7 Extensions

In this section, we investigate the generality of the baseline model by considering two extended versions. First, we modify the model, going from fully endogenous growth to semi-endogenous growth, which eliminates scale effects. Second, we introduce physical capital as a factor input into the production of both intermediate goods and innovations. Our analytical results show that the first-best optimal design for the patent instruments is robust to these realistic extensions. However, the quantitative results differ in terms of the magnitudes of welfare comparisons between the decentralized equilibrium and the optimal outcomes.

7.1 Semi-Endogenous Growth

In our baseline setting, the population size is normalized to unity and the equilibrium growth rate of technology depends on the size of research labor, implying that scale effects are present, the same as in the early endogenous Schumpeterian growth models (e.g., Grossman and Helpman (1991) and Aghion and Howitt (1992)). Nonetheless, Jones (1995) and the subsequent studies argue that in modern industrialized economies scale effects are not consistent with the observations of growth. Hence, following Segerstrom (1998) and Chu and Furukawa (2011), we reexamine optimal patent policy and its implications in a quality-ladder model with semi-endogenous growth that removes scale effects, which is done by incorporating population growth and a fishing-out effect on innovations.

The population in this economy (or the representative household) now grows at a constant rate $n \in (0, \rho)$, and its size evolves according to $\dot{N}_t = nN_t$. Consequently, the labor-market-clearing condition becomes $L_t + L_{x,t} + L_{r,t} = N_t$ if each individual is still endowed with one unit of time. The household's preference is given by

$$U = \int_0^\infty e^{-(\rho-n)t} \left[\ln C_t + \phi \ln (L_t/N_t) \right] dt, \tag{35}$$

where C_t is the per capita consumption. Furthermore, the law of motion for per capita assets is $\dot{V}_t = (R_t - n) V_t + W_t (1 - L_t/N_t) - E_t - T_t$, where $E_t \equiv P_t C_t$ and T_t are the nominal consumption expenditures and the lump-sum tax for each individual, respectively. Other notations are the same as before.

Suppose that the arrival rate of innovations is subject to a fishing-out effect: innovations become more difficult to produce as the aggregate technology advances, and the formulation is given by $\lambda_t = (\varphi/Z_t) L_{r,t}$. Therefore, along the BGP, the growth rate of technology requires a constant arrival rate of innovations, implying that this steady-state (equilibrium) growth rate equals the population growth rate, namely, $g = \lambda \ln z = n$. In this case, the steady-state growth rate of technology becomes independent of the population size, and thus, scale effects do not occur. Additionally, in Appendix B, we derive the steady-state equilibrium labor allocations as follows:

$$L_t = \phi \alpha \mu L_{x,t}, \tag{36}$$

$$L_{x,t} = \left[1 + \phi \alpha \mu + \frac{\alpha \lambda (\mu - 1)}{\sigma} \left(\frac{1-s}{\rho - n + \lambda} + \frac{s\lambda}{(\rho - n + \lambda)^2} \right) \right]^{-1} N_t, \tag{37}$$

$$L_{r,t} = \frac{\alpha\lambda(\mu-1)}{\sigma}\left[\frac{1-s}{\rho-n+\lambda}+\frac{s\lambda}{(\rho-n+\lambda)^2}\right]L_{x,t}, \tag{38}$$

where $\lambda = n/\ln z$.

As for the social optimum shown in Appendix B, maximizing (35) subject to the constraints $C_t = Z_t L_{x,t}/N_t$, $L_t + L_{x,t} + L_{r,t} = N_t$, and $\dot{Z}_t = \varphi \ln z L_{r,t}$ yields the optimal labor allocations along the BGP:

$$L_t^* = \phi L_{x,t}^*, \tag{39}$$

$$L_{x,t}^* = \left(1+\phi+\frac{n}{\rho}\right)^{-1} N_t, \tag{40}$$

$$L_{r,t}^* = \frac{n}{\rho} L_{x,t}^*. \tag{41}$$

Comparing the steady-state equilibrium and the social optimum indicates that this extended model features the two similar allocative distortions as in the baseline model. Specifically, comparing the ratio of $L_t/L_{x,t}$ between the steady-state equilibrium in (36) and the social optimum in (39) reveals that the distortion in the relative supply of labor (i.e., the first layer of distortion) is eliminated by optimal patent breadth remaining at $\mu^* = 1/\alpha$, which is still determined by the exogenous subsidy rate for intermediate-goods production. Moreover, using this result and comparing (37)-(38) and (40)-(41) generates the optimal profit-division rule, as follows:

$$s^* = \left(1+\frac{n/\rho}{(1-n/\rho)\ln z}\right)\left[1-\frac{\sigma}{1-\alpha}\left(\ln z\left(1-\frac{n}{\rho}\right)+\frac{n}{\rho}\right)\right]. \tag{42}$$

In the absence of scale effects on economic growth, s^* continues to decrease in z, α, and σ, as in the baseline model. However, (42) implies that n/ρ has both a positive and a negative effect on s^*, and that which effect dominates is ambiguous. Intuitively, the profit-division rule pins down the relative allocation of research labor against other labor inputs, which helps remove the distortion on R&D (i.e., the second layer of distortion). Thus, we investigate this relation by comparing the ratio of R&D-production labor between the steady-state equilibrium and the social optimum. Combining (38) and (41) and rearranging it yields

$$\frac{L_{r,t}/L_{x,t}}{L_{r,t}^*/L_{x,t}^*} = \frac{\frac{1-\alpha}{\sigma}\left[\frac{1-s}{\ln z(\rho/n-1)+1}+\frac{s}{(\ln z(\rho/n-1)+1)^2}\right]}{n/\rho}, \tag{43}$$

where both the numerator and the denominator increase as n/ρ increases. On the one hand, for a relatively small $\ln z$, a higher n/ρ tends to increase $\frac{L_{r,t}/L_{x,t}}{L_{r,t}^*/L_{x,t}^*}$;[1] compared to in the social optimum, in the steady-state equilibrium, too much R&D labor is allocated to overcome the fishing-out effect caused by the increasing complexity of innovations. Thus, s^* rises to offset this impact by depressing the equilibrium R&D.[2] On the other hand, for a relatively large $\ln z$, a higher n/ρ would decrease $\frac{L_{r,t}/L_{x,t}}{L_{r,t}^*/L_{x,t}^*}$; there is too much manufacturing labor allocated in equilibrium. In this case, s^* declines

[1] See Appendix B for the level of $\ln z$ at which the positive effect of n/ρ on s^* dominates the negative one.

[2] Notice that in the numerator of (43), the marginal change of $\frac{-s}{\ln z(\rho/n-1)+1}$ with respect to s outweighs that of $\frac{s}{(\ln z(\rho/n-1)+1)^2}$. Hence, a decrease (increase) in the numerator requires a higher (lower) s.

to stimulate R&D, which promotes innovations and maintains the steady-state equilibrium growth.

We then perform a quantitative analysis to evaluate the welfare differences between the decentralized equilibrium and the optimal outcomes in this extension. For simplicity, our analysis focuses on steady-state welfare.[3] Denote $l_{x,t} \equiv L_{x,t}/N_t$, $l_{r,t} = L_{r,t}/N_t$, and $l_t \equiv L_t/N_t$ as the per capita level of production labor, of R&D labor, and of leisure, respectively, all of which are stationary in the steady-state equilibrium allocations. Imposing the balanced growth on the household's lifetime utility (35) yields the steady-state equilibrium welfare function as follows

$$U = \frac{1}{\rho - n}\left[\ln C_0 + \phi \ln l + \frac{n}{\rho - n}\right], \tag{44}$$

where $C_0 = Z_0 l_x$, $Z_0 = \varphi \ln z l_r N_0/n$, and the exogenous term N_0 will be dropped. The parameters are calibrated by the same values as before, and we set $n = 0.0216$, which is close to the long-run average growth rate of the labor force in the US (roughly 0.02).[4] Thus, the first-best level of patent breadth is still given by $\mu^* = 4/3$, and the first-best profit-division rule is maintained at $s^* = 0.5$. In addition, the second-best outcomes are obtained by optimizing s under $\mu = 1.3$ in Table 5-(1) and by optimizing μ under $s = 0.8$ in Table 5-(2), respectively.[5] [6]

It can be seen that the qualitative patterns of the welfare differences do not differ considerably from those in the baseline model: (a) optimizing a mix of patent instruments yields more welfare improvements than optimizing either of them individually; (b) optimizing μ is much more welfare-enhancing than optimizing s because the majority of the welfare gains in both Tables 5-(1) and 5-(2) are obtained by adjusting μ. Moreover, the welfare effects of these patent instruments become more significant. In contrast to the limited impact of optimizing s in the baseline analysis (as shown by ξ_2 in Table 1-(3)), under the current setting, the welfare gain of using the second-best optimal s^{**} rises to 0.26% for the market equilibrium in which $\mu = 1.3$ and $s = 0.15$ and to 1.04% on average. Also, within the same ranges of the calibrated values, the welfare gains moving from the second-best outcomes to the first-best outcome enlarge drastically under semi-endogenous growth (i.e., the average is 2.89% for s^{**} and 0.53% for μ^{**}) compared to under fully endogenous growth (i.e., the average is 0.029% for s^{**} and 0.001% for μ^{**}). This amplification may be because the

[3] A more complete welfare analysis should take into consideration the dynamic transition of the household's utility from the initial steady state to the final one; however, such an analysis is much more challenging both analytically and numerically in this class of models. Therefore, the current welfare analysis follows the usual treatment in the literature to focus on steady-state welfare when deriving second-best patent policy and computing the welfare differences between the decentralized equilibrium and the optimal outcomes, given that this approach does not usually affect the numerical results by too much despite of neglecting the transitional welfare changes. See, for example, Acemoglu and Akcigit (2012) and Chu, Cozzi, and Galli (2012).

[4] Data source: the Bureau of Labor Statistics.

[5] In this numerical exercise, the steady-state welfare function relies on the household's lifetime utility given by (44). Under semi-endogenous growth, the welfare effect of patent instruments mainly depends on the channel of R&D labor, which dominates the welfare effect through the channels of consumption production and leisure. Thus, if μ is too small (large), which makes the equilibrium R&D level low (high), the welfare improvement of optimizing a coordination of patent instruments will be overstated (undermined). To balance these impacts on allocations, we focus on a medium range of $\mu \in [1.24, 1.3333]$ with a high level of s in Table 5-(2) and choose the steady-state equilibrium value of $\mu = 1.3$ in Table 5-(1), respectively, so that reasonable comparisons can be made against the baseline setting.

[6] Notice that the second-best policy instruments in this exercise are given by corner solutions. Specifically, we obtain the second-best optimal $s^{**} = 0$ in Table 5-(1) and the second-best optimal $\mu^{**} = 1.3333$ in Table 5-(2), respectively.

steady-state welfare function is now measured in terms of the representative household rather than for an individual; the population of the former grows at the rate of n, increasing the discount factor in the welfare function (44).

Table 5: The welfare differences for semi-endogenous growth.

(1) $\mu = 1.3$

s	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	average
ξ_1	3.13	3.23	3.41	3.60	3.78	3.98	4.17	4.36	4.56	4.76	3.93
ξ_2	0.26	0.35	0.53	0.71	0.90	1.08	1.27	1.46	1.65	1.84	1.04
ξ	2.87	2.87	2.88	2.88	2.89	2.89	2.90	2.90	2.91	2.92	2.89

(2) $s = 0.8$

μ	1.24	1.25	1.26	1.27	1.28	1.29	1.30	1.31	1.32	1.33	average
ξ_1	14.82	12.66	10.69	8.89	7.25	5.74	4.36	3.09	1.92	0.84	6.70
ξ_2	14.24	12.10	10.14	8.35	6.71	5.22	3.84	2.58	1.41	0.34	6.17
ξ	0.57	0.56	0.55	0.54	0.54	0.53	0.52	0.51	0.51	0.50	0.53

Notes: ξ_1, ξ_2, and ξ denote the welfare gains in percentage between the equilibrium and the first-best outcome, between the equilibrium and the second-best outcome, and between the second-best outcome and the first-best outcome, respectively.

7.2 Physical Capital Accumulation

In this subsection, we follow Chu (2009) to consider an extension of the semi-endogenous growth model in the previous analysis by incorporating both capital and labor as factor inputs into the production of intermediate goods and innovations.

The population size of the representative household is N_t growing at the rate $n \in (0, \rho)$, and the household's utility function is still given by (35). However, the final goods now can be either consumed by the household or invested in physical capital accumulation. To ensure balanced growth (which will be described below), we choose final goods as the numeraire, implying that $P_t = 1$. Therefore, the law of motion for per capita asset becomes $\dot{A}_t = (R_t - n) A_t + W_t (1 - L_t/N_t) - C_t - T_t$, in which A_t denotes the value of risk-free financial assets in the form of patents and physical capital owned by each household member. The familiar Euler equation derived from the household's optimization yields the growth rate of per capita consumption, such that $g_c \equiv \dot{C}_t/C_t = R_t - \rho$.

The final-goods production follows (5), but the demand for intermediate-goods in industry i becomes $X_t(i) = Y_t/P_t(i)$. The intermediate-goods production function is altered to $X_t(i) = z^{q_t(i)} K_{x,t}^{\theta}(i) L_{x,t}^{1-\theta}(i)$, where $K_{x,t}(i)$ is the capital inputs for producing intermediate goods i and θ is the capital share in production. Using cost minimization, the marginal cost of production for the current leader in industry i is $MC_t(i) = \left(\alpha/z^{q_t(i)}\right) (Q_t/\theta)^{\theta} (W_t/(1-\theta))^{1-\theta}$, where Q_t is the rental price of capital and $1-\alpha$ is the fixed rate of subsidy for intermediate-goods production that applies to the cost of both capital and labor. Using the definition of patent breadth μ_t, the monopolistic price is given by (9), and the leader's profit is $\Pi_t(i) = (1 - 1/\mu_t) P_t(i) X_t(i) = (1 - 1/\mu_t) Y_t$. The factor payments for labor and capital are $\alpha W_t L_{x,t} = ((1-\theta)/\mu_t) Y_t$ and $\alpha Q_t K_{x,t} = (\theta/\mu_t) Y_t$, respectively.

With the infringement of sequential innovations along a quality ladder and the profit-division rule s_t for the licensing agreement between the entrant and the incumbent, the no-arbitrage conditions for the values of the second-most recent innovation and the most recent innovation are given by (11) and (12), respectively. In the R&D sector, the arrival rate of innovations for an R&D firm j is now a function of capital and labor such that $\lambda_t(j) = \bar{\varphi}_t K_{r,t}^{\theta}(j) L_{r,t}^{1-\theta}(j)$, where $\bar{\varphi}$ represents R&D productivity that the R&D firm takes as given. Hence, in the symmetric equilibrium, the first-order conditions for an R&D firm's profit maximization are given by $(1-\theta)V_{1,t}\bar{\varphi}_t\left(K_{r,t}/L_{r,t}\right)^{-\theta} = \sigma W_t$ and $\theta V_{1,t}\bar{\varphi}_t\left(K_{r,t}/L_{r,t}\right)^{\theta-1} = \sigma Q_t$, where $1-\sigma$ is the fixed rate of subsidy for R&D that applies to the cost of both capital and labor. To remove the scale effects, the R&D productivity takes a formulation such that $\bar{\varphi} = \varphi(K_{r,t}^{\theta} L_{r,t}^{1-\theta})^{\gamma-1}/Z_t^{1-\beta}$, where $\gamma \in (0,1)$ captures the negative duplication externality and $\beta \in (-\infty, 1)$ captures the knowledge spillovers externality, respectively. Consequently, the aggregate-level arrival rate of innovations is given by $\lambda_t = \varphi(K_{r,t}^{\theta} L_{r,t}^{1-\theta})^{\gamma}/Z_t^{1-\beta}$, and the growth rate of technology is $g_z \equiv \dot{Z}_t/Z_t = \lambda_t \ln z$.

In the decentralized equilibrium, the final-goods market clears such that $Y_t = C_t N_t + I_t$, where I_t denotes the level of capital investment. The capital market and the labor market clear, such that $K_t = K_{x,t} + K_{r,t}$ and $N_t = L_t + L_{x,t} + L_{r,t}$, respectively. The capital stock evolves according to $\dot{K}_t = Y_t - C_t N_t - \delta K_t$, where δ is the depreciation rate that satisfies the no-arbitrage condition for capital such that $R_t = Q_t - \delta$. Using the intermediate-goods demand and (5) yields the aggregate production function of the final goods, such that $Y_t = Z_t K_{x,t}^{\theta} L_{x,t}^{1-\theta}$. On the balanced growth path, each labor allocation grows at the rate n. Then, the growth rate of technology is given by $g_z = \gamma(\theta g_k + (1-\theta)n)(1-\beta)^{-1}$, where g_k denotes the capital growth rate. Also, the growth rate of consumption per capita is $g_c = g_y - n$, where g_y denotes the growth rate of final goods. Imposing the BGP on the aggregate production function Y_t yields $g_k = g_y = n + g_z/(1-\theta)$, which implies $g_z = n[(1-\beta)/\gamma - \theta/(1-\theta)]^{-1}$. Thus, the steady-state arrival rate of innovations is given by $\lambda = g_z/\ln z$, and the steady-steady real interest rate is $R_t = R = \rho + g_c = \rho + g_y - n$.

Denote the rate of capital investment by $i_t \equiv I_t/Y_t$. In Appendix C, we derive the steady-state equilibrium labor allocations as follows:

$$L_t = (1-i)\frac{\phi\alpha\mu}{1-\theta} L_{x,t}, \tag{45}$$

$$L_{x,t} = \left[1 + (1-i)\frac{\phi\alpha\mu}{1-\theta} + \frac{\alpha\lambda(\mu-1)}{\sigma}\left(\frac{1-s}{\rho-n+\lambda} + \frac{s\lambda}{(\rho-n+\lambda)^2}\right)\right]^{-1} N_t, \tag{46}$$

$$L_{r,t} = \frac{\alpha\lambda(\mu-1)}{\sigma}\left[\frac{1-s}{\rho-n+\lambda} + \frac{s\lambda}{(\rho-n+\lambda)^2}\right] L_{x,t}, \tag{47}$$

and the steady-state equilibrium R&D share of factor inputs and the rate of capital investment as follows:

$$\frac{m}{1-m} = \frac{\alpha\lambda(\mu-1)}{\sigma}\left[\frac{1-s}{\rho-n+\lambda} + \frac{s\lambda}{(\rho-n+\lambda)^2}\right], \tag{48}$$

$$i = \frac{\theta}{\alpha\mu}\left(1 + \frac{m}{1-m}\right)\frac{g_k+\delta}{R+\delta}, \tag{49}$$

where $m = m_k = m_l$, $m_k \equiv K_{r,t}/K_t$, and $m_l \equiv L_{r,t}/(N_t - L_t)$.

As for the first-best allocations, the social planner chooses leisure L_t, the ratio of R&D allocation m_t, and the rate of capital investment i_t to maximize the households' lifetime utility, subject to the constraints of final-goods production, labors, capital accumulation, and technology. The derivation is shown in Appendix C. This optimization yields the optimal R&D share of factor inputs given by

$$\frac{m^*}{1-m^*} = \frac{L^*_{r,t}}{L^*_{x,t}} = \frac{K^*_{r,t}}{K^*_{x,t}} = \frac{\gamma g_z}{\rho - n + (1-\beta) g_z}, \tag{50}$$

the optimal rate of capital investment given by

$$i^* = \theta \left(1 + \frac{m^*}{1-m^*}\right) \frac{g_k + \delta}{R + \delta}, \tag{51}$$

and the optimal ratio of leisure and production labor given by

$$L^*_t = (1 - i^*) \frac{\phi}{1-\theta} L^*_{x,t}. \tag{52}$$

Notice that the mix of μ and s suffices to equate $m/(1-m)$ and i_t under the steady-state equilibrium (i.e., (48) and (49)) to $m^*/(1-m^*)$ and i^*_t under the social optimum (i.e., (50) and (51)). Hence, comparing $L_t/L_{x,t}$ in (44) and $L^*_t/L^*_{x,t}$ in (52) reveals that optimal patent breadth continues to be $\mu^* = 1/\alpha$. In addition, using this result and comparing (48) and (50) yields the optimal profit-division rule such that

$$s^* = \frac{1}{1 - 1/\Phi} \left[1 - \frac{\sigma}{1-\alpha} \left(\frac{\gamma}{(1-1/\Phi)/\ln z + (1-\beta)/\Phi}\right)\right], \tag{53}$$

where $\Phi \equiv \ln z [(1-\beta)/\gamma - \theta/(1-\theta)](\rho/n - 1) + 1$. Obviously, s^* in (53) reduces to the one without capital accumulation in (42) when $\beta = 0$, $\gamma = 1$, and $\theta = 0$.

Comparing the steady-state equilibrium and the social optimum indicates that this extended model features four layers of allocative distortions: (a) the distortion on the relative supply of labor, given by L/L_x; (b) the distortion on the relative allocation of R&D in labor, given by L_r/L_x; (c) the distortion on the relative allocation of R&D in capital, given by K_r/K_x; and (d) the distortion on the investment for capital accumulation, given by i. However, the above result demonstrates that the optimal coordination of patent instruments is sufficient for correcting all these distortions and reaching the first-best outcome. The reason is straightforward. Along the BGP, the R&D share of factor inputs is identical under both labor and capital, implying that the distortions (b) and (c) coincide with each other and can be removed simultaneously. Furthermore, as shown in Chu (2009), the discrepancy between the steady-state rate of capital investment and the first-best rate (i.e., distortion (d)) stems from the markup μ in addition to the difference between the equilibrium R&D share of capital and its socially optimal share (i.e., distortion (c)). More importantly, the discrepancy between the steady-state ratio of leisure and production labor and its first-best counterpart (i.e., distortion (a)) stems from the markup μ in addition to the difference between the steady-state rate of capital investment and the first-best rate (i.e., distortion (d)). In other words, once the wedges in allocations caused by the markup and the R&D share of factor inputs (i.e., $m/(1-m)$) are eliminated, all of the above four distortions are simultaneously remedied; thus, the first-best outcome can be restored. This objective is achievable by employing patent breadth together with the profit-division rule to adjust the effects of μ and $m/(1-m)$.

As for the comparative statics of the optimal profit-sharing rule, observing (53) shows that s^* continues to decrease in α and σ, as in the prior models. In addition, combining (47) and (50) yields the labor allocation in R&D relative to manufacturing, such that

$$\frac{L_{r,t}/L_{x,t}}{L^*_{r,t}/L^*_{x,t}} = \frac{1-\alpha}{\sigma}\left[\frac{1-s}{\tilde{\Phi}\ln z(\rho/n-1)+1} + \frac{s}{(\tilde{\Phi}\ln z(\rho/n-1)+1)^2}\right]\left[\frac{\tilde{\Phi}(\rho/n-1)+1-\beta}{\gamma}\right], \quad (54)$$

where $\tilde{\Phi} \equiv (1-\beta)/\gamma - \theta/(1-\theta)$. Therefore, like the similar reasoning applied in (43), s^* is strictly decreasing in z, but it could be increasing or decreasing in n/ρ, β, and θ, depending on the size of $\ln z$.

We denote $l_t \equiv L_t/N_t$, $l_{x,t} \equiv L_{x,t}/N_t$, and $k_{x,t} \equiv K_{x,t}/N_t$ similarly as in Section 7.1. To quantify the welfare comparisons between the decentralized equilibrium and the optimal outcomes in this extension, imposing balanced growth on (35) yields the steady-state equilibrium welfare function given by

$$U = \frac{1}{\rho-n}\left[\ln C_0 + \phi \ln l + \frac{g_c}{\rho-n}\right], \quad (55)$$

where $C_0 = (1-i)Z_0(1-m)^\theta k_0^\theta l_x^{1-\theta}$, $Z_0 = [\varphi \ln z(N_0 m^\theta k_0^\theta l_r^{1-\theta})^\gamma/g_z]^{1/(1-\beta)}$, $g_c = g_z/(1-\theta) = [(1-\beta)/\gamma - \theta/(1-\theta)]^{-1} n/(1-\theta)$, and the exogenous terms N_0 and k_0 will be dropped. Here, we continue to focus on steady-state welfare and then derive the second-best outcomes by optimizing s under $\mu = 1.3$ in Table 6-(1) and by optimizing μ under $s = 0.8$ in Table 6-(2), respectively (See Footnote 3). The parameters are again calibrated as previously, except that we set $n = 0.01$ to match the long-run average growth rate of the US population.[7] There are four structural parameters $\{\theta, \delta, \gamma, \beta\}$ that are new in this numerical exercise. First, we set the annual depreciation rate δ on physical capital and the capital-share parameter θ to their conventional values of 0.08 and 0.3, respectively. As for $\{\gamma, \beta\}$, we use $R = 0.08$ as the market level of real interest rate, which is in line with the historical rate of return on the US stock market. Accordingly, maintaining the first-best value of s^* at 0.5 as in our previous analysis, the calibrated value of the R&D duplication externality γ and that of the knowledge spillovers externality β are approximately 0.98 and 0.23, respectively. The first-best level of patent breadth μ^* remains at 4/3.

As shown in Table 6, the qualitative patterns of the numerical results are analogous to those in the previous analysis.[8] It is worthwhile to note that the size of welfare gains in this exercise is even much larger than those in the previous models. For example, in Table 6-(1) where s is optimized given the level of μ, starting off with the market equilibrium in which $\mu = 1.3$ and $s = 0.15$, ξ_1, ξ_2, and ξ increases from 0.033%, 0.004%, and 0.029% in the baseline model (or 3.13%, 0.26%, 2.87% in semi-endogenous growth without physical capital) to 6.99%, 0.44%, and 6.55% with physical capital. Furthermore, compared to the setting of no capital in the last subsection, optimizing s in the presence of physical capital accumulation can better enhance welfare for all equilibrium levels of s, with the upper bound of the welfare gain increasing from 1.84% to 3.19% and the average increasing from 1.04% to 1.78%. The range of these welfare sizes thus falls in line with Chu (2009).

[7]Data source: World Development Indicators. Although the choice of n is changed in this exercise, the implied growth rate of technology is $g_z = 0.028$, which is only slightly lower than the counterpart in the previous subsection (i.e., 0.0216).

[8]The second-best policy instruments in this subsection are still given by corner solutions, yielding the second-best optimal $s^{**} = 0$ in Table 6-(1) and the second-best optimal $\mu^{**} = 1.3333$ in Table 6-(2), respectively.

This pattern on the changes in the welfare sizes is also true for the scenario where μ is optimized given the level of s, as shown in Table 6-(2).

The reasons for the above changes are twofold. First, the current model follows the foregoing semi-endogenous growth model to use a utility function stemming from the representative household, which is embedded with a larger discount factor than the one for an individual. Second, four layers of distortions exist in this model as mentioned. Specifically, in addition to the distortions in labor allocations $\{L, L_x, L_r\}$, distortions in the rate of capital investment i and the R&D share of capital inputs $\{K_r, K_x\}$ are introduced. Because of these extra distortions, the steady-state welfare level in the current model tends to be considerably lower than in the first-best outcome and in the second-best outcomes, and optimizing one patent instrument alone or a mix of them effectively remedies two more layers of distortions in this variant than in the previous ones. Thus, it is not surprising that welfare improvements in this exercise turn out to be more remarkable when appropriate policy interventions are executed.

Table 6: The welfare differences for semi-endogenous growth with physical capital accumulation.

(1) $\mu = 1.3$

s	0.15	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	average
ξ_1	6.99	7.15	7.48	7.81	8.14	8.49	8.83	9.19	9.55	9.91	8.42
ξ_2	0.44	0.60	0.90	1.21	1.53	1.85	2.18	2.51	2.84	3.19	1.78
ξ	6.55	6.56	6.58	6.60	6.62	6.64	6.66	6.68	6.70	6.72	6.63

(2) $s = 0.8$

μ	1.24	1.25	1.26	1.27	1.28	1.29	1.3	1.31	1.32	1.33	average
ξ_1	31.82	27.11	22.84	18.96	15.41	12.17	9.19	6.45	3.92	1.58	14.24
ξ_2	30.73	26.06	21.82	17.97	14.46	11.24	8.28	5.56	3.06	0.73	13.29
ξ	1.09	1.05	1.02	0.99	0.96	0.93	0.90	0.88	0.86	0.84	0.95

Notes: ξ_1, ξ_2, and ξ denote the welfare gains in percentage between the equilibrium and the first-best outcome, between the equilibrium and the second-best outcome, and between the second-best outcome and the first-best outcome, respectively.

Appendix B

In this appendix, we derive the labor allocations in the steady-state equilibrium and in the social optimum, respectively, under semi-endogenous growth. Additionally, we show the comparative statics of the optimal profit-division rule in (42).

Steady-State and Optimal Labor Allocations

With the arrival rate of innovations being subject to the fishing-out effect, normalizing W_t to unity implies that V_t, $V_{1,t}$, and $V_{2,t}$ all grow at the rate n along the balanced growth path. Therefore, using (11) and (12), we obtain that $V_{2,t} = \frac{s\Pi_t}{R_t - n + \lambda}$ and $V_{1,t} = \left[\frac{1-s}{R_t - n + \lambda} + \frac{s\lambda}{(R_t - n + \lambda)^2}\right]\Pi_t$. Moreover, combining the monopoly profit (i.e., $\Pi_t = (1 - 1/\mu)N_t E_t$), the FOC for the R&D labor (i.e., $V_{1,t} = \sigma L_{r,t}/\lambda$), and the labor-market-clearing condition (i.e., $L_t + L_{x,t} + L_{r,t} = N_t$), a few steps of simplification yield $\frac{V_{1,t}}{\Pi_t} = \frac{\sigma(1/E_t - 1/(\alpha\mu) - \phi)}{\lambda(1-1/\mu)}$, where we also use the leisure-consumption relation (i.e., $W_tL_t = \phi E_tN_t$) and the production-labor share of output (i.e., $\alpha W_tL_{x,t} = E_tN_t/\mu$). These conditions imply that E_t is constant, given that labor inputs grow at the rate n along the BGP.

Thus, we can derive the relationship between R_t and E_t as follows:

$$\frac{1-s}{R_t-n+\lambda}+\frac{s\lambda}{(R_t-n+\lambda)^2}=\frac{\sigma(1/E_t-1/(\alpha\mu)-\phi)}{\lambda(1-1/\mu)}. \tag{B.1}$$

From (B.1), we know that R_t is an increasing function of E_t, so the Euler equation implies that $\dot{E}_t/E_t$ is also increasing in E_t. In addition, when $E_t \to 0$, $R_t \to n-\lambda$ and $\dot{E}_t/E_t < 0$ because $n < \rho$. Then, the steady-state equilibrium value of E_t must be positive. Hence, setting the Euler equation to zero yields $E = \frac{\alpha\sigma\mu}{\alpha\lambda(\mu-1)[(\rho-n)(1-s)+\lambda]/(\rho-n+\lambda)^2+\sigma(1+\alpha\mu\phi)}$, which implies $R_t = \rho$ along the BGP.

Consequently, from the production-labor share of output and the FOC for R&D labor, (B.1) implies $\frac{L_{r,t}}{L_{x,t}} = \frac{\alpha\lambda(\mu-1)}{\sigma}\left[\frac{1-s}{\rho-n+\lambda}+\frac{s\lambda}{(\rho-n+\lambda)^2}\right]$. Making use of the leisure-consumption relation, it is easy to derive the steady-state equilibrium labor allocations in (36)-(38).

As for the optimal labor allocations, the social planner maximizes (35) subject to the constraints of consumption production, technology, and labors, yielding the following current-value Hamiltonian:

$$H_t = \ln\frac{Z_t(1-m_t)(N_t-L_t)}{N_t}+\phi\ln\frac{L_t}{N_t}+\eta_t\left[m_t(N_t-L_t)\varphi\ln z\right], \tag{B.2}$$

where $m_t = L_{r,t}/(N_t-L_t)$ is the ratio of R&D labor and η_t is the costate variable associated with the law of motion for technology. Therefore, the FOCs for L_t and m_t are given, respectively, as follows:

$$\frac{\partial H_t}{\partial L_t}=0 \Rightarrow -\frac{1}{N_t-L_t}+\frac{\phi}{L_t}=\eta_t m_t\varphi\ln z; \tag{B.3}$$

$$\frac{\partial H_t}{\partial m_t}=0 \Rightarrow \frac{1}{1-m_t}=\eta_t(N_t-L_t)\varphi\ln z; \tag{B.4}$$

$$\frac{\partial H_t}{\partial Z_t}=\frac{1}{Z_t}=\eta_t(\rho-n)-\dot{\eta}_t. \tag{B.5}$$

Applying $1-m_t = L_{x,t}/(N_t-L_t)$ in (B.3) and (B.4) yields (39). Then, manipulating (B.5) and using the fact that $\dot{Z}_t = nZ_t$ along the BGP yields a differential equation such that $\dot{\eta}_tZ_t+\eta_t\dot{Z}_t = \rho\eta_tZ_t-1$. Integrating this equation with respect to time implies that $\eta_tZ_t = 1/\rho$. Using this result with (B.4) and the definition of $\dot{Z}_t$ yields (41). Finally, combining (39), (41), and the labor-market-clearing condition yields (40).

Comparative Statics of the Optimal Profit-Division Rule

First, we find out the range of n/ρ and $\ln z$ to ensure that s^* in (42) lies between 0 and 1, that is, $n/\rho < \frac{1-\alpha}{\sigma}$ and $\ln z \in (\ln z_1, \ln z_0)$, where $\ln z_1 \equiv \frac{-(n/\rho)+\sqrt{(1-\alpha)(n/\rho)/\sigma}}{1-n/\rho}$ is the boundary for s^* to be less than 1 and $\ln z_0 \equiv \frac{1-\alpha-\sigma(n/\rho)}{\sigma(1-n/\rho)}$ is the boundary for s^* to be greater than 0, respectively.

Next, from (42), s^* decreases as z, σ, and/or α increases. Nevertheless, n/ρ has both a positive and a negative effect on s^*. To see the conditions under which the positive effect dominates the negative one, taking the derivative of s^* with respect to n/ρ yields

$$\frac{\partial s^*}{\partial(n/\rho)} = \frac{\sigma(\ln z - 1)^2 - \frac{\alpha+\sigma-1}{(1-n/\rho)^2}}{(1-\alpha)\ln z}, \tag{B.6}$$

where the denominator is positive. Thus, whether or not $\partial s^*/\partial(n/\rho)$ is positive depends on the sign of the numerator, which is determined by the sign of $\alpha+\sigma-1$ and the magnitude of $\ln z$. There are two cases to be considered.

(1) Suppose $\alpha+\sigma-1>0$. Then, the numerator in (B.6) can be viewed as a function of $\ln z$, and the roots of the numerator are $\ln z^{\pm} \equiv 1 \pm \frac{\sqrt{\alpha+\sigma-1}}{\sqrt{\sigma}(1-n/\rho)}$. To guarantee that $\ln z^- > 0$, we assume $n/\rho < 1-\sqrt{\frac{\alpha+\sigma-1}{\sigma}} \equiv \overline{\chi}$, where $\overline{\chi} < \frac{1-\alpha}{\sigma}$. Also, it is easy to verify that $\ln z^+ > \ln z_0 > \max\{\ln z^-, \ln z_1\}$ and that $\ln z^- \gtreqless \ln z_1$ if and only if $n/\rho \lesseqgtr \frac{\alpha+2\sigma-1-2\sqrt{\sigma(\alpha+\sigma-1)}}{1-\alpha} \equiv \underline{\chi}$, where $0<\underline{\chi}<\overline{\chi}$. Thus, comparing $\ln z^-$ and $\ln z_1$ implies the following four possibilities for the effect of n/ρ on s^*:
(i) when $n/\rho \in (\underline{\chi}, \overline{\chi})$, the set of $\ln z$ that makes s^* bounded between 0 and 1 and increasing in n/ρ is empty;
(ii) when $n/\rho \in (\underline{\chi}, \overline{\chi})$, the set of $\ln z$ that makes s^* bounded between 0 and 1 and decreasing in n/ρ is $(\ln z_1, \ln z_0)$;
(iii) when $n/\rho \in (0, \underline{\chi})$, the set of $\ln z$ that makes s^* bounded between 0 and 1 and increasing in n/ρ is $(\ln z_1, \ln z^-)$; and
(iv) when $n/\rho \in (0, \underline{\chi})$, the set of $\ln z$ that makes s^* bounded between 0 and 1 and decreasing in n/ρ is $(\ln z^-, \ln z_0)$.
In other words, the numerator and $\partial s^*/\partial(n/\rho)$ are positive (or negative) if $\ln z$ is sufficiently small (i.e., $\ln z < \ln z^-$ in (iii)) (or large (i.e., $\ln z > \ln z_1 > \ln z^-$ in (ii) and $\ln z > \ln z^-$ in (iv)).

(2) Suppose $\alpha+\sigma-1<0$. Then, the numerator in (B.6) is positive, so s^* is strictly increasing in n/ρ, regardless of the level of $\ln z$. However, this case does not apply when subsidies to intermediate goods and research are relatively small (namely, α and σ are relatively large), which is more likely to occur.

Appendix C

In this appendix, under the extended version of the blocking-patents model with physical capital accumulation, we derive the allocations of labor and capital in addition to the rate of capital investment in the steady-state equilibrium and in the first-best outcome, respectively.

The rate of capital investment is denoted by $i_t \equiv I_t/Y_t$. As for the steady-state equilibrium allocations, on the balanced growth path, using (11) and (12) yields $V_{1,t} = \left[\frac{1-s}{R_t-g_y+\lambda} + \frac{s\lambda}{(R_t-g_y+\lambda)^2}\right]\Pi_t$, where $R_t = g_c+\rho = g_y-n+\rho$. Moreover, combining this condition with the capital factor payment and the FOC for capital in R&D (i.e., $\alpha Q_t K_{x,t} = (\theta/\mu_t)\,Y_t$ and $\theta V_{1,t}\bar{\varphi}_t\,(K_{r,t}/L_{r,t})^{\theta-1} = \sigma Q_t$) yields (48), implying that $m_{k,t}$ is stationary, given that μ, s, and λ are constant along the BGP. Then, using capital accumulation $i_t = (g_k+\delta)K_t/Y_t$, the capital factor payment, and the no-arbitrage condition for capital (i.e., $Q_t = R_t+\delta$) yields (49), implying that i_t depends on m and is thus stationary.

Next, using the labor factor payment (i.e., $\alpha W_t L_{x,t} = ((1-\theta)/\mu_t)\,Y_t$), the FOC for labor in

R&D (i.e., $(1-\theta)V_{1,t}\bar{\varphi}_t\left(K_{r,t}/L_{r,t}\right)^{-\theta}=\sigma W_t$), and the relation between $V_{1,t}$ and Π_t yields (46), which implies that $m_{l,t}$ is stationary and confirms $m=m_k=m_l$. Furthermore, the final-goods resource constraint implies $(1-i_t)Y_t=C_tN_t$. Combining this constraint together with the labor factor payment and the leisure-consumption condition (i.e., $W_tL_t=\phi C_tN_t$) yields (45). Substituting (45) and (46) into the labor-market-clearing condition yields (47).

As for the first-best allocations, the social planner chooses L_t, m_t, and i_t to maximize the households' utility subject to the constraints of $Y_t=Z_tK_{x,t}^{\theta}L_{x,t}^{1-\theta}$, $\dot{K}_t=i_tY_t-\delta K_t$, $N_t=L_t+L_{x,t}+L_{r,t}$, and $\dot{Z}_t=\varphi\ln z(K_{r,t}^{\theta}L_{r,t}^{1-\theta})^{\gamma}Z_t^{\beta}$, yielding the following current-value Hamiltonian:

$$
\begin{aligned}
H_t=\ln(1-i_t)\frac{Z_t(1-m_t)K_t^{\theta}(N_t-L_t)^{1-\theta}}{N_t}+\phi\ln\frac{L_t}{N_t}&+\zeta_{k,t}\left[i_tZ_t(1-m_t)K_t^{\theta}(N_t-L_t)^{1-\theta}-\delta K_t\right]\\
&+\zeta_{z,t}\left[Z_t^{\beta}m_t^{\gamma}K_t^{\gamma\theta}(N_t-L_t)^{\gamma(1-\theta)}\varphi\ln z\right],
\end{aligned}
\tag{C.1}
$$

where $\zeta_{k,t}$ and $\zeta_{v,t}$ are the costate variables associated with the law of motion for capital and for technology, respectively. Therefore, the FOCs for L_t, m_t, and i_t are given, respectively, as follows:

$$\frac{\partial H_t}{\partial L_t}=0\Rightarrow\frac{\phi}{1-\theta}\frac{N_t-L_t}{L_t}=1+\zeta_{k,t}i_tY_t+\zeta_{z,t}\gamma g_zZ_t; \tag{C.2}$$

$$\frac{\partial H_t}{\partial m_t}=0\Rightarrow\zeta_{z,t}\gamma g_zZ_t\frac{1-m_t}{m_t}=1+\zeta_{k,t}i_tY_t; \tag{C.3}$$

$$\frac{\partial H_t}{\partial i_t}=0\Rightarrow(1-i_t)\zeta_{k,t}Y_t=1; \tag{C.4}$$

$$\frac{\partial H_t}{\partial K_t}=(\rho-n)\zeta_{k,t}-\dot{\zeta}_{k,t}\Rightarrow\theta\left(1+\zeta_{k,t}i_tY_t+\zeta_{z,t}\gamma g_zZ_t\right)=(R_t+\delta)\zeta_{k,t}K_t; \tag{C.5}$$

$$\frac{\partial H_t}{\partial Z_t}=(\rho-n)\zeta_{z,t}-\dot{\zeta}_{z,t}\Rightarrow1+\zeta_{k,t}i_tY_t=\left[\rho-n+(1-\beta)g_z\right]\zeta_{z,t}Z_t, \tag{C.6}$$

where we use the fact that $g_{\zeta_k}=-g_y=-g_c+n$ in (C.5) and that $g_{\zeta_z}=-g_z$ in (C.6) along the BGP. Combining (C.3) and (C.6) immediately yields (50). In addition, substituting (C.3) and (C.4) into (C.5) and applying the law of motion for capital yields (51). Finally, combining (C.2)-(C.4) and using the definition of $m^*=L_{r,t}^*/(N_t-L_t^*)$ yields (52).

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