}}. \tag{27}$$

Using island market clearing in equation (8), sectoral output and investment can be used to compute consumption on each island. Finally, to determine sectoral labor, use consumer equations (1) and (2) to derive the relation $\varphi = \frac{W_i}{C_j}(1 - L_i)$, which implies that

$$W_i = C_i \varphi + W_i L_i. \tag{28}$$

From the labor-choice condition in equation (3), we have

$$W_i L_i = P_i Q_i^{\frac{1}{\sigma}} \Omega_{2,i}^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\kappa}}-1} a_{il} L_i^{1-\frac{1}{\kappa}}, \tag{29}$$

which can be plugged back into equation (28) to determine the wage. The steady-state value of L_i follows directly. Finally, equation (27) can be used to solve for a_{il}, and consumer equation (1) can be used to determine λ_i.

B Linearized Model: Derivation and Proofs

We derive linearized equilibrium conditions for the model when intermediate production is Cobb-Douglas in all inputs, sectoral weights in final-good production are symmetric, capital depreciates fully each period, labor is supplied inelastically, and preferences take a CRRA-form with an elasticity of intertemporal substitution equal to τ.

The linearized first-order condition of the consumer in sector i is

$$\hat{c}_{i,t} = -\tau \hat{\lambda}_{i,t}. \tag{30}$$

Intermediate production is characterized by the linearized production function

$$\hat{q}_{i,t} = \hat{\theta}_{i,t} + \alpha_{ik}\hat{k}_{i,t} + \sum_{j=1}^{N} \alpha_{ij}\hat{x}_{ij,t}, \tag{31}$$

where the parameters α_{ik} denote the capital share of output in sector i and α_{ij} the share of good j used in the production of sector i. We assume that $\alpha_{ik} + \sum_{j=1}^{N} \alpha_{ij} = 1 - \alpha_l < 1$, so that the share of inelastically supplied labor is positive (equivalently, that the economy exhibits decreasing returns to scale.) To simplify summation statements, we adopt the convention that $\hat{x}_{ij,t} = 0$ whenever $\alpha_{ij} = 0$.

The firm's optimal choice of input $\hat{x}_{ij,t}$ is given by

$$\hat{p}_{j,t} = \hat{p}_{i,t} + \hat{q}_{i,t} - \hat{x}_{ij,t}. \tag{32}$$

Linearizing the intertemporal optimality condition of the firm, and using the consumer's first-order condition to substitute out $\hat{\lambda}_{i,t}$ yields

$$-\frac{1}{\tau} E_t^i \left[\hat{c}_{i,t} - \hat{c}_{i,t+1}\right] = E_t^i \left[\hat{p}_{i,t+1} + \hat{q}_{i,t+1} - \hat{k}_{i,t+1}\right]. \tag{33}$$

Final-good aggregation with symmetric weights implies that

$$\hat{y}_t = \frac{1}{N} \sum_{i=1}^{N} \hat{z}_{i,t}, \tag{34}$$

with $\hat{z}_{i,t}$ demanded according to

$$\hat{z}_{i,t} = \hat{y}_t - \zeta \hat{p}_{i,t}. \tag{35}$$

Sectoral market clearing implies

$$\hat{q}_{i,t} = s_{iz}\hat{z}_{i,t} + (1 - s_{iz}) \sum_{j=1}^{N} \eta_{ji}\hat{x}_{ji,t}, \tag{36}$$

$$\hat{p}_{i,t} + \hat{q}_{i,t} = s_{ic}\hat{c}_{i,t} + s_{ik}\hat{k}_{i,t+1} + (1 - s_{ic} - s_{ik}) \sum_{j=1}^{N} \omega_{ij}(\hat{p}_{j,t} + \hat{x}_{ij,t}), \tag{37}$$

where s_{iz} is the steady-state share of sector i output devoted to final-good production, η_{ji}

is the fraction of sector i good's usage as an intermediate devoted to sector j, s_{ic} and s_{ik} are the shares of gross value of sectoral output dedicated to consumption and investment, respectively, and ω_{ij} is the fraction of sector i's payments for intermediates going to sector j.

B.1 Aggregate Representations

We first derive a matrix-representation of the sectoral economy's equilibrium conditions. To begin, rearrange the firm's input choice in equation (32) to derive an expression for the $\hat{x}_{ij,t}$,

$$\hat{x}_{ij,t} = \hat{p}_{i,t} + \hat{q}_{i,t} - \hat{p}_{j,t}. \tag{38}$$

Substituting this expression into (31) delivers the following expression for the vector of sectoral gross output,

$$\mathbf{q}_t = \boldsymbol{\theta}_t + (\Phi_x - IO)\mathbf{p}_t + \Phi_x \mathbf{q}_t + \Phi_k \mathbf{k}_t. \tag{39}$$

In (39) above, bold-face type represents the vector $\mathbf{x}_t \equiv [\hat{x}_{1,t}, \hat{x}_{2,t}, ..., \hat{x}_{N,t}]'$ for any variable $\hat{x}_{i,t}$,Φ_k is a diagonal matrix with entries α_{ik}, and Φ_x is a diagonal matrix whose entries are the row-sums of the input-output matrix, IO, $\alpha_{ix} \equiv \sum_{j=1}^{N} \alpha_{ij}$.

Lemma. *Together, equations (36) and (38) imply*

$$\mathbf{p}_t + \mathbf{q}_t = \Psi(\mathbf{z}_t + \mathbf{p}_t), \tag{40}$$

where $\Psi \equiv \Phi_z(I - IO')^{-1}$ *and* Φ_z *is a diagonal matrix with entries* $s_{iz} = z_i/q_i = 1/q_i$ *. Moreover, the rows of* Ψ *sum to one.*

Proof. Substituting (38) into (36) yields

$$\hat{q}_{i,t} = s_{iz}\hat{z}_{i,t} + (1 - s_{iz}) \sum_{j=1}^{N} \eta_{ji}(\hat{p}_{j,t} + \hat{q}_{j,t} - \hat{p}_{i,t}). \tag{41}$$

Writing (41) in matrix form, we have

$$\mathbf{q}_t = \Phi_z \mathbf{z_t} + M_1(\mathbf{p}_t + \mathbf{q}_t) - M_2 \mathbf{p}_t. \tag{42}$$

Notice first that $M_2 = (I - \Phi_z)$, since the $\hat{p}_{i,t}$ in the sum in (41) are not indexed by j and $\sum_{j=1}^{N} \eta_{ji} = 1$. Second notice that, from steady-state share relations, we have

$$M_1 = \begin{bmatrix} \frac{x_{11}}{q_1} & \frac{x_{21}}{q_1} & \cdots & \frac{x_{j1}}{q_1} \\ \frac{x_{12}}{q_2} & \frac{x_{22}}{q_2} & & \\ \vdots & & \ddots & \\ \frac{x_{1i}}{q_i} & & & \frac{x_{ji}}{q_i} \end{bmatrix} = \begin{bmatrix} \frac{\alpha_{11}q_1}{q_1} & \frac{\alpha_{21}q_2}{q_1} & \cdots & \frac{\alpha_{j1}q_j}{q_1} \\ \frac{\alpha_{12}q_1}{q_2} & \frac{\alpha_{22}q_2}{q_2} & & \\ \vdots & & \ddots & \\ \frac{\alpha_{1i}q_1}{q_i} & & & \frac{\alpha_{ji}q_j}{q_i} \end{bmatrix} = \Phi_z IO' \Phi_z^{-1} \tag{43}$$

where we have used our steady-state result that $z_i = 1$ so that $s_{iz} = 1/q_i$. Hence (42) becomes

$$(I - \Phi_z IO' \Phi_z^{-1})(\mathbf{q}_t + \mathbf{p}_t) = \Phi_z(\mathbf{z_t} + \mathbf{p_t}). \tag{44}$$

Solving for $\mathbf{q}_t + \mathbf{p}_t$ and simplifying yields the result.

To see that the rows of Ψ sum to one, notice that since the entries of Φ_z^{-1} are just the values q_i, the row sums of $\Psi^{-1} = (I - IO')\Phi_z^{-1}$ can be written as

$$q_i - \sum_{j=1}^{n} \alpha_{ji} q_j = z_i = 1. \tag{45}$$

The inverse of any matrix with unit rows sums has, itself, unit row sums.

□

Combining demand and aggregation equations of the final-good sector, (34) and (35), yields

$$\mathbf{z}_t = A_v \mathbf{z}_t - \zeta \mathbf{p}_t, \tag{46}$$

where the matrix A_v is defined as the $N \times N$ matrix with constant entries $1/N$. For future reference, let a_v be the first row of A_v, i.e., the row vector that averages the columns of any matrix it pre-multiplies. Equation (46) can be rearranged as

$$\mathbf{p}_t + \mathbf{z}_t = A_v \mathbf{z}_t + (1-\zeta)\mathbf{p}_t. \tag{47}$$

Combining this with (40) and using the result that $\Psi A_v = A_v$ due to the unit row sums of Ψ, we get

$$\mathbf{p}_t + \mathbf{q}_t = A_v \mathbf{z}_t + (1-\zeta)\ \Psi \mathbf{p_t}. \tag{48}$$

Again using (32), the island-level resource constraint in equation (37) reduces to

$$\mathbf{p}_t + \mathbf{q}_t = \phi_c \mathbf{c}_t + (1-\phi_c)\mathbf{k}_{t+1}, \tag{49}$$

where ϕ_c represents the share of consumption in value-added production, which is common across sectors. Finally, from (33) we have the equation describing intertemporal choice in the economy,

$$\mathbf{k}_{t+1} = E_t^f [\frac{1}{\tau}(\mathbf{c}_t - \mathbf{c}_{t+1}) + \mathbf{p}_{t+1} + \mathbf{q}_{t+1}]. \tag{50}$$

Lemma 1. *Suppose that $\rho^A = \rho_i^\varsigma = \rho$. Then, there exists a vector of weights, w, and a scalar, $\tilde{\alpha}$, such that whenever*

$$(1-\zeta) w \Psi \mathbf{p}_t = 0, \qquad \forall t, \tag{51}$$

the full-information economy has a (single-sector) aggregate representation given by

$$\tilde{y}_t = \tilde{\theta}_t + \tilde{\alpha}\tilde{k}_t, \tag{52}$$

$$\tilde{y}_t = \phi_c \tilde{c}_t + (1-\phi_c)\tilde{k}_{t+1}, \tag{53}$$

$$\tilde{k}_{t+1} = E_t \left[\frac{1}{\tau}(\tilde{c}_t - \tilde{c}_{t+1}) + \tilde{y}_{t+1} \right], \tag{54}$$

$$\tilde{\theta}_{t+1} = \rho\tilde{\theta}_t + \sigma\tilde{\epsilon}_{t+1}, \tag{55}$$

with $\tilde{y}_t \equiv a_v \mathbf{z}_t$*,* $\tilde{c}_t \equiv w\mathbf{c}_t$*,* $\tilde{k}_t \equiv w\mathbf{k}_t$*,* $\tilde{\theta}_t \equiv a_v H^{-1}\boldsymbol{\theta}_t$ *, and* $H \equiv (1-\zeta^{-1})(I-\Phi_x)\Psi + \zeta^{-1}(I - IO)$*.*

Proof. Rearrange equation (39) to provide an explicit expression for $\mathbf{q}_t$,

$$\mathbf{q}_t = (I-\Phi_x)^{-1}\boldsymbol{\theta}_t + (I-\Phi_x)^{-1}(\Phi_x - IO)\mathbf{p}_t + (I-\Phi_x)^{-1}\Phi_k\mathbf{k}_t. \tag{56}$$

Similarly, rearrange (40) to get

$$\mathbf{q}_t = (\Psi - I)\mathbf{p}_t + \Psi\mathbf{z}_t. \tag{57}$$

Equate the right hand sides of (56) and (57) to get

$$(\Psi - I)\mathbf{p}_t + \Psi\mathbf{z}_t = (I-\Phi_x)^{-1}\boldsymbol{\theta}_t + (I-\Phi_x)^{-1}(\Phi_x - IO)\mathbf{p}_t + (I-\Phi_x)^{-1}\Phi_k\mathbf{k}_t. \tag{58}$$

From (46) we have

$$\mathbf{p}_t = \frac{1}{\zeta}(A_v - I)\mathbf{z}_t. \tag{59}$$

Substituting this expression for $\mathbf{p_t}$ into (58) and solving for $\mathbf{z}_t$, we have

$$\mathbf{z}_t = H^{-1}\left(\boldsymbol{\theta}_t + \Phi_k\mathbf{k}_t\right), \tag{60}$$

where $H \equiv (I-\Phi_x)\left[(\Psi - I)\frac{1}{\zeta}(A_v - I) + \Psi - (1-\Phi_x)^{-1}(\Phi_x - IO)\frac{1}{\zeta}(A_v - I)\right]$. Recalling that $\Psi A_v = A_v$, the matrix H simplifies to

$$H = (1-\zeta^{-1})(I-\Phi_x)\Psi + \zeta^{-1}(I - IO). \tag{61}$$

Next, use (60) to eliminate $\mathbf{z}_t$ from the right-hand side of equations (46) and (59),

$$\mathbf{z}_t = A_v H^{-1}\left(\boldsymbol{\theta}_t + \Phi_k\mathbf{k}_t\right) - \zeta\mathbf{p}_t \tag{62}$$

$$\mathbf{p}_t = \frac{1}{\zeta}(A_v - I)H^{-1}\left(\boldsymbol{\theta}_t + \Phi_k\mathbf{k}_t\right). \tag{63}$$

Notice that, by the construction of A_v, the rows of the matrix $A_v H^{-1}$ are all identical. Hence, equation (62) shows that equilibrium value added production is equal to the sum of one term that is constant across all sectors — depending on productivity and initial capital stocks — and a second term that depends each sector's own price.

Using the next period version of (48) to eliminate $\mathbf{q}_{t+1} + \mathbf{p}_{t+1}$ from the the intertemporal condition (50), we arrive at

$$\mathbf{k}_{t+1} = E_t^f\left[\frac{1}{\tau}(\mathbf{c}_t - \mathbf{c}_{t+1}) + A_v\mathbf{z}_{t+1} + (1-\zeta)\Psi\mathbf{p}_{t+1}\right]. \tag{64}$$

Define the scalar $\tilde{\alpha}$ such that $\tilde{\alpha}a_v = a_v H^{-1}\Phi_k A_v$. Such a scalar exists because the right hand side above is, by construction, a row-vector of constant entries. Multiplying both sides of equation (64) by the weighting vector $w \equiv \frac{1}{\tilde{\alpha}} a_v H^{-1} \Phi_k$ we have

$$w\mathbf{k}_{t+1} = E_t^f \left[\frac{1}{\tau} w(\mathbf{c}_t - \mathbf{c}_{t+1}) + a_v \mathbf{z}_{t+1} + (1-\zeta) w \Psi \mathbf{p}_{t+1} \right]. \tag{65}$$

In the above equation, $\mathbf{z}_t$ is premultiplied by a_v because the w was defined to ensure $wA_v = a_v$.

Define $\tilde{y}_t \equiv a_v \mathbf{z}_t$, $\tilde{c}_t \equiv w\mathbf{c}_t$, $\tilde{k}_t \equiv w\mathbf{k}_t$, and $\tilde{\theta}_t \equiv \tilde{\alpha} w \Phi_k^{-1} \boldsymbol{\theta}_t = a_v H^{-1} \boldsymbol{\theta}_t$. Whenever $(1-\zeta) w \Psi \mathbf{p}_t = 0$, the intertemporal condition (65) reduces to

$$\tilde{k}_{t+1} = E_t \left[\frac{1}{\tau} (\tilde{c}_t - \tilde{c}_{t+1}) + \tilde{y}_{t+1} \right]. \tag{66}$$

Premultiplying equation (62) by a_v yields

$$\tilde{y}_t = \tilde{\theta}_t + \tilde{\alpha} \tilde{k}_t \tag{67}$$

where we have used $a_v \mathbf{p}_t = 0$. Using (48) and the presumption that $(1-\zeta) w \Psi \mathbf{p}_t = 0$, we have

$$w(\mathbf{p}_t + \mathbf{q}_t) = \tilde{y}_t. \tag{68}$$

Hence, premultiplying the resource constraint in (49) by w yields

$$\tilde{y}_t = \phi_c \tilde{c}_t + (1-\phi_c) \tilde{k}_{t+1}. \tag{69}$$

Lastly, because all components of productivity have the same autocorrelation parameter, they can be summed to yield the auto-regressive process in (55).

□

Equation (51) can be satisfied in several ways to yield an aggregate representation. The most direct path is given by Corollary 1:

Corollary 1. *The economy has an aggregate representation whenever $\zeta = 1$.*

Proof. The result follows from observing that the left-hand side of (51) is premultiplied by $\zeta - 1$. □

Corollary 2. *If the input-output matrix of the economy is circulant, then the full-information economy has a (single-sector) aggregate representation that is identical to a representative agent economy whose equilibrium conditions are given by*

$$\hat{y}_t = (1-\alpha_x)^{-1} \hat{\theta}_t + (1-\alpha_x)^{-1} \alpha_k \hat{k}_t, \tag{70}$$

$$\hat{y}_t = (1-\alpha_x)^{-1} \alpha_l \hat{c}_t + (1-\alpha_x)^{-1} \alpha_k \hat{k}_t, \tag{71}$$

$$\hat{k}_{t+1} = E_t^f \left[\frac{1}{\tau} E_t^f (\hat{c}_t - \hat{c}_{t+1}) + \hat{y}_{t+1} \right], \tag{72}$$

$$\hat{\theta}_{t+1} = \rho \hat{\theta}_t + \hat{\epsilon}_{t+1}, \tag{73}$$

where α_x is the row sum of the IO matrix, $\alpha_k = 1 - \alpha_l - \alpha_x$ is the capital share of the sectoral economy, and each aggregate variable $\hat{v}_t \equiv \frac{1}{N} \sum_{i=1}^{N} \hat{v}_{i,t}$, with $\hat{v}_{i,t}$ being the corresponding sectoral variable.

Proof. Define $\hat{q}_t = \frac{1}{N}\sum_{i=1}^{N} q_{i,t}$ and define $\hat{c}_t, \hat{k}_t$ and $\hat{x}_t$ analogously. Observe that $a_v M$ is a constant vector for any matrix M with constant column sums, and hence, for circulant matrices. From equations (17) and (18), it follows that in this case the diagonal matrix Φ_z contains constant, non-zero values and so can be treated as a scalar in matrix multiplication. The circulant nature of IO similarly implies that Φ_x and Φ_k may also be treated as scalars α_x and α_k. Using these results, multiply equation (56) by a_v to find

$$\hat{q}_t = (1-\alpha_x)^{-1}\hat{\theta}_t + (1-\alpha_x)^{-1}\alpha_k \hat{k}_t \tag{74}$$

where we use the implication of the constant column sums of IO that $a_v IO \mathbf{p}_t = \alpha_x a_v \mathbf{p_t}$, while $a_v \mathbf{p}_t = 0$ by the assumption of the numeraire. Observe that given the assumption of a circulant input-output structure, $A_v \Psi = A_v$. From equation (48), we therefore have $a_v \mathbf{q_t} = a_v \mathbf{z_t} = \hat{y}_t$, so that combining with (74) yields equation (70).

Next, use the intermediate-good optimality condition in (32) to eliminate x_{ij} from equation (37):

$$\hat{p}_{i,t} + \hat{q}_{i,t} = s_{ic}\hat{c}_{i,t} + s_{ik}k_{i,t+1} + (1 - s_{ic} - s_{ik}) \sum_{j=1}^{N} \omega_{ij}(\hat{p}_{i,t} + \hat{q}_{i,t}). \tag{75}$$

Rewrite equation (75) in matrix form using the fact that $s_{ic} = \alpha_l$, $s_{ik} = \alpha_k$, and $\sum_{j=1}^{N} \omega_{ij} = 1$.

$$\mathbf{p}_t + \mathbf{q}_t = \alpha_l \mathbf{c}_t + \alpha_k \mathbf{k}_{t+1} + \alpha_x(\mathbf{p}_t + \mathbf{q_t}). \tag{76}$$

Multiplying by a_v, and solving for $\hat{q}_t = \hat{y}_t$ yields

$$\hat{y}_t = (1-\alpha_x)^{-1}\alpha_l \hat{c}_t + (1-\alpha_x)^{-1}\alpha_k \hat{k}_t, \tag{77}$$

which corresponds to equation (71).

Finally, equation (72) follows directly from summing the log-linear Euler equation (33).

□

B.2 Sectoral Irrelevance

Proposition 1. *Suppose that $\tau = \infty$, $\zeta = 1$ and $\rho^A = \rho_i^{\varsigma} = \rho$. Then (i) market consistent information, $\hat{\Omega}_t^{i,MC}$, is both aggregate and action informative and (ii) the* only *equilibrium of the diverse-information model with information sets $\hat{\Omega}_t^{i,MC}$ is that of the full information economy.*

Proof. To show part (i), we need to solve for the full information equilibrium of the economy. Substituting (62) into the intertemporal Euler equation (64) and using that $\zeta = 1$ and $H = I - IO$ yields

$$\mathbf{k}_{t+1} = E_t[A_v \mathbf{z}_{t+1}] = A_v (I - IO)^{-1} (\rho I + \Phi_k \mathbf{k}_{t+1}), \tag{78}$$

where we have also used that the full information forecast of $\boldsymbol{\theta}_{t+1} = \rho \boldsymbol{\theta}_t$. Solving for $\mathbf{k}_{t+1}$, we have

$$\mathbf{k}_{t+1} = \Lambda \boldsymbol{\theta}_t, \tag{79}$$

where

$$\Lambda \equiv \rho(I - A_v(I - IO)^{-1}\Phi_k)^{-1}A_v(I - IO)^{-1}. \tag{80}$$

Notice that equation (80) implies that

$$\Lambda = c\rho A_v(I - IO)^{-1} \tag{81}$$

for a constant c. To see this, first note that the matrix $(I - A_v(I - IO)^{-1}\Phi_k)$ is a constant row sum matrix, and therefore its inverse is as well. The result then follows from the observation that premultiplying a constant row matrix by a constant row-sum matrix is equivalent to multiplying the former by a constant.[1]

Having solved for the full information equilibrium, we want to show that market consistent information reveals sector i's optimal capital choice. Using equation (63), we have

$$\mathbf{p}_t = (A_v - I)(I - IO)^{-1}(\boldsymbol{\theta}_t + \Phi_k\Lambda\boldsymbol{\theta}_{t-1}). \tag{82}$$

Pre-multiply (82) by $(I - IO)$ and rearrange to get

$$(I - IO)\mathbf{p}_t + \boldsymbol{\theta}_t = (I - IO)A_v(I - IO)^{-1}\boldsymbol{\theta}_t + (I - IO)(A_v - I)(I - IO)^{-1}\Phi_k\Lambda\boldsymbol{\theta}_{t-1}. \tag{83}$$

Using our solution for $\mathbf{k}_{t+1}$ and equation (48), we also have

$$\mathbf{q}_t + \mathbf{p_t} = A_v(I - IO)^{-1}(\boldsymbol{\theta}_t + \Phi_k\Lambda\boldsymbol{\theta}_{t-1}). \tag{84}$$

Notice that equation (31) implies that $\hat{q}_{i,t}$ is observable to agents in sector i, since optimal time-t input and output choices of sector i are a function of sector i's market consistent information.

Equations (83) and (84) can be used to establish that market information is action informative. The left hand sides of both equations are functions of market consistent information. Hence, market consistent information is equivalent to observing two signals, $s^p_{i,t} \equiv \hat{p}_{i,t} - \sum \alpha_{ij}\hat{p}_{j,t} - \hat{\theta}_{i,t}$ and $s^q_{i,t} \equiv \hat{q}_{i,t} + \hat{p}_{i,t}$.

By construction of Φ_x, we have $(I - IO)A_v = (I - \Phi_x)A_v$, while by construction of $\tilde{\alpha}$, we have that $A_v(I - IO)^{-1}\Phi_k A_v = \tilde{\alpha}A_v$. Using these results, equations (83) and (84) imply that

$$s^p_{i,t} = (1 - \alpha_{ix})\tilde{\theta}_t + [(1 - \alpha_{ix})\tilde{\alpha} - \alpha_{ik}]\,\rho c\tilde{\theta}_{t-1} \tag{85}$$

$$s^q_{i,t} = \tilde{\theta}_t + \tilde{\alpha}\rho c\tilde{\theta}_{t-1}. \tag{86}$$

From the above, it is clear that $s^p_{i,t} - (1 - \alpha_{ix})s^q_{i,t} = -\alpha_{ik}\rho c\tilde{\theta}_{t-1}$, so that agents can always recover last period's notional aggregate productivity state from their private signals. But then agents can immediately infer this period's notional aggregate state, $\tilde{\theta}_t$, from either signal. Since, we have already shown that optimal capital choice depends only on the history of $\tilde{\theta}_t$, both action and aggregate informativeness are established.

Action-informativeness is sufficient for the full information equilibrium to be an equilibrium of the information economy. To prove part (ii) of the theorem, that the full information equilibrium is the *only* equilibrium of the information economy, we adopt

[1] Comparing the definitions of c and $\tilde{\alpha}$, it is clear that $c = 1/(1 - \tilde{\alpha})$.

the information-error representation of Chahrour and Ulbricht (2018). From the results in that paper, any equilibrium of the incomplete information economy can be written as an equilibrium of the full information economy in which the expectational equations are perturbed by an additional informational error.

Introducing a vector of information errors $\mathbf{v}_t$, we can rewrite the intertemporal Euler equation in (64) as

$$\mathbf{k}_{t+1} = E_t^f[A_v \mathbf{z}_t] + \mathbf{v}_t. \tag{87}$$

The only restriction on the expectation errors in $\mathbf{v_t}$ is that each must be orthogonal to the information in the corresponding sector's information set, i.e. $cov(v_{i,t}, \hat{\Omega}_t^{i,mc}) = 0$. We will show that imposing this restriction is enough to ensure that the variance of expectation errors is zero, i.e. in equilibrium firms cannot make any mistake relative to the full information economy.

Substituting (62) and using that $\zeta = 1$ and $H = I - IO$ yields,

$$\mathbf{k}_{t+1} = E_t[A_v \mathbf{z}_{t+1}] = A_v (I - IO)^{-1} (\rho \theta_t + \Phi_k \mathbf{k}_{t+1}) + \mathbf{v}_t. \tag{88}$$

Solving for $\mathbf{k}_{t+1}$, we have

$$\mathbf{k}_{t+1} = \Lambda \theta_t + \beta_0 \mathbf{v}_t, \tag{89}$$

where

$$\beta_0 \equiv (I - A_v (I - IO)^{-1} \Phi_k)^{-1}. \tag{90}$$

Using the Woodbury matrix identity, we can write this as

$$\beta_0 = I + A_v W \Phi_k \tag{91}$$

where $W \equiv (I - IO - \Phi_k A_v)^{-1}$.

Next, we show that

$$A_v W = (I - A_v (I - IO)^{-1} \Phi_k)^{-1} A_v (I - IO)^{-1}. \tag{92}$$

Post multiply both sides by $(I - IO)$ to get

$$A_v (I - IO - \Phi_k A_v)^{-1} (I - IO) = (I - A_v (I - IO)^{-1} \Phi_k)^{-1} A_v. \tag{93}$$

Rewrite the left hand side of (93) as

$$A_v (I - (I - IO)^{-1} \Phi_k A_v)^{-1} = (I - A_v (I - IO)^{-1} \Phi_k)^{-1} A_v. \tag{94}$$

Expand the left hand side of (94):

$$A_v (I + (I - IO)^{-1} \Phi_k A_v + (I - IO)^{-1} \Phi_k A_b (I - IO)^{-1} \Phi_k A_v + ...). \tag{95}$$

Expand the right hand side of (94):

$$(I + A_v (I - IO)^{-1} \Phi_k + A_v (I - IO)^{-1} \phi_k A_v (I - IO)^{-1} \Phi_k + ...) A_v. \tag{96}$$

Distributing the A_v that pre/post multiplies equations (95/96) and using the fact that $A_v I = I A_v$, it is clear that the two expansions match, establishing (92).

Since the right hand side of (92), $(I - A_v(I - IO)^{-1}\Phi_k)^{-1} A_v (I - IO)^{-1} = \rho^{-1}\Lambda$, we have that in any information equilibrium

$$\mathbf{k}_{t+1} = \Lambda\theta_t + \rho^{-1}\Lambda\mathbf{\Phi_k}\mathbf{v}_t + \mathbf{v}_t. \tag{97}$$

Using this result and (48), we have

$$\mathbf{q}_t + \mathbf{p_t} = A_v(I - IO)^{-1}(\theta_t + \Phi_k\Lambda\theta_{t-1} + \Phi_k(I + \rho^{-1}\Lambda\Phi_k)\mathbf{v}_{t-1}) \tag{98}$$

Letting $\tilde{v}_t \equiv w\mathbf{v}_t$, the rows of (97) and (98) correspond to two "signals", each observable by agents in sector i.[2]

$$s^k_{i,t} = \frac{\rho}{1-\tilde{\alpha}}\tilde{\theta}_t + \frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{v}_t + v_{i,t} \tag{99}$$

$$s^q_{i,t} = \tilde{\theta}_t + \frac{\tilde{\alpha}}{1-\tilde{\alpha}}\rho\tilde{\theta}_{t-1} + \frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{v}_{t-1} \tag{100}$$

Rationality thus requires (at least) the three following restrictions on the covariances of informational errors:

$$cov(s^k_{i,t}, v_{i,t}) = \frac{\rho}{1-\tilde{\alpha}}cov(\tilde{\theta}_t, v_{i,t}) + \frac{\tilde{\alpha}}{1-\tilde{\alpha}}cov(\tilde{v}_t, v_{i,t}) + cov(v_{i,t}, v_{i,t}) = 0 \tag{101}$$

$$cov(s^k_{i,t-1}, v_{i,t}) = \frac{\rho}{1-\tilde{\alpha}}cov(\tilde{\theta}_{t-1}, v_{i,t}) + \frac{\tilde{\alpha}}{1-\tilde{\alpha}}cov(\tilde{v}_{t-1}, v_{i,t}) + cov(v_{i,t-1}, v_{i,t}) = 0 \tag{102}$$

$$cov(s^q_{i,t}, v_{i,t}) = cov(\tilde{\theta}_t, v_{i,t}) + \frac{\tilde{\alpha}\rho}{1-\tilde{\alpha}}cov(\tilde{\theta}_{t-1}, v_{i,t}) + \frac{\tilde{\alpha}}{1-\tilde{\alpha}}cov(\tilde{v}_{t-1}, v_{i,t}) = 0 \tag{103}$$

Rearranging (102), we have

$$\frac{\rho}{1-\tilde{\alpha}}cov(\tilde{\theta}_{t-1}, v_{i,t}) = -\frac{\tilde{\alpha}}{1-\tilde{\alpha}}cov(\tilde{v}_{t-1}, v_{i,t}) - cov(v_{i,t-1}, v_{i,t}). \tag{104}$$

Substitute this into (103) and solve for $cov(\tilde{\theta}_t, v_{i,t})$ to get

$$\begin{aligned} cov(\tilde{\theta}_t, v_{i,t}) &= \frac{\tilde{\alpha}^2}{1-\tilde{\alpha}}cov(\tilde{v}_{t-1}, v_{i,t}) + \tilde{\alpha}cov(v_{i,t-1}, v_{i,t}) - \frac{\tilde{\alpha}}{1-\tilde{\alpha}}cov(\tilde{v}_{t-1}, v_{i,t}) \\ &= \tilde{\alpha}cov(v_{i,t-1}, v_{i,t}) - \tilde{\alpha}cov(\tilde{v}_{t-1}, v_{i,t}). \end{aligned} \tag{105}$$

Use the expression from (105) to eliminate $cov(\tilde{\theta}_t, v_{i,t})$ from (101)

$$0 = \frac{\tilde{\alpha}}{1-\tilde{\alpha}}\left(cov(\tilde{v}_t, v_{i,t}) - \rho cov(\tilde{v}_{t-1}, v_{i,t})\right) + \left(cov(v_{i,t}, v_{i,t}) + \frac{\rho\tilde{\alpha}}{1-\tilde{\alpha}}cov(v_{i,t-1}, v_{i,t})\right). \tag{106}$$

[2]Since $k_{i,t+1}$ is an action taken by sector i agents, it must be observable to them. We could have similarly derived the following results with $s^p_{i,t}$, rather than $s^k_{i,t}$, as we did in proving part (i) of the theorem, but the algebra is more involved.

We want to show that the only way to satisfy (106) is with $cov(v_{i,t}, v_{i,t}) = 0$. To see this, define $T_{i,1} \equiv cov(\tilde{v}_t, v_{i,t}) - \rho cov(\tilde{v}_{t-1}, v_{i,t})$ and $T_{i,2} = cov(v_{i,t}, v_{i,t}) + \frac{\rho\tilde{\alpha}}{1-\tilde{\alpha}} cov(v_{i,t-1}, v_{i,t})$. Sum equation (106) using the weights in w. Notice that

$$T_1 \equiv \sum_i w_i T_{i,1} = cov(\tilde{v}_t, \tilde{v}_t) - \rho cov(\tilde{v}_{t-1}, \tilde{v}_t) \geq 0, \tag{107}$$

since the autocovariance of a variable can never exceed its covariance. Hence

$$T_2 \equiv \sum_i w_i T_{i,2} \leq 0. \tag{108}$$

Under our assumption of one period revelation, $cov(v_{i,t-1}, v_{i,t}) = 0$, and the terms $T_{i,2}$ are guaranteed to be weakly positive.[3] Hence, the only way (108) can hold is if each of the covariances $cov(v_{i,t}, v_{i,t}) = var(v_{i,t}, v_{i,t})$ is zero.

□

B.3 Aggregate Irrelevance

Proposition 2. *Suppose that $\rho^A = \rho_i^S = \rho$, $\mu_i = \mu$, the input-output matrix is circulant, and that*

$$\hat{\Omega}_t^i = \left\{ \hat{\Omega}_t^{i,MC}, \{\hat{\theta}_{t-h}\}_{h=0}^{\infty} \right\}. \tag{109}$$

Then any symmetric equilibrium of the diverse information economy has the same aggregate dynamics as the full information equilibrium.

Corollary 3. *Any equilibrium of the model with $\hat{\Omega}_t^i = \left\{ \hat{\Omega}_t^{i,MC}, \{\hat{\theta}_{t-h}\}_{h=0}^{\infty} \right\}$ is also an equilibrium of the model with $\hat{\Omega}_t^i = \left\{ \hat{\Omega}_t^{i,MC}, \{\hat{v}_{t-h}\}_{h=0}^{\infty} \right\}$, where $\{\hat{v}_{t-h}\}_{h=0}^{\infty}$ is aggregate informative.*

We begin the proof of Proposition 2 with a lemma.

Lemma 2. *Relative prices do not depend on the aggregate shock $\hat{\theta}_t$.*

Proof. In any equilibrium, the price of the good in sector i can be written as a weighted sum of past sectoral shocks:

$$\begin{aligned} \hat{p}_{i,t} &= \sum_{\tau=0}^{\infty} \sum_{j=1}^{N} \gamma_{ij,\tau} \hat{\theta}_{j,t-\tau} \\ &= \sum_{\tau=0}^{\infty} \sum_{j=1}^{N} \gamma_{ij,\tau} (\hat{\theta}_{j,t-\tau} - \hat{\theta}_{t-\tau}) + \sum_{\tau=0}^{\infty} \hat{\theta}_{t-\tau} \left(\sum_{i=1}^{N} \gamma_{ij,\tau} \right), \end{aligned} \tag{110}$$

[3] Indeed, even without lagged revelation, $T_{i,2}$ are positive so long as $\frac{\rho\tilde{\alpha}}{1-\tilde{\alpha}} < 1$, for which a sufficient condition is that the labor share exceeds the capital share.

where the coefficients $\gamma_{ij,\tau}$ are generic coefficients in the MA representation of $\hat{p}_{i,t}$. Summing this expression across the (symmetric) sectors and dividing by N yields

$$0 \equiv \hat{p}_t = \sum_{\tau=0}^{\infty}\sum_{j=1}^{N}(\hat{\theta}_{j,t-\tau} - \hat{\theta}_{t-\tau})\left(\frac{1}{N}\sum_{i=1}^{N}\gamma_{ij,\tau}\right) + \sum_{\tau=0}^{\infty}\hat{\theta}_{t-\tau}\left(\frac{1}{N}\sum_{j=1}^{N}\gamma_{ij,\tau}\right)$$
$$= 0 + \sum_{\tau=0}^{\infty}\hat{\theta}_{t-\tau}\left(\frac{1}{N}\sum_{i=1}^{N}\gamma_{ij,\tau}\right) \tag{111}$$

where the second line follows from the fact that, by symmetry, $\left(\frac{1}{N}\sum_{i=1}^{N}\gamma_{ij,\tau}\right)$ is constant for all j and from the definition of $\hat{\theta}_t$. Since the last equation must hold for any sequence of $\theta_{t-\tau}$, however, it immediately follows that $\left(\frac{1}{N}\sum_{i=1}^{N}\gamma_{ij,\tau}\right) = 0, \forall\tau$, so that $p_{i,t}$ may only depend on the deviations of productivity from the average, $\theta_{j,t-\tau} - \theta_{t-\tau}$ and not independently on the average. □

Corollary 4. *Suppose that the information set of firms in sector i consists of market-consistent information and $\hat{\theta}_t$. Then, sector i's expectations of any price at any future horizon must be a function only of the histories of $(\hat{\theta}_{i,t} - \hat{\theta}_t), p_{i,t}$, and $\{p_{j,t}, \forall j \text{ s.t } a_{ij} > 0\}$.*

Proof. This holds because relative prices and aggregate outcomes are orthogonal at all horizons. □

Corollary 5. *Suppose that the information set of firms in sector i consists of market-consistent information and $\hat{\theta}_t$. Then the average expectations regarding any future price are zero, i.e., $\sum_{i=1}^{N} E_t^i[\hat{p}_{i,t+\tau}] = \sum_{i=1}^{N} E_t^i[\hat{p}_{i+1,t+\tau}] = \sum_{i=1}^{N} E_t^i[\hat{p}_{i+2,t+\tau}] = ... = 0, \forall\tau$.*

Proof. Since expectations of future prices depend symmetrically on a set of mean-zero objects, sums of those expectations must be zero. □

We now prove Proposition 2:

Proof. Our goal is to prove that

$$\frac{1}{N}\sum_{i=1}^{N} E_t^i[\hat{x}_{i,t+1}] = E_t^f[\hat{x}_{t+1}], \tag{112}$$

for any variable $\hat{x}_{i,t+1}$. If this is true, then individual Euler equations can be summed to yield the aggregate full-information Euler in equation (72) and the conclusion follows.

The action of a firm in sector i can be written

$$\hat{x}_{i,t} = \sum_{\tau=0}^{\infty}\left(\tilde{\varphi}_{1,\tau}\hat{\theta}_{i,t-\tau} + \tilde{\varphi}_{2,\tau}\hat{\theta}_{t-\tau} + \sum_{k=0}^{N-1}\tilde{\nu}_{k,\tau}\hat{p}_{i+k,t-\tau}\right) \tag{113}$$

where $\tilde{\nu}_{k,\tau} = 0$ for all k such that $a_{i,i+k} = 0$. Since we have assumed a circulant matrix, $a_{i,i+k} = 0$ will be true for all i if it is true for any i.

Summing across sectors, the final term in the summation cancels due to symmetry. Thus, the average action is given by

$$\hat{x}_t = \sum_{\tau=0}^{\infty} \left(\tilde{\varphi}_{1,\tau} + \tilde{\varphi}_{2,\tau}\right) \hat{\theta}_{t-\tau}, \tag{114}$$

and the one-period-ahead full-information expectation is given by

$$E_t^f\left[\hat{x}_{t+1}\right] = \left(\sum_{\tau=1}^{\infty} \left(\tilde{\varphi}_{1,\tau} + \tilde{\varphi}_{2,\tau}\right) \hat{\theta}_{t+1-\tau}\right) + \left(\tilde{\varphi}_{1,0} + \tilde{\varphi}_{2,0}\right) \rho \hat{\theta}_t. \tag{115}$$

The one-period-ahead expectation of a firm in sector i is then given by

$$\begin{aligned} E_t^i[\hat{x}_{i,t+1}] = &\left(\sum_{\tau=1}^{\infty} \tilde{\varphi}_{1,\tau}\hat{\theta}_{i,t+1-\tau} + \tilde{\varphi}_{2,\tau}\hat{\theta}_{t+1-\tau} + \sum_{k=0}^{N-1} \tilde{\nu}_{k,\tau}\hat{p}_{i+k,t+1-\tau}\right) + \\ &\tilde{\varphi}_{1,0}\rho\hat{\theta}_{i,t} + \tilde{\varphi}_{2,0}\rho\hat{\theta}_t + \sum_{k=1}^{N-1} \tilde{\nu}_{k,0} E_t^i[p_{i+k,t+1}.] \end{aligned} \tag{116}$$

Averaging across sectors and using the result in Corollary 5 to eliminate terms depending on prices delivers the desired result:

$$\frac{1}{N}\sum_{i=1}^{N} E_t^i[\hat{x}_{i,t+1}] = \left(\sum_{\tau=1}^{\infty} \left(\tilde{\varphi}_{1,\tau} + \tilde{\varphi}_{2,\tau}\right) \hat{\theta}_{t+1-\tau}\right) + \left(\tilde{\varphi}_{1,0} + \tilde{\varphi}_{2,0}\right) \rho \hat{\theta}_t = E_t^f[\hat{x}_{t+1}]. \tag{117}$$

$\square$

Proposition 3. *Suppose that $\zeta = 1$, μ_i, ρ_i^ς and σ_i are common across sectors, the input-output matrix is circulant, and the information set of firms is $\hat{\Omega}_t^{i,MC}$. Then the equilibrium of the diverse-information model is equivalent to that of the single-sector economy in which the representative agent observes $\hat{\theta}_t$, but not the decomposition between common and sectoral shocks.*

Proof. Using (60) and (64), we have for each sector i

$$\begin{aligned} k_{i,t+1} &= E_t^i[a_v \mathbf{z}_{t+1}] \\ &= E_t^i[a_v(I - IO)^{-1}\theta_{t+1}] + a_v(I - IO)^{-1}\Phi_k \mathbf{k}_t \\ &= E_t^i[(1-\alpha_x)^{-1}\hat{\theta}_{t+1}] + +a_v(I - IO)^{-1}\Phi_k \mathbf{k}_t \end{aligned}$$

where the second term has dropped out of expectations due to our assumption that past shocks, and therefore past capital, are observed. Recall that the composite signal

$$s_{i,t}^p \equiv \hat{p}_{i,t} - \sum \alpha_{ij}\hat{p}_{j,t} - \hat{\theta}_{i,t} \tag{118}$$

is a sufficient statistic for current productivity (given knowledge of past shocks) and this information can be used to forecast future productivity.

To prove that expectations $E_t^i[\hat{\theta}_{t+1}] = E_t[\hat{\theta}_{t+1}|\hat{\theta}_t]$ then requires showing that sectoral information does not reveal more about the future notional aggregate state than is already revealed by $s_{i,t}^p$. Since all endogenous variables depend on (common) past information and sectoral productivities, this will be true if no linear combination of sectoral productivities provides additional forecasting power for $\tilde{\theta}_{t+1}$ beyond the information contained in $\tilde{\theta}_t$. But the symmetry of sectoral productivities ensures that the optimal forecast update for an agent who observes all $\{\hat{\theta}_{i,t}\}$ depends only on the equally-weighted average of productivities, precisely what is revealed by $s_{i,t}^p$.

□

B.4 Derivation of Steady-State Investment Complementarities

In this section, we want to show that

$$\hat{k}_1^* = \hat{k}_{1,ss} + \frac{\phi_k}{1+\phi_k}\Delta. \tag{119}$$

where

$$\phi_k \equiv \frac{1}{2}\left(\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k}\right)\left(\frac{(1+\alpha_x)^2}{(1-\alpha_x)^2\zeta + 4\alpha_x}\right). \tag{120}$$

Assuming only two symmetric sectors and dropping time subscripts and productivity shocks, equations (31), (32), (35), and (36) become, respectively,

$$\hat{q}_i = \alpha_k\hat{k}_i + \alpha_x\hat{x}_{ij} \tag{121}$$

$$\hat{p}_j = \hat{p}_i + \alpha_k\hat{k}_i + (\alpha_x - 1)\,\hat{x}_{ij} \tag{122}$$

$$\hat{z}_i = -\zeta\hat{p}_i + \frac{1}{2}\sum_j \hat{z}_j \tag{123}$$

$$\hat{q}_i = \alpha_x\hat{x}_{ji} + (1-\alpha_x)\hat{z}_i. \tag{124}$$

Moreover, since we are considering steady-state, consumption drops from the intertemporal relation in equation (33). Substituting out for the production functions yields

$$\hat{p}_i = -\left(\alpha_k - 1\right)\hat{k}_i - \alpha_x\hat{x}_{ij}. \tag{125}$$

Now, combine equations (122) and (123) to find

$$(\hat{z}_i - \hat{z}_j) = \zeta\left(\alpha_k\hat{k}_i + (\alpha_x - 1)\,\hat{x}_{ij}\right). \tag{126}$$

Since the above equation holds for all i and j, we have that

$$2(\hat{z}_1 - \hat{z}_2) = \zeta\left[\alpha_k(\hat{k}_1 - \hat{k}_2) + (\alpha_x - 1)\,(\hat{x}_{12} - \hat{x}_{21})\right] \tag{127}$$

Equation (127) can be solved for the difference $(\hat{x}_{12} - \hat{x}_{21})$:

$$(\hat{x}_{12} - \hat{x}_{21}) = \frac{2}{\zeta(\alpha_x - 1)}(\hat{z}_1 - \hat{z}_2) - \frac{\alpha_k}{\alpha_x - 1}(\hat{k}_1 - \hat{k}_2). \tag{128}$$

Equations (121) and (124) can be combined to yield

$$\hat{z}_i = \frac{\alpha_k}{1-\alpha_x}\hat{k}_i + \frac{\alpha_x}{1-\alpha_x}(\hat{x}_{ij} - \hat{x}_{ji}), \tag{129}$$

which implies that

$$(\hat{z}_1 - \hat{z}_2) = \frac{\alpha_k}{1-\alpha_x}(\hat{k}_1 - \hat{k}_2) + \frac{2\alpha_x}{1-\alpha_x}(\hat{x}_{12} - \hat{x}_{21}). \tag{130}$$

Combine equations (128) and (130) to find that

$$(\hat{z}_i - \hat{z}_j) = \phi_1(\hat{k}_i - \hat{k}_j), \tag{131}$$

where $\phi_1 \equiv \frac{\alpha_k(1+\alpha_x)}{(1-\alpha_x)^2+4\alpha_x/\zeta}$. Rearranging equation (123) yields

$$p_1 = -\frac{1}{2\zeta}(\hat{z}_1 - \hat{z}_2). \tag{132}$$

Plugging equation (131) back into equation (132) yields

$$\hat{p}_1 = -\frac{1}{2\zeta}\phi_1(\hat{k}_1 - \hat{k}_2). \tag{133}$$

Price aggregation requires that $p_1 = -p_2$. Using this result, equations (125) and (122) together imply that

$$\hat{p}_1 = (1-\alpha_k)\hat{k}_1 + \frac{\alpha_x}{1-\alpha_x}(\hat{p}_2 - \hat{p}_1 - \alpha_k\hat{k}_1) \tag{134}$$

$$= (1-\alpha_k)\hat{k}_1 + \frac{\alpha_x}{1-\alpha_x}(-2\hat{p}_1 - \alpha_k\hat{k}_1) \tag{135}$$

Solving for p_1 yields

$$p_1 = \phi_2 k_1, \tag{136}$$

where $\phi_2 \equiv \frac{1-\alpha_x-\alpha_k}{1+\alpha_x}$. Finally, combining equations (133) and (136) yields the expression

$$\hat{k}_1 = \frac{1}{2\zeta}\frac{\phi_1}{\phi_2}(\hat{k}_2 - \hat{k}_1), \tag{137}$$

so that

$$\phi_k \equiv \frac{1}{2\zeta}\frac{\phi_1}{\phi_2} = \frac{1}{2}\frac{\alpha_k}{1-\alpha_k-\alpha_x}\frac{(1+\alpha_x)^2}{(1-\alpha_x)^2\zeta + 4\alpha_x}. \tag{138}$$

Specializing to the Cobb-Douglass production function,

$$F(K, L, X) = K^{\tilde{\alpha}_k(1-\alpha_x)}L^{(1-\tilde{\alpha}_k)(1-\alpha_x)}X^{\alpha_x} \tag{139}$$

and using the relationship $\alpha_k = \tilde{\alpha}_k(1-\alpha_x)$ yields expression (120).

C Solution Method

A substantial literature has arisen in recent years for solving models of dispersed information, including Kasa et al. (2004); Hellwig and Venkateswaran (2009); Baxter et al. (2011); Nimark (2011); Rondina and Walker (2017); and Huo and Takayama (2018). These techniques are not applicable here, because they assume information symmetry across all agent types and/or a large number (or continuum) of agents. In these environments, agents are shown to care only about their own expectation of the states, the economy-wide average expectation of the same states, the average expectation of the average expectation, and so on. In contrast, with a finite number of sectors, we must keep track of a complete structure of each agent-type's expectation of other agent-type's expectation *for each level of expectation.* Concretely, firms in sector 1 must follow the expectations of firms in sector 2 and firms in sector 3 separately, as the dependence of their optimal choice on these two sectors is not identical.

The linearized equations in our model can be rearranged to take the form

$$0 = \sum_{j=0}^{N} \left(\begin{bmatrix} \mathbf{A}_1^i & \mathbf{A}_2^i \end{bmatrix} E_t^j \begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1^i & \mathbf{B}_2^i \end{bmatrix} E_t^j \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{bmatrix} \right), \tag{140}$$

where $j = 0$ denotes the full-information set. The vector of endogenous choice variables, $\mathbf{y}_t$, has dimension $n_y \times 1$, and the vector of predetermined states, $\mathbf{x}_t$, is of dimension $n_x \times 1$. The state vector $\mathbf{x}_t$ is decomposed into a vector $\mathbf{x}_t^1$ of n_x^1 endogenous state variables and a vector $\mathbf{x}_t^2$ of n_x^2 exogenous state variables, which follow the autoregressive process

$$\mathbf{x}_{t+1}^2 = \rho \mathbf{x}_t^2 + \tilde{\eta} \epsilon_{t+1} \tag{141}$$

where ρ is a square matrix of dimension n_x^2. The column vector of n_ϵ exogenous shocks ϵ_t is assumed to be i.i.d. with identity covariance matrix.

In general, the solution to such a model is an MA(∞) process. Atolia and Chahrour (2014) show how to approximate the solution to such models as an ARMA(1,K) under the assumption that past shocks become common knowledge in period $K+1$. This approach generalizes the one taken by Townsend (1983). Nimark (2011) discusses some theoretical requirements for a related approach with such approximations to be valid, although such theoretical details have yet to be fully expounded for our current environment. The (approximate) solution to the model can then be written as

$$\mathbf{x}_{t+1} = \mathbf{h}_x \mathbf{x}_t + \sum_{\kappa=0}^{K} \mathbf{h}_\kappa \epsilon_{t-\kappa} + \eta \epsilon_{t+1} \tag{142}$$

$$\mathbf{y}_t = \mathbf{g}_x \mathbf{x}_t + \sum_{\kappa=0}^{K} \mathbf{g}_\kappa \epsilon_{t-\kappa}. \tag{143}$$

Formulating the model solution in this way ensures that the matrices $\mathbf{h}_x$ and $\mathbf{g}_x$ do not depend on the information assumption; they are the transition and observation matrices implied by the solution to the (linearized) full-information model. Thus the presence of incomplete (and heterogeneous) information is captured completely by the MA terms in equations (142) and (143). Atolia and Chahrour (2014) provide a numerical approach for finding the matrices $\mathbf{h}_\kappa$ and $\mathbf{g}_\kappa$, which we employ in our calibration exercises above.

Atolia and Chahrour (2014) also discuss an alternative approximation for such models in which agents have "finite recall," and include in their information sets only their observations for the most recent K periods. This alternative approach prevents agents' inferences from putting arbitrarily large weights on shocks far in the past, and thus prevents agents from perfect inference when the observables are non fundamental in the shocks, as in Graham and Wright (2010) and Rondina and Walker (2017). Our claim in the text that such non-fundamental equilibria do not appear is based on the observation that the imperfect recall and delayed-but-complete revelation approaches to approximation converge to the same dynamics for sufficiently large horizons K.

D Details on Estimated Productivity Process

Tables 1 through 3 below provide additional details regarding the estimated values for the factor model estimated on the Jorgenson et al. (2013) sectoral TFP data.

Table 1: Estimated Autocorrelations

	Median	**Mode**	**SD**
aggregate tfp	0.95	0.95	0.03
sectoral mean	0.95	0.95	0.03
agriculture, hunting, forestry and fishing	0.87	0.87	0.04
mining and quarrying	0.98	0.98	0.01
food, beverages and tobacco	0.96	0.95	0.02
textiles, leather and footwear	0.92	0.91	0.04
wood and of wood and cork	0.97	0.97	0.01
pulp, paper, printing and publishing	0.98	0.98	0.01
chemical, rubber, plastics and fuel	0.85	0.81	0.17
coke, refined petroleum and nuclear fuel	0.96	0.96	0.02
chemicals and chemical products	0.98	0.98	0.01
rubber and plastics	0.90	0.90	0.04
other non-metallic mineral	0.86	0.84	0.09
basic metals and fabricated metal	0.95	0.95	0.03
machinery, nec	0.98	0.98	0.01
electrical and optical equipment	1.00	1.00	0.00
transport equipment	0.92	0.92	0.03
manufacturing nec; recycling	0.96	0.96	0.02
post and telecommunications	0.97	0.97	0.01
construction	1.00	1.00	0.00
sale, maintenance and repair of motor vehicles and	0.92	0.92	0.03
wholesale trade and commission trade, except of mo	0.94	0.94	0.03
retail trade, except of motor vehicles and motorcy	0.93	0.92	0.03
hotels and restaurants	0.99	0.99	0.01
transport and storage	0.94	0.94	0.02
post and telecommunications	0.97	0.97	0.01
financial intermediation	0.98	0.98	0.01
real estate, renting and business activities	0.98	0.98	0.01
real estate activities	0.97	0.97	0.01
renting of m&eq and other business activities	0.94	0.94	0.02
public admin and defense; compulsory social securi	0.97	0.97	0.02
education	0.98	0.98	0.01
health and social work	0.97	0.96	0.02
other community, social and personal services	0.98	0.98	0.01

Table 2: Estimated Standard Deviations

	Median	**Mode**	**SD**
aggregate tfp	0.01	0.01	0.00
sectoral mean	0.03	0.03	0.00
agriculture, hunting, forestry and fishing	0.05	0.05	0.01
mining and quarrying	0.04	0.04	0.00
food, beverages and tobacco	0.04	0.04	0.00
textiles, leather and footwear	0.03	0.03	0.00
wood and of wood and cork	0.03	0.03	0.00
pulp, paper, printing and publishing	0.02	0.02	0.00
chemical, rubber, plastics and fuel	0.02	0.02	0.01
coke, refined petroleum and nuclear fuel	0.15	0.16	0.02
chemicals and chemical products	0.03	0.03	0.00
rubber and plastics	0.03	0.03	0.00
other non-metallic mineral	0.03	0.03	0.00
basic metals and fabricated metal	0.02	0.02	0.00
machinery, nec	0.04	0.04	0.00
electrical and optical equipment	0.04	0.04	0.00
transport equipment	0.04	0.04	0.00
manufacturing nec; recycling	0.03	0.03	0.00
post and telecommunications	0.02	0.02	0.00
construction	0.02	0.02	0.00
sale, maintenance and repair of motor vehicles and	0.04	0.04	0.00
wholesale trade and commission trade, except of mo	0.03	0.03	0.00
retail trade, except of motor vehicles and motorcy	0.03	0.03	0.00
hotels and restaurants	0.02	0.02	0.00
transport and storage	0.02	0.02	0.00
post and telecommunications	0.02	0.02	0.00
financial intermediation	0.03	0.03	0.00
real estate, renting and business activities	0.01	0.01	0.00
real estate activities	0.02	0.02	0.00
renting of m&eq and other business activities	0.02	0.02	0.00
public admin and defense; compulsory social securi	0.02	0.02	0.00
education	0.02	0.02	0.00
health and social work	0.02	0.02	0.00
other community, social and personal services	0.01	0.01	0.00

Table 3: Estimated Weight on Aggregate

	Median	**Mode**	**SD**
sectoral mean	1.27	1.26	0.45
agriculture, hunting, forestry and fishing	0.63	0.63	0.60
mining and quarrying	2.32	2.26	0.47
food, beverages and tobacco	1.14	1.14	0.62
textiles, leather and footwear	-0.18	-0.16	0.40
wood and of wood and cork	-0.66	-0.61	0.29
pulp, paper, printing and publishing	1.67	1.69	0.37
chemical, rubber, plastics and fuel	5.92	5.91	0.64
coke, refined petroleum and nuclear fuel	13.85	13.40	1.49
chemicals and chemical products	4.31	4.33	0.67
rubber and plastics	1.97	1.98	0.42
other non-metallic mineral	1.60	1.60	0.47
basic metals and fabricated metal	1.09	1.10	0.45
machinery, nec	0.80	0.80	0.63
electrical and optical equipment	-0.41	-0.34	0.44
transport equipment	1.84	1.82	0.58
manufacturing nec; recycling	1.07	1.07	0.43
post and telecommunications	-0.46	-0.44	0.30
construction	0.58	0.58	0.32
sale, maintenance and repair of motor vehicles and	1.60	1.60	0.59
wholesale trade and commission trade, except of mo	1.16	1.17	0.52
retail trade, except of motor vehicles and motorcy	1.40	1.41	0.46
hotels and restaurants	0.27	0.28	0.34
transport and storage	0.51	0.52	0.32
post and telecommunications	-0.44	-0.42	0.31
financial intermediation	-0.45	-0.43	0.32
real estate, renting and business activities	-0.13	-0.12	0.20
real estate activities	-0.30	-0.30	0.24
renting of m&eq and other business activities	0.08	0.09	0.23
public admin and defense; compulsory social securi	-0.23	-0.23	0.34
education	0.16	0.15	0.39
health and social work	-0.20	-0.21	0.28
other community, social and personal services	0.11	0.12	0.25

References

Atolia, M. and R. Chahrour (2014). A Generalized Solution Method For Rational Expectations Models With Diverse Information Sets. C, C

Baxter, B., L. Graham, and S. Wright (2011). Invertible and Non-invertible Information Sets in Linear Rational Expectations Models. *Journal of Economic Dynamics and Control 35*(3), 295–311. C

Chahrour, R. and R. Ulbricht (2018). Robust predictions for dsge models with incomplete information. Workig paper. B.2

Graham, L. and S. Wright (2010). Information, Heterogeneity and Market Incompleteness. *Journal of Monetary Economics 57*(2), 164 – 174. C

Hellwig, C. and V. Venkateswaran (2009). Setting the Right Prices For the Wrong Reasons. *Journal of Monetary Economics 56*(Supplement 1), S57 – S77. C

Huo, Z. and N. Takayama (2018). Rational Expectations Models With Higher Order Beliefs. C

Jorgenson, D. W., M. S. Ho, and J. D. Samuels (2013). A Prototype Industry-Level Production Account For the United States, 1947-2010. Working Paper. D

Kasa, K., T. Walker, and C. Whiteman (2004). Asset Pricing With Heterogeneous Beliefs: A Frequency-Domain Approach. *Unpublished Paper*. C

Nimark, K. (2011). Dynamic Higher Order Expectations. Economics Working Papers 1118, Department of Economics and Business, Universitat Pompeu Fabra. C, C

Rondina, G. and T. B. Walker (2017). Information Equilibria in Dynamic Economies With Dispersed Information. Working paper, Indiana University. C, C

Townsend, R. M. (1983). Forecasting the Forecasts of Others. *The Journal of Political Economy 91*(4), 546–588. C