Online Appendix: Gradualism and Liquidity Traps

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June 2018

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A Interest-rate smoothing regimes

A.1 Interest-rate smoothing

The Lagrange problem of the central bank with an IRS objective at period t is given by

$$\begin{aligned} V_t^{CB}(r_t^n, i_{t-1}) = \max_{\pi_t, y_t, i_t} \Bigg[& -\frac{1}{2}\left[(1-\alpha)\left(\pi_t^2 + \lambda y_t^2\right) + \alpha(i_t - i_{t-1})^2\right] + \beta E_t V_{t+1}^{CB}(r_{t+1}^n, i_t) \\ & + \phi_t^{PC}(\pi_t - \beta E_t \pi_{t+1}(r_{t+1}^n, i_t) - \kappa y_t) \\ & + \phi_t^{IS}(y_t - E_t y_{t+1}(r_{t+1}^n, i_t) + \sigma(i_t - E_t \pi_{t+1}(r_{t+1}^n, i_t) - r_t^n)) \\ & + \phi_t^{ZLB} i_t \Bigg] \end{aligned} \tag{A.1}$$

where the central banker takes the value and policy functions next period as given. The FONC are

$$(1-\alpha)\pi_t - \phi_t^{PC} = 0 \tag{A.2}$$

$$(1-\alpha)\lambda y_t + \kappa\phi_t^{PC} - \phi_t^{IS} = 0 \tag{A.3}$$

$$\begin{aligned} & \alpha(i_t - i_{t-1}) - \beta\frac{\partial E_t V_{t+1}^{CB}(r_{t+1}^n, i_t)}{\partial i_t} + \beta\frac{\partial E_t \pi(r_{t+1}^n, i_t)}{\partial i_t}\phi_t^{PC} \\ & + \left(\frac{\partial E_t y(r_{t+1}^n, i_t)}{\partial i_t} + \sigma\frac{\partial E_t \pi(r_{t+1}^n, i_t)}{\partial i_t} - \sigma\right)\phi_t^{IS} - \phi_t^{ZLB} = 0 \end{aligned} \tag{A.4}$$

as well as the complementary slackness conditions and the NKPC and IS equation. Combining the first two conditions, we get

$$(1-\alpha)(\lambda y_t + \kappa\pi_t) = \phi_t^{IS} \tag{A.5}$$

Furthermore, note that

$$\frac{\partial V_t^{CB}(r_t^n, i_{t-1})}{\partial i_{t-1}} = \alpha(i_t - i_{t-1}) \tag{A.6}$$

We can then consolidate the third optimality condition to obtain an interest-rate target criterion

$$\begin{aligned} 0 = & \alpha(1+\beta)i_t - \alpha i_{t-1} - \beta\alpha E_t i(r_{t+1}^n, i_t) \\ & + \beta(1-\alpha)\frac{\partial E_t \pi(r_{t+1}^n, i_t)}{\partial i_t}\pi_t + (1-\alpha)\left(\frac{\partial E_t y(r_{t+1}^n, i_t)}{\partial i_t} + \sigma\frac{\partial E_t \pi(r_{t+1}^n, i_t)}{\partial i_t}\right)(\lambda y_t + \kappa\pi_t) \\ & - (1-\alpha)\sigma(\lambda y_t + \kappa\pi_t) - \phi_t^{ZLB} \end{aligned} \tag{A.7}$$

A.2 Shadow interest-rate smoothing

The value of the central bank with a shadow interest-rate smoothing (SIRS) objective is given by

$$V_t^{CB,SIRS} = u^{CB,SIRS}(\pi_t, y_t, i_t, i_{t-1}^*) + \beta E_t V_{t+1}^{CB,SIRS} \tag{A.8}$$

where the central bank's contemporaneous objective function, $u^{CB,SIRS}(\cdot,\cdot,\cdot,\cdot)$, is given by

$$u^{CB,SIRS}(\pi_t, y_t, i_t, i^*_{t-1}) = -\frac{1}{2}\left[(1-\alpha)\left(\pi_t^2 + \lambda y_t^2\right) + \alpha(i_t - i^*_{t-1})^2\right] \tag{A.9}$$

Each period t, the central bank with a SIRS objective first chooses the shadow nominal interest rate in order to maximize the value today subject to the behavioral constraints of the private sector, with the value and policy functions at time $t+1$—$V^{CB,SIRS}_{t+1}(\cdot,\cdot)$, $y_{t+1}(\cdot,\cdot)$, $\pi_{t+1}(\cdot,\cdot)$—taken as given:

$$i^*_t = \text{argmax}_x \qquad u^{CB,SIRS}(\pi(x), y(x), x, i^*_{t-1}) + \beta E_t V^{CB,SIRS}_{t+1}(r^n_{t+1}, x) \tag{A.10}$$

with

$$\begin{aligned} y(x) =& E_t y_{t+1}(r^n_{t+1}, x) - \sigma(x - E_t\pi_{t+1}(r^n_{t+1}, x) - r^n_t) \\ \pi(x) =& \kappa y(x) + \beta E_t \pi_{t+1}(r^n_{t+1}, x) \end{aligned} \tag{A.11}$$

The actual policy rate i_t is given by

$$i_t = \max(i^*_t, 0) \tag{A.12}$$

That is, the actual policy rate today is zero when $i^*_t < 0$, and it is equal to i^*_t when $i^*_t \geq 0$.

The central bank's value today is given by

$$V^{CB,SIRS}_t(r^n_t, i^*_{t-1}) = u^{CB,SIRS}(\pi_t, y_t, i_t, i^*_{t-1}) + \beta E_t V^{CB,SIRS}_{t+1}(r^n_{t+1}, i^*_t) \tag{A.13}$$

where inflation and the output gap are given by

$$\begin{aligned} y_t =& E_t y_{t+1}(r^n_{t+1}, i^*_t) - \sigma(i_t - E_t\pi_{t+1}(r^n_{t+1}, i^*_t) - r^n_t) \\ \pi_t =& \kappa y_t + \beta E_t \pi_{t+1}(r^n_{t+1}, i^*_t) \\ i_t \geq& 0 \end{aligned} \tag{A.14}$$

A Markov-Perfect equilibrium with a SIRS objective is defined as a set of time-invariant value and policy functions $\{V^{CB,SIRS}(\cdot),\ \pi(\cdot),\ y(\cdot),\ i^*(\cdot),\ i(\cdot)\}$ that solves the problem of the central bank above, together with the value function $V(\cdot)$ that is consistent with $\pi(\cdot)$ and $y(\cdot)$.

B Calibration of the process of the natural real rate shock

To calibrate the process of the natural real rate shock in the simple model, we follow the procedure used by Adam and Billi (2006). Specifically, we construct conditional output gap and inflation expectations. We then plug these expectations along with actual values for the output gap and inflation into the consumption Euler equation

$$y_t = \tilde{E}_t y_{t+1} - \sigma(i_t - i - \tilde{E}_t\pi_{t+1}) + d_t, \tag{B.1}$$

where $\tilde{E}_t y_{t+1}$ and $\tilde{E}_t \pi_{t+1}$ are conditional expectations, i is the mean of the policy rate and d_t is

the equation residual. We then identify the natural real rate shock as

$$r_t^n - r^n = \frac{1}{\sigma} d_t. \tag{B.2}$$

We use quarterly data for the U.S. economy from 1984-Q1 to 2016-Q4. For the inflation rate we use the quarterly percentage change of the GDP implicit price deflator. The output gap is constructed as the log difference between real GDP and the CBO estimate of real potential GDP (both in billions of chained 2009 U.S. dollars). For the policy rate we use the quarterly average of the effective federal funds rate. All data series are obtained from FRED. We then subtract the respective sample mean from the three constructed data series.

Following Adam and Billi (2006), we use actual future values of the output gap and inflation for the conditional expectations. We then estimate an AR(1) model for $r_t^n - r^n$ using OLS, and obtain $\rho_r = 0.851$ (standard error: 0.045) and $\sigma_r^2 = 0.1588/100^2$ (standard error: $0.0165/100^2$), or $\sigma_r = 0.399/100$.

C Numerical algorithm and solution accuracy for the simple model

We use the policy function iteration algorithm described below to solve the simple model for the various monetary policy regimes.

C.1 Numerical algorithm

We approximate the policy functions for the inflation rate, output and the policy rate with a finite elements method using collocation. For the basis functions we use cubic splines. The algorithm uses fixed-point iteration and proceeds in the following steps (here exemplified for the IRS regime):

1. Construct the collocation nodes. The nodes are chosen such that they coincide with the spline breakpoints. Use a Gaussian quadrature scheme to discretize the normally distributed innovation to the natural real rate shock.

2. Start with a guess for the basis coefficients.

3. Use the current guess for the basis coefficients to approximate the expectation terms.

4. Solve the system of equilibrium conditions for inflation, output and the policy rate at the collocation nodes, assuming that the zero lower bound is not binding. For those nodes where the zero bound constraint is violated solve the system of equilibrium conditions associated with a binding zero bound.

5. Update the guess for the basis coefficients. If the new guess is sufficiently close to the old one, the algorithm has converged. Otherwise, go back to step 3.

C.2 Solution accuracy

We assess the solution accuracy by evaluating the residual functions associated with, the New Keynesian Phillips curve ($R_{PC,t}$), the consumption Euler equation ($R_{EE,t}$) and the target criterion (A.7) ($R_{TC,t}$) along a simulated equilibrium path with a length of 100,000 periods. For each equation, the residual function is defined as the absolute value of the difference between the left-hand side and the right-hand side of the equation. Table 1 reports the average and the maximum of these residuals for the optimized interest-rate smoothing regime.

Table 1: **Solution accuracy: Simple model with** $\alpha = 0.029$

	$\text{Mean}\left[\log_{10}(\text{R}_{k,t})\right]$	$\text{Max}\left[\log_{10}(\text{R}_{k,t})\right]$
$k = PC$: Sticky-price error	−6.54	−4.50
$k = EE$: Euler equation error	−5.46	−3.08
$k = TC$: Target criterion error	−7.66	−5.16

D Numerical algorithm and solution accuracy for the quantitative model

D.1 First-order necessary conditions for central bank's problem

Including private-sector equilibrium conditions (equation (20) - (24)), first-order necessary conditions for the central bank's maximization problem are enumerated as follows:

$$0 = -(1-\alpha)\lambda y_t + \phi_{1,t} - \kappa_w\left(\frac{1}{\sigma}+\eta\right)\phi_{2,t}, \tag{D.1}$$

$$0 = -(1-\alpha)\lambda_w(\pi_t^w - \iota_w\pi_{t-1}^p) + \phi_{2,t} + \phi_{4,t}, \tag{D.2}$$

$$\begin{aligned}0 =& -(1-\alpha)(\pi_t^p - \iota_p\pi_{t-1}^p) + \beta(1-\alpha)\iota_p(E_t\pi_{t+1}^p - \iota_p\pi_t^p) + \beta(1-\alpha)\lambda_w\iota_w(E_t\pi_{t+1}^w - \iota_w\pi_t^p)\\ &- \phi_{1,t}\left(\frac{\partial E_t y_{t+1}}{\partial \pi_t^p} + \sigma\frac{\partial E_t\pi_{t+1}^p}{\partial \pi_t^p}\right) - \phi_{2,t}\beta\left(\frac{\partial E_t\pi_{t+1}^w}{\partial \pi_t^p} - \iota_w\right) - \beta\iota_w E_t\phi_{2,t+1}\\ &+ \phi_{3,t}\left(1-\beta\left(\frac{\partial E_t\pi_{t+1}^p}{\partial \pi_t^p} - \iota_p\right)\right) - \beta\iota_p E_t\phi_{3,t+1} - \phi_{4,t},\end{aligned} \tag{D.3}$$

$$\begin{aligned}0 =& -\phi_{1,t}\left(\frac{\partial E_t y_{t+1}}{\partial w_t} + \sigma\frac{\partial E_t\pi_{t+1}^p}{\partial w_t}\right) + \phi_{2,t}\left(\kappa_w - \beta\frac{\partial E_t\pi_{t+1}^w}{\partial w_t}\right) - \phi_{3,t}\left(\kappa_p + \beta\frac{\partial E_t\pi_{t+1}^p}{\partial w_t}\right)\\ &- \phi_{4,t} + \beta E_t\phi_{4,t+1},\end{aligned} \tag{D.4}$$

$$\begin{aligned}0 =& -\alpha(i_t - i_{t-1}) + \beta\alpha(E_t i_{t+1} - i_t) + \phi_{1,t}\left(\sigma - \frac{\partial E_t y_{t+1}}{\partial i_t} - \sigma\frac{\partial E_t\pi_{t+1}^p}{\partial i_t}\right)\\ &- \phi_{2,t}\beta\frac{\partial E_t\pi_{t+1}^w}{\partial i_t} - \phi_{3,t}\beta\frac{\partial E_t\pi_{t+1}^p}{\partial i_t} + \phi_{5,t},\end{aligned} \tag{D.5}$$

where $\phi_{1,t}$ - $\phi_{5,t}$ are Lagrangian multipliers for equation (**??**) - (**??**), respectively.

D.2 Solution method

There are total of five state variables, which we denote by $\mathbb{S}_t \ni [u_t, r_t^n, \pi_{t-1}^p, w_{t-1}, i_{t-1}]$. The problem is to find a set of policy functions, $\{\pi^p(\mathbb{S}_t)$, $\pi^w(\mathbb{S}_t)$, $y(\mathbb{S}_t)$, $w(\mathbb{S}_t)$, $i(\mathbb{S}_t)$, $\phi_1(\mathbb{S}_t)$, $\phi_2(\mathbb{S}_t)$, $\phi_3(\mathbb{S}_t)$, $\phi_4(\mathbb{S}_t)$, and $\phi_5(\mathbb{S}_t)\}$ that solves the following system of *functional* equations:

$$\begin{aligned}
\pi^p(\mathbb{S}_t) - \iota_p \pi_{t-1}^p &= \kappa_p w(\mathbb{S}_t) + \beta \left(E_t \pi^p(\mathbb{S}_{t+1}) - \iota_p \pi^p(\mathbb{S}_t) \right) + u_t, && \text{(D.6)} \\
\pi^w(\mathbb{S}_t) - \iota_w \pi_{t-1}^p &= \kappa_w \left(\left(\frac{1}{\sigma} + \eta \right) y(\mathbb{S}_t) - w(\mathbb{S}_t) \right) + \beta \left(E_t \pi^w(\mathbb{S}_{t+1}) - \iota_w \pi^p(\mathbb{S}_t) \right), && \text{(D.7)} \\
\pi^w(\mathbb{S}_t) &= w(\mathbb{S}_t) - w_{t-1} + \pi^p(\mathbb{S}_t), && \text{(D.8)} \\
y(\mathbb{S}_t) &= E_t y(\mathbb{S}_{t+1}) - \sigma \left(i(\mathbb{S}_t) - E_t \pi^p(\mathbb{S}_{t+1}) - r_t^n \right), && \text{(D.9)} \\
i(\mathbb{S}_t) &\geq i_{ELB}. && \text{(D.10)}
\end{aligned}$$

$$\begin{aligned}
0 =& -(1-\alpha)\lambda y(\mathbb{S}_t) + \phi_1(\mathbb{S}_t) - \kappa_w \left(\frac{1}{\sigma} + \eta \right) \phi_2(\mathbb{S}_t), && \text{(D.11)} \\
0 =& -(1-\alpha)\lambda_w (\pi^w(\mathbb{S}_t) - \iota_w \pi_{t-1}^p) + \phi_2(\mathbb{S}_t) + \phi_4(\mathbb{S}_t), && \text{(D.12)} \\
0 =& -(1-\alpha)(\pi^p(\mathbb{S}_t) - \iota_p \pi_{t-1}^p) + \beta(1-\alpha)\iota_p (E_t \pi^p(\mathbb{S}_{t+1}) - \iota_p \pi^p(\mathbb{S}_t)) \\
& + \beta(1-\alpha)\lambda_w \iota_w (E_t \pi^w(\mathbb{S}_{t+1}) - \iota_w \pi^p(\mathbb{S}_t)) \\
& - \phi_1(\mathbb{S}_t) \left(\frac{\partial E_t y(\mathbb{S}_{t+1})}{\partial \pi^p(\mathbb{S}_t)} + \sigma \frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial \pi^p(\mathbb{S}_t)} \right) - \phi_2(\mathbb{S}_t)\beta \left(\frac{\partial E_t \pi^w(\mathbb{S}_{t+1})}{\partial \pi^p(\mathbb{S}_t)} - \iota_w \right) - \beta \iota_w E_t \phi_2(\mathbb{S}_{t+1}) \\
& + \phi_3(\mathbb{S}_t) \left(1 - \beta \left(\frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial \pi^p(\mathbb{S}_t)} - \iota_p \right) \right) - \beta \iota_p E_t \phi_3(\mathbb{S}_{t+1}) - \phi_4(\mathbb{S}_t), && \text{(D.13)} \\
0 =& -\phi_1(\mathbb{S}_t) \left(\frac{\partial E_t y(\mathbb{S}_{t+1})}{\partial w(\mathbb{S}_t)} + \sigma \frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial w(\mathbb{S}_t)} \right) + \phi_2(\mathbb{S}_t) \left(\kappa_w - \beta \frac{\partial E_t \pi^w(\mathbb{S}_{t+1})}{\partial w(\mathbb{S}_t)} \right) \\
& - \phi_3(\mathbb{S}_t) \left(\kappa_p + \beta \frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial w(\mathbb{S}_t)} \right) \\
& - \phi_4(\mathbb{S}_t) + \beta E_t \phi_4(\mathbb{S}_{t+1}), && \text{(D.14)} \\
0 =& -\alpha(i(\mathbb{S}_t) - i_{t-1}) + \beta\alpha(E_t i(\mathbb{S}_{t+1}) - i(\mathbb{S}_t)) + \phi_1(\mathbb{S}_t) \left(\sigma - \frac{\partial E_t y(\mathbb{S}_{t+1})}{\partial i(\mathbb{S}_t)} - \sigma \frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial i(\mathbb{S}_t)} \right) \\
& - \phi_2(\mathbb{S}_t)\beta \frac{\partial E_t \pi^w(\mathbb{S}_{t+1})}{\partial i(\mathbb{S}_t)} - \phi_3(\mathbb{S}_t)\beta \frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial i(\mathbb{S}_t)} + \phi_5(\mathbb{S}_t), && \text{(D.15)}
\end{aligned}$$

Following the idea of Christiano and Fisher (2000), we decompose these policy functions into two parts using an indicator function: one in which the policy rate is allowed to be less than 0, and the other in which the policy rate is assumed to be 0. That is, for any variable Z,

$$Z(\cdot) = I_{\{R(\cdot)\geq 0\}} Z_{NZLB}(\cdot) + (1 - I_{\{R(\cdot)\geq 0\}}) Z_{ZLB}(\cdot). \tag{D.16}$$

The problem then becomes finding a set of a *pair* of policy functions, $\{\left[\pi_{NZLB}^p(\cdot),\ \pi_{ZLB}^p(\cdot)\right]$,

$\{[\pi^w_{NZLB}(\cdot), \pi^w_{ZLB}(\cdot)], \{[y_{NZLB}(\cdot), y_{ZLB}(\cdot)], [w_{NZLB}(\cdot), w_{ZLB}(\cdot)], [i_{NZLB}(\cdot), i_{ZLB}(\cdot)], [\phi_{1,NZLB}(\cdot), \phi_{1,ZLB}(\cdot)], [\phi_{2,NZLB}(\cdot), \phi_{2,ZLB}(\cdot)], [\phi_{3,NZLB}(\cdot), \phi_{3,ZLB}(\cdot)], [\phi_{4,NZLB}(\cdot), \phi_{4,ZLB}(\cdot)]$, and $[\phi_{5,NZLB}(\cdot) \phi_{5,ZLB}(\cdot)]\}$ that solves the system of functional equations above. This approach of Christiano and Fisher (2000) can achieve a given level of accuracy with a considerable less number of grid points relative to the standard approach.

The time-iteration method aims to find the values for the policy and value functions consistent with the equilibrium conditions on a finite number of grid points within the pre-determined grid intervals for the model's state variables. Let $X(\cdot)$ be a vector of policy functions that solves the functional equations above and let $X^{(0)}$ be the initial guess of such policy functions.[1] At the s-th iteration, given the approximated policy function $X^{(s-1)}(\cdot)$, we solve the system of nonlinear equations given by equations (D.6)-(D.15) to find today's $\pi^p_t, \pi^w_t, y_t, w_t, i_t, \phi_{1,t}, \phi_{2,t}, \phi_{3,t}, \phi_{4,t}$, and $\phi_{5,t}$ at each grid point. In solving the system of nonlinear equations, we use Gaussian quadrature (with 10 Gauss-Hermite nodes) to discretize and evaluate the expectation terms in the Euler equation, the price and wage Phillips curves, and expectational partial derivative terms.

The values of the policy function that are not on any of the grid points are interpolated or extrapolated linearly. The values of the partial derivatives of the policy functions not on any of the grid points are approximated by the slope of the policy functions evaluated from the adjacent two grid points. That is, for any variable X and Z,

$$\frac{\partial X(\delta_{t+1,t})}{\partial Z_t} = \frac{X(\delta_{t+1}, Z'') - X(\delta_{t+1}, Z')}{Z'' - Z'}. \tag{D.17}$$

where Z' and Z'' are two adjacent grid points to Z_t such that $Z' < Z_t < Z''$. When Z_t is outside the grid interval, the partial derivative is approximated by the slope evaluated at the edge of the grid interval.

The system is solved numerically by using a nonlinear equation solver, `dneqnf`, provided by the IMSL Fortran Numerical Library. If the updated policy functions are sufficiently close to the previously approximated policy functions, then the iteration ends. Otherwise, using the former as the guess for the next period's policy functions, we iterate on this process until the difference between the guessed and updated policy functions is sufficiently small ($\left\|vec(X^s(\delta) - X^{s-1}(\delta))\right\|_\infty <$ 1E-12 is used as the convergence criteria). The solution method can be extended to models with multiple (non-perfectly correlated) exogenous shocks and with multiple endogenous state variables in a straightforward way.

D.3 Solution accuracy

In this section, we report the accuracy of our numerical solutions for the quantitative model. Following Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015) and Maliar and Maliar (2015), we evaluate the residuals functions along a simulated equilibrium path. The length of the simulation is 100,000.

[1] For all models and all variables, we use flat functions at the deterministic steady-state values as the initial guess.

For the quantitative model, there are six key residual functions of interest. The first three residual functions, denoted by $R_{1,t}$, $R_{2,t}$, and $R_{3,t}$, are associated with the sticky-price equation, the sticky-wage equation, and the Euler equation, respectively (equations (**??**), (**??**), and (**??**)). The last three residual functions, denoted by $R_{4,t}$, $R_{5,t}$, and $R_{6,t}$, are associated with the first-order conditions of the central bank's optimization problem with respect to price inflation, real wage, and the policy rate, respectively (equations (D.3), (D.4), and (D.5)). For each equation, the residual function is defined as the absolute value of the difference between the left-hand side and the right-hand side of the equation. Table 2 shows the average and the maximum of the six residual functions over the 100,000 simulations.

The size of the residuals are comparable to those reported in other numerical works on the New Keynesian model with the ELB constraint, such as Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015), Hills, Nakata, and Schmidt (2016), Hirose and Sunakawa (2015), and Maliar and Maliar (2015).

Table 2: **Solution accuracy: Quantitative model with** $\alpha = 0.37$

	$\text{Mean}\left[\log_{10}(\text{R}_{k,t})\right]$	$\text{Max}\left[\log_{10}(\text{R}_{k,t})\right]$
$k = 1$: Sticky-price error	-6.46	-4.86
$k = 2$: Sticky-wage error	-6.05	-4.40
$k = 3$: Euler equation error	-4.10	-2.25
$k = 4$: Error in the FONC w.r.t price inflation	-5.07	-4.12
$k = 5$: Error in the FONC w.r.t real wage	-3.90	-3.41
$k = 6$: Error in the FONC w.r.t. policy rate	-2.94	-2.78

E Sensitivity of results with respect to the calibration of the shock processes

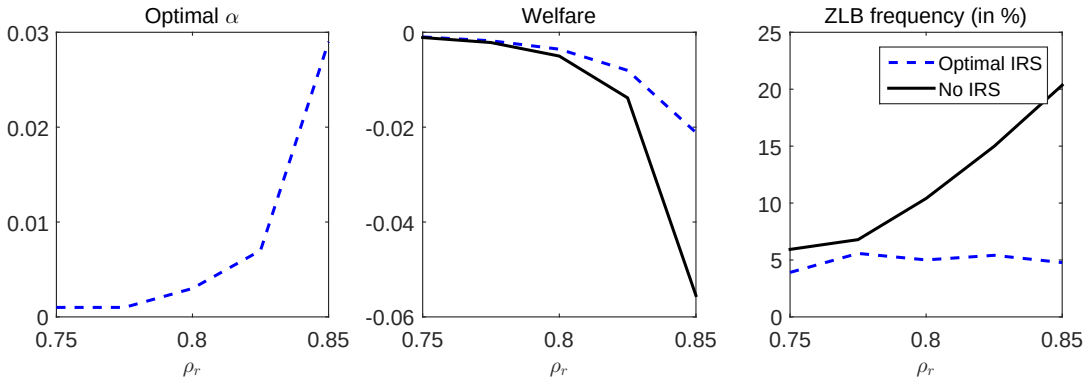
This section documents how the optimal relative weight on the interest-rate smoothing objective, the welfare gains from IRS, and the frequency of a binding ZLB constraint depend on the calibration of the shock processes. The first subsection summarizes results for the simple model of Section 2, and the second subsection for the quantitative model of Section 5.

E.1 Simple model

Figure 1 shows how the calibration of the persistence parameter of the natural real rate shock process, ρ_r, affects the optimal relative weight on the IRS objective (left panel), welfare under the optimal IRS regime and under the standard discretionary regime (middle panel) and the frequency of a binding ZLB constraint under the optimal IRS regime and under the standard discretionary regime (right panel). All other parameters, including the standard deviation of the innovation to the natural real rate shock remain unchanged. Results are only shown for the model with the ZLB, as in the model without the ZLB the efficient equilibrium can be replicated by the standard

discretionary regime.

Figure 1: The role of the persistence of the natural rate shock in the simple model with ZLB

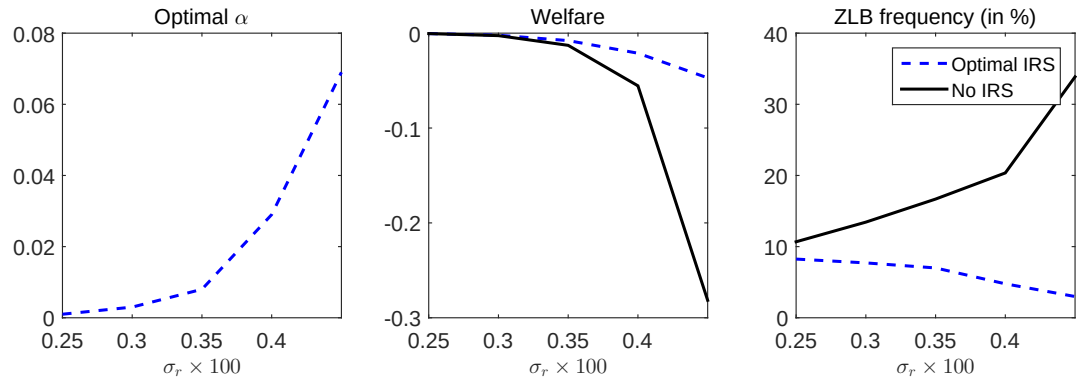


Note: The baseline calibration is $\rho_r = 0.85$. The welfare measure is defined in equation (10) of the paper.

The optimal relative weight on the IRS objective and the welfare gain from optimal IRS—represented by the distance between the blue dashed line and the solid black line in the middle panel—are both increasing in ρ_r. These two results are intuitive since all else equal, the ZLB is binding more often the higher the persistence of the natural real rate shock. This can be seen from the results for the frequency of ZLB events under the standard discretionary regime (solid black line in the right panel). Finally, we find that in the case of the optimized IRS regime, ρ_r has only small (and non-monotonic) effects on the frequency of ZLB events. This reflects the fact that the optimal relative weight on the IRS objective itself is varying with ρ_r.

Figure 2 shows how results depend on the calibration of the standard deviation of the natural real rate shock innovation, σ_r.

Figure 2: The role of the volatility of the natural rate shock in the simple model with ZLB



Note: The baseline calibration is $\sigma_r = 0.40/100$. The welfare measure is defined in equation (10) of the paper.

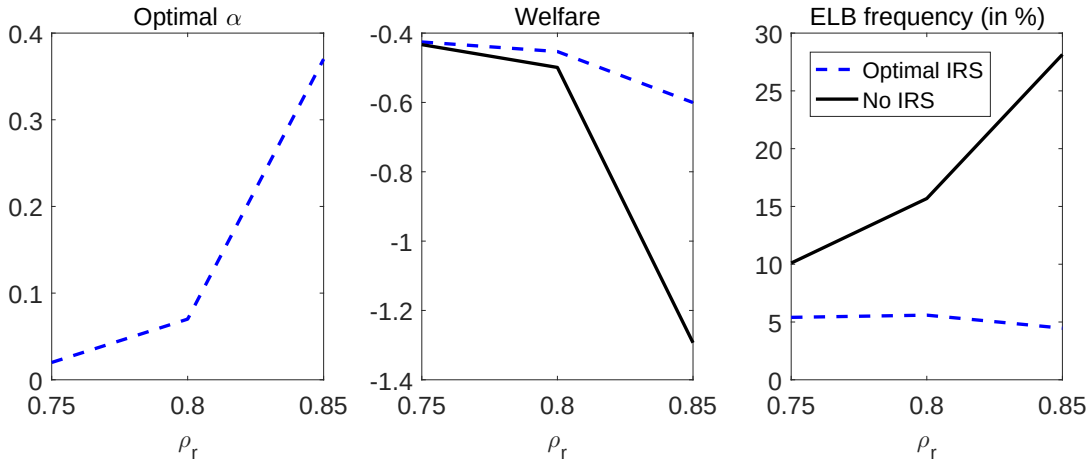
Results are qualitatively similar to those obtained for the persistence parameter ρ_r. In particular, both the optimal relative weight on the IRS objective and the relative welfare gain from optimal IRS are increasing in the standard deviation of the natural real rate shock innovation. The only notable difference is that under the optimal IRS regime, the frequency of a binding ZLB constraint

is strictly declining in σ_r.

E.2 Quantitative model

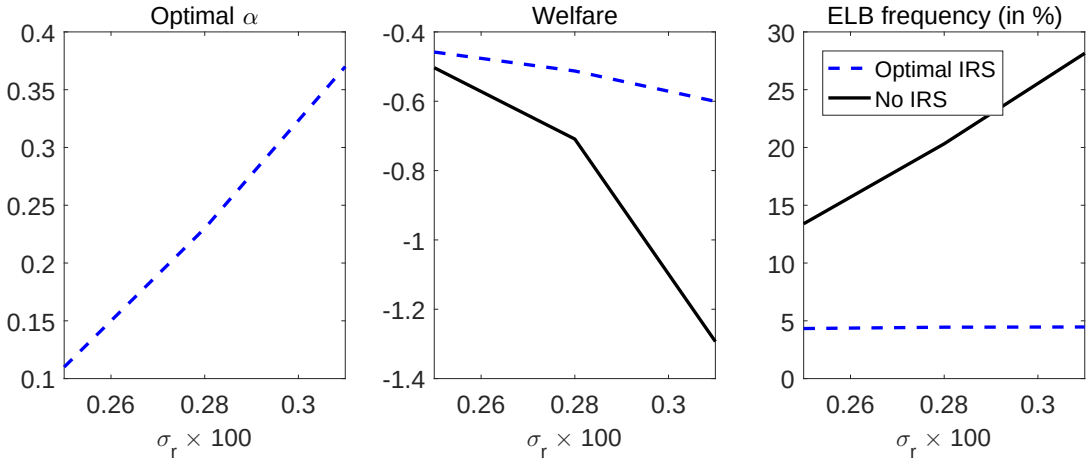
Next, consider the quantitative model. Figure 3 shows how the calibration of the persistence parameter of the natural real rate shock process, ρ_r, affects the optimal relative weight on the IRS objective (left panel), welfare under the optimal IRS regime and under the standard discretionary regime (middle panel) and the frequency of a binding ZLB constraint under the optimal IRS regime and under the standard discretionary regime (right panel). All other parameters, including the standard deviation of the innovation to the natural real rate shock remain unchanged.

Figure 3: The role of the persistence of the natural rate shock in the quantitative model with ELB



Note: The baseline calibration is $\rho_r = 0.85$. The welfare measure is defined in equation (31) of the paper.

Figure 4: The role of the volatility of the natural rate shock in the quantitative model with ELB



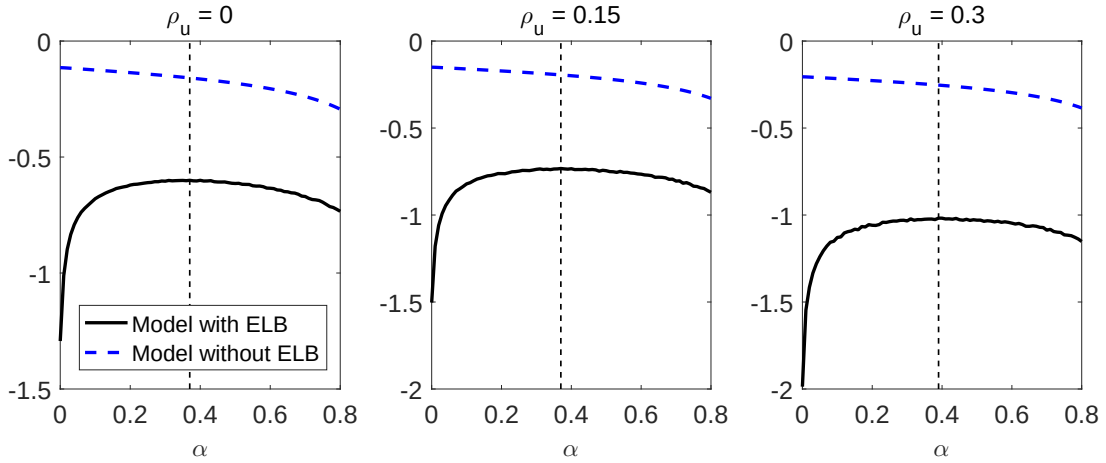
Note: The baseline calibration is $\sigma_r = 0.31/100$. The welfare measure is defined in equation (31) of the paper.

Results are qualitatively similar to those reported for the simple model. In particular, the optimal relative weight on the IRS objective is strictly increasing in ρ_r. The same holds true for

the relative welfare gain from optimal IRS. Figure 4 shows how results depend on the calibration of the standard deviation of the natural real rate shock innovation, σ_r. The optimal α and the relative welfare gains from optimal IRS are strictly increasing in σ_r.

Finally, we explore how results depend on the calibration of the cost-push shock. The baseline version of the quantitative model features only temporary cost-push shocks. We now allow for some persistence in the shock. Figure 5 plots social welfare in the quantitative model as a function of the central bank's relative weight on the interest-rate smoothing objective α for three alternative values of the AR(1) parameter ρ_u.

Figure 5: Welfare in the quantitative model for alternative degrees of cost-push shock persistence



Note: The figure shows how welfare as defined in equation(31) of the paper varies with the relative weight α on the interest-rate smoothing objective. The vertical black dashed line indicates the optimal weight on the smoothing objective in the model with ELB. The optimal weight is zero in the model without the ELB.

In the model without a lower bound on nominal interest rates, it is optimal to put zero weight on the IRS objective. This holds true for all three considered values of ρ_u. While the presence of cost-push shocks makes it desirable to have some endogenous inertia, in the quantitative model such inertia is already induced through the presence of nominal wage rigidities and the presence of partial indexation of prices and wages to past inflation. In the model with the lower bound, the optimal relative weight on the IRS objective is increasing in the persistence of the cost-push shock, but the effect is quantitatively small.

F Interest-rate feedback rule in the quantitative model

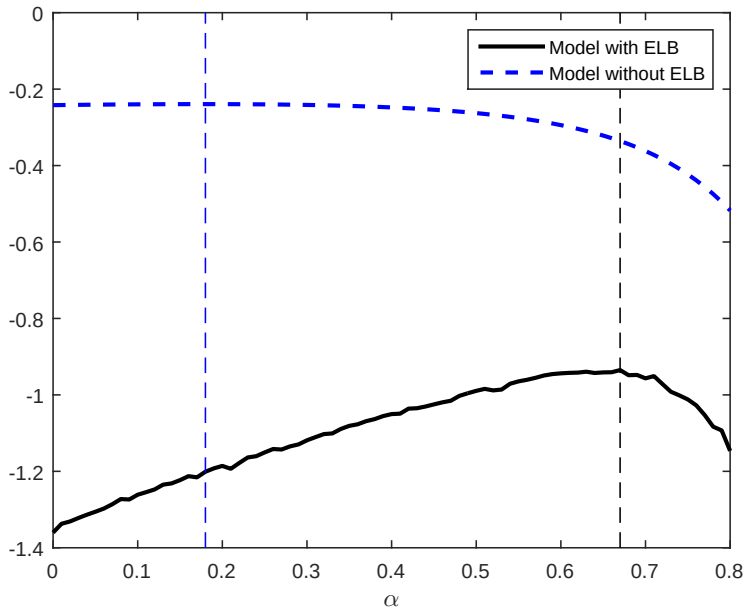
This section explores how the ELB constraint affects the optimal degree of interest-rate smoothing in the quantitative model when monetary policy is governed by a Taylor-type interest-rate rule rather than optimal discretionary policy. The policy rule takes the following form

$$i_t = \max\left[i_{ELB}, \alpha i_{t-1} + (1-\alpha)(r_t^n + \phi_\pi \pi_t)\right], \tag{F.1}$$

where α now represents the response coefficient to the lagged policy rate. We set $\phi_\pi = 3$.

Figure 6 shows how the degree of gradualism α in the policy rule affects welfare in the quantitative model with and without the ELB constraint—indicated by black solid and blue dashed lines, respectively.

Figure 6: Welfare under an interest-rate feedback rule in the quantitative model



Note: The figure shows how welfare as defined in equation (31) of the paper varies with the response coefficient α in interest-rate rule (F.1). The vertical dashed black line indicates the optimized response coefficient with the ELB constraint, and the vertical blue dashed line indicates the optimized response coefficient without the ELB.

Without the ELB constraint, the optimal α is 0.18. Since the economy is buffeted by cost-push shocks, allowing the interest-rate rule to respond to the lagged policy rate increases welfare. However, quantitatively, the welfare improvement is very small. Accounting for the ELB constraint raises the optimal α to 0.67. In the model with the ELB constraint, the welfare gains from optimal interest-rate smoothing are quantitatively non-negligible. All in all, the analysis of the interest-rate feedback rule confirms our main findings on the desirability of gradualism in the presence of a lower bound on nominal interest rates.

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