Baby Busts and Baby Booms: The Fertility Response to Shocks in Dynastic Models Online Appendix

Larry E. Jones

University of Minnesota and Federal Reserve Bank of Minneapolis

Alice Schoonbroodt

The University of Iowa and CPC

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Contents

O.1 Proof of Proposition 1

Initial Condition: There is an initial age distribution of the population given by (N_0^2, N_0^1) where N_0^2 is the number of initial old workers (i.e., their age in period $t = 0$ is $a = 3$). We normalize by assuming that $N_0^2 = 1$.

Preferences of Initial Old: Similarly to equation (1), we can define the continuation utility for a person who was a young adult in period t (i.e., he was born in period $t - 1$) looking forward from the point when he reaches age $a = 2$. Of particular interest is the continuation utility of an adult of age 2 in period 0 – i.e., the initial old. These are the only agents in the model who care about all agents of all ages in all periods. Sequentially substituting V and U for later generations and grouping terms by period instead of generations, utility for the initial old, U_0^2, is given by:

$$U_0^2 = E_0 \sum_{t=0}^{\infty} (\phi\beta)^t \left[\sum_{a=1}^{2} \phi^{2-a} \left[\Pi_{k=-1}^{t-a} g(n_k) \right] u(c_t^a) \right] \tag{O.1}$$

Assuming $g(n) = n^\eta$, simplification occurs in equation (O.1) because $g(n_t)g(n_{t+1}) = g(n_t n_{t+1})$. To see this, let N_t^a be the number of descendants of the initial age 2 agent, that are of age a in period t. Then the laws of motion for population are given by:

$$\begin{aligned} N_t^1 &= n_{t-1} N_{t-1}^1 \text{ is the number of births in period } t-1; \\ N_t^2 &= N_{t-1}^1; \\ N_t^a &= 0 \text{ for } a > 2; \\ N_0^2 &= 1 \qquad \text{given;} \\ N_0^1 &= n_{-1} \quad \text{given.} \end{aligned} \tag{O.2}$$

Using the functional form assumption for g and the laws of motion for population, U_0^T can be written as:

$$U_0^2 = E_0 \sum_{t=0}^{\infty} (\phi\beta)^t \left[\sum_{a=1}^{2} \phi^{2-a} g(N_t^a) u(c_t^a) \right] \tag{O.3}$$

Hence, as is standard in dynastic models à la Barro-Becker, the utility of the initial old can be written in terms of the number of descendants of age a in period t, N_t^a.

Feasibility from the Initial Old's Point of View: Multiplying feasibility per old worker in period t, equation (3), by the number of old worker descendants of the initial old, N_t^2, we get:

$$\sum_{a=1}^{2} N_t^a c_t^a + \theta_t(w_t^1) N_{t+1}^1 \leq \sum_{a=1}^{2} w_t^a N_t^a \equiv W_t \tag{O.4}$$

where N_{t+1}^1 is the total number of births in period t. If the costs are in terms of time, $\theta_t = b w_t^1$, there is an additional constraint, namely that $0 \leq b N_{t+1}^1 \leq N_t^1$.

Dynastic Planner's Problem and Equivalence with Equilibrium: Consider the Dynastic Planner's Problem, $P(\{N_0^a\}, s_0)$, which is to choose $\left\{\{c_t^a(s^t)\}_{a=1}^3, N_{t+1}^1(s^t)\right\}_{t=0}^{\infty}$ to maximize utility in (O.3) subject to feasibility in (O.4) and laws of motion of population in (O.2) where $s^t = (s_0, s_1, ..., s_t)$ is the history of shocks up to and including period t.

The solution to this problem has the feature that the resulting allocation can also be supported dynamically as the equilibrium of a game in which in each period, the oldest individuals make choices for all variables for all of their descendants. The reason for this is that the Dynastic Planner discounts the utility from consumption and children in period, t, at $(\phi\beta)^t$, but trades off all the choices within period t in the same way as age $a = 2$ agents in period t.

The equilibrium of this game can further be decentralized through state contingent transfers as described in the main text. The reason for this is that, the only disagreement between age $a = 1$ children and age $a = 2$ parents is consumption of the parent c_t^2. Given a budget (net of transfers) when young, age $a = 2$ parents and age $a = 1$ children agree on how to split it between consumption at age $a = 1$, c_t^1, and births, n_t.

Hence, since parents can impose a (possibly negative) transfer on their children when they are both adults after the current shock is observed and thanks to our timing assumption, the equilibrium allocation in which every person decides on their own consumption and fertility in addition to the transfer is equivalent to the Dynastic Planner's allocation in aggregates.[1]

Procyclical Fertility and Endogenous Oscillations: Next, we study the properties of the solution to the Planner's Problem outlined above. In particular, we derive comparative statics of current fertility with respect to productivity shocks and last period's fertility. To do this, we first simplify the problem to one with only one state variable. We then take first-order conditions and analyze comparative statics across steady states/balanced growth paths therein.

Since both, c_t^a and N_t^a are choice variables, the feasible set is not convex as written.

[1]See Schoonbroodt and Tertilt (2014) for a detailed proof of this equivalence in a deterministic setting.

Let $C_t^a = N_t^a c_t^a$. Then, given our functional form assumptions, utility in equation (O.3) can be rewritten as:

$$U_0^2 = E_0 \sum_{t=0}^{\infty} (\phi\beta)^t \left[\sum_{a=1}^{2} \phi^{2-a} (N_t^a)^{\eta+\sigma-1} \frac{(C_t^a)^{1-\sigma}}{1-\sigma} \right] \tag{O.5}$$

and feasibility as

$$\sum_{a=1}^{2} C_t^a + \theta_t(w_t^1) N_{t+1}^1 \leq \sum_{a=1}^{2} w_t^a N_t^a \equiv W_t \tag{O.6}$$

where utility is now increasing and concave under both parameter configurations, AI. and AII., and the constraint set is convex.[2]

Denote by $V(N^1, N^2; s)$ the maximized value obtained in the problem $P(\{N_0^a\}, s_0)$ when the initial conditions are $(N_0^1, N_0^2) = (N^1, N^2)$ and $s_0 = s$. Because of our assumptions on the functional forms for the utility function, it is straightforward to show that the value function is homogeneous of degree η in (N^1, N^2), i.e., $V(\lambda N^1, \lambda N^2; s) = \lambda^\eta V(N^1, N^2; s)$. The problem $P(\{N_0^a\}, s_0)$ is therefore a standard, stationary dynamic program as long as s is first-order Markov. Because of this result, we can characterize the solution through Bellman's Equation.

Under the additional assumptions that $\eta = 1-\sigma$, N does not enter the period utility function except in aggregate consumption. In this case, it is straightforward to see that $C^1 = \phi^{\frac{1}{\sigma}} C^2$ and V satisfies:

$$V(N^1, N^2; s) \equiv \max_{N^{a\prime}, C^2, N^b} \left[(\phi^{\frac{1}{\sigma}} + 1) \frac{(C^2)^{1-\sigma}}{1-\sigma} + \beta E\left[V(N^{1\prime}, N^{2\prime}; s') | s \right] \right] \tag{O.7}$$

subject to:

$(\phi^{\frac{1}{\sigma}} + 1) C^2 + \theta N^{1\prime} \leq s \sum_{a=1}^{2} w^a N^a$;
$N^{2\prime} = N^1$.

Assuming further that shocks are *i.i.d.* and using the homogeneity property of V, the Bellman Equation in (O.7) is equivalent to one where fertility per young person,

[2] To gain some intuition about the working of the model, notice that if $\eta = 1-\sigma$, then N does not enter the period utility function except in aggregate consumption, and hence, if people are productive for only one period ($w^a = 0$, $a \neq 1$), N plays exactly the same role in this model as k does in a stochastic Ak model. See Jones and Manuelli (1990) and Rebelo (1991), seminal papers on this model. Similar to Ak models, aggregate consumption, C, grows at the same rate as N. However, in the absence of shocks, per capita consumption is constant. There is one important twist to the Ak-analogy. This is that, at least in the case where child-rearing is modeled as a time cost, the cost of the investment good is also stochastic. In that case, since $\theta_t = bw_t^1$, periods when productivity is high are also periods when children – the analog of the investment good in the Ak model – are expensive. Other than that, the analogy is very close.

$n' = N^{1\prime}/N^1$, is the choice variable and last period's fertility per person, the current stock of young to old workers $n = \frac{N^1}{N^2}$ is the state variable. That is, let $\widehat{V}(n;s) = V(N^1/N^2, 1; s)/(\phi^{\frac{1}{\sigma}} + 1)$, then (O.7) becomes:

$$\widehat{V}(n;s) \equiv \max_{n'} \left[\left(\frac{s\left[w^1 n + w^2\right] - \theta(s) n' n}{\phi^{\frac{1}{\sigma}} + 1} \right)^{1-\sigma} /(1-\sigma) + \phi\beta n^{1-\sigma} E\left[\widehat{V}(n'; s')\right] \right]. \tag{O.8}$$

Taking the first-order condition with respect to n' and rearranging gives:

$$LHS(n') \equiv \frac{\theta(s)}{(\phi^{\frac{1}{\sigma}} + 1) E\hat{V}_1(n', s')} = \left(\frac{s\left[w^1 + \frac{w^2}{n}\right] - \theta(s) n'}{\phi^{\frac{1}{\sigma}} + 1} \right)^{\sigma} \beta \equiv RHS(n'). \tag{O.9}$$

$LHS(n')$ is increasing in n' since $\hat{V}$ is concave, with $LHS(0) = 0$ if $E\widehat{V}_1(0, s) = \infty$. Also, $RHS(n')$ is decreasing in n' with a positive intercept at $n' = 0$. Thus, there is a unique solution.

To see the behavior of n' as a function of the shock, we consider the two extreme cases, $\theta(s) = \theta$, a goods cost, and a time cost, $\theta(s) = bsw^1$. In the first case, $\theta(s) = \theta$, RHS shifts up when s goes up while LHS is unchanged. Thus, n' is increasing in s – fertility is procyclical. This is a pure income effect which is larger, the larger σ, i.e. the desire to smooth consumption. Also, it follows that n' is linear in s. On the other hand, when $\theta(s) = bsw^1$, both LHS and RHS shift up when s increases. The effect on RHS is still the pure income effect, which tends to increase fertility and is increasing in σ, through s^σ. The effect on LHS is the substitution effect because when s is high, fertility is temporarily more expensive which tends to decrease fertility. This effect is linear in s. If $\sigma > 1$, the income effect dominates, i.e. RHS shifts up by more than LHS. Thus, again n' is increasing in s – fertility is procyclical. If, $\sigma < 1$ the substitution effect dominates – fertility is countercyclical.

To see how current fertility (per young adult), n', depends on past fertility, n, notice that RHS shifts down when the current state, n, increases while LHS is independent of n. Thus n' decreases when n increases, generating cycles. This crucially depends on $w^2 > 0$. If there is only one period of working life (as in standard Barro-Becker models), this effect is zero.

Small Cohorts get Larger Transfers: Next, we show that in the case of multiple periods of productive life, the transfers that decentralize the Planner's solution are decreasing in last period's fertility, n. From the individual's budget constraint in equation (2), transfers per age $a = 1$ person are given by $t^1 = c^1 + \theta n'(n, s) - sw^1$, where we have used the fact that savings, x_{t+1}, are zero in equilibrium. From the Planner's feasibility in equation (O.6) and using $C^1 = \phi^{\frac{1}{\sigma}} C^2$, we can solve for c^1. That is, $c^1 = \frac{C^1}{N^1} = \frac{\phi^{\frac{1}{\sigma}} C^2}{N^1} = \phi^{\frac{1}{\sigma}} \frac{s(N^1 w^1 + N^2 w^2) - \theta(s) N^{1\prime}}{(\phi^{\frac{1}{\sigma}}+1)N^1} = \phi^{\frac{1}{\sigma}} \frac{s\left(w^1 + \frac{w^2}{n}\right) - \theta(s) n'(n,s)}{\phi^{\frac{1}{\sigma}}+1}$. Hence,

$$t^1 = \frac{1}{\phi^{\frac{1}{\sigma}} + 1} \left(\phi^{\frac{1}{\sigma}} \frac{s w^2}{n} + \theta(s) n'(n,s) - s w^1 \right).$$

As long as $w^2 > 0$, the first and second terms are decreasing in n (since $n'(n,s)$ is decreasing in n in this case) and hence, so is t^1. In other words, small cohorts (low n) get larger transfers. If $w^2 = 0$ then both terms are independent on n and so are transfers.

This concludes the proof of Proposition 1.

O.2 Solving the quantitative model

Using the basics laid out in Section 4 of the main text, this section describes how we solve the model using Dynare++.

O.2.1 Representative Firm

The representative firm solves:

$$\max_{\{K_t^d, L_t^d\}} s_t F(K_t^d, \gamma^t L_t^d) - r_t K_t^d - w_t L_t^d \tag{O.10}$$

which leads to the conditions in equations (14).

O.2.2 Household

Since n_t, h_t and t^2_{t+2} are all choice variables but multiplied in the parent's budget constraints, the constraints in equations (12) are not convex. With a simple change in variables for all future generations, however, we can recover a concave utility subject to a convex constrained set. Furthermore, as is standard in growth models, we will work with detrended variables, dividing by γ^t.

O.2.2.1 Change of variables

Let $C^a_{t,j+a} = N^a_{t,j+a} c^a_{j+a}$, $X^a_{t,j+a+1} = N^a_{t,j+a} x^a_{j+a+1}$, $H^a_{t,j+a} = h_j N^a_{t,j+a}$, and $T^2_{t,j+2} = N^2_{t,j+2} t^2_{j+2}$ for $j \geq t$, $a = 1, ..T$. Note that, since everyone survives from age $a = 1$ to $a = 2$, $H^1_{t,t+1} = H^2_{t,t+2}$. However, $H^1_{t,t+2} \neq H^2_{t,t+2}$, which is why we are keeping the age superscript. Note also that $w^a_{t+a}(h_t) N^a_{t,t+a} = A^a h_t^{\psi} N^a_{t,t+a} = A^a (H^a_{t,t+a})^{\psi} (N^a_{t,t+a})^{1-\psi} w_{t+a}$, $a = 1, 2$. When choosing $T^2_{t,t+3}$ and $H^1_{t,t+1}$, generation t takes the effect on their children's budget into account. Thus, we explicitly write out the budget constraint for children at age $a = 1$and $a = 2$ multiplied by $N^1_{t,t+1} = N^2_{t,t+2}$ in the problem for households of generation t. Given our utility assumptions, the laws of motion for descendants and the definitions above, generation t's problem can be rewritten as:

$$\max_{\substack{c^a_{t+a-1}, x^1_{t+1}, x^2_{t+2}, \\ N^1_{t,t+1}, H^1_{t,t+1}, T^2_{t,t+2}}} E_t \sum_{a=1}^{T} \beta^{a-1} \Pi_{i=1}^{a-1} u(c^a_{t+a-1}) + \sum_{j=t+1}^{\infty} (\phi\beta)^{j-t} \left[\sum_{a=1}^{T} \beta^{a-1} (N^a_{t,j+a-1})^{\eta+\sigma-1} u(C^a_{t,j+a-1}) \right]$$

subject to

$$\begin{aligned}
& c^1_t + \theta_t N^1_{t,t+1} + \nu_t H^1_{t,t+1} + x^1_{t+1} = w^1_t(h_{t-1})(1 - b_t N^1_{t,t+1} - \zeta_t H^1_{t,t+1})) \\
& c^2_{t+1} + x^2_{t+2} = w^2_{t+1}(h_{t-1}) + R_{t+1} x^1_{t+1} + t^2_{t+1} \\
& c^3_{t+2} + T^2_{t,t+2} = R_{t+2} x^2_{t+2} \\
& C^1_{t,t+1} + \theta_{t+1} N^1_{t,t+2} + \gamma X^1_{t,t+2} \\
& \qquad = A^1 (H^1_{t,t+1})^{\psi} (N^1_{t,t+1})^{1-\psi} w_{t+1} (1 - [b_{t+1} N^1_{t,t+2} + \zeta_{t+1} H^2_{t,t+2}]/N^1_{t,t+1}) + R_{t+2} X^1_{t,t+2} \\
& C^2_{t,t+2} + \gamma X^2_{t,t+3} = A^2 (H^1_{t,t+1})^{\psi} (N^1_{t,t+1})^{1-\psi} w_{t+2} + R_{t+2} X^1_{t,t+2} + T^2_{t,t+2}
\end{aligned} \tag{O.11}$$

O.2.2.2 Detrending

Reinterpreting all variables (including goods costs of children and of human capital), except population, human capital and the interest rate as detrended variables, i.e. $x_t = \frac{\widehat{x}_t}{\gamma^t}$, the problem becomes:

$$\max_{\substack{c^a_{t+a-1}, x^1_{t+1}, x^2_{t+2}, \\ N^1_{t,t+1}, H^1_{t,t+1}, T^2_{t,t+2}}} E_t \sum_{a=1}^{T} (\beta\gamma^{1-\sigma})^{a-1} u(c^a_{t+a-1}) + \sum_{j=t+1}^{\infty} \left(\phi\beta\gamma^{1-\sigma}\right)^{j-t} \left[\sum_{a=1}^{T} (\beta\gamma^{1-\sigma})^{a-1} (N^a_{t,j+a-1})^{\eta+\sigma-1} u(C^a_{t,j+a-1}) \right]$$

subject to

$$\begin{aligned}
& c^1_t + (\theta + \nu h_t) N^1_{t,t+1} + \gamma x^1_{t+1} = w^1_t(h_{t-1})(1 - N^1_{t,t+1}(b + \zeta h_t)) \\
& c^2_{t+1} + \gamma x^2_{t+2} = w^2_{t+1}(h_{t-1}) + R_{t+1} x^1_{t+1} + t^2_{t+1} \\
& c^3_{t+2} + T^2_{t,t+2} = R_{t+2} x^2_{t+2} \\
& C^1_{t,t+1} + (\theta + \nu h_{t+1}) N^1_{t,t+2} + \gamma X^1_{t,t+2} \\
& \qquad = A^1 (H^1_{t,t+1})^{\psi} (N^1_{t,t+1})^{1-\psi} w_{t+1} (1 - [b N^1_{t,t+2} + \zeta H^1_{t,t+2}]/N^1_{t,t+1}) \\
& C^2_{t,t+2} + \gamma X^2_{t,t+3} = A^2 (H^1_{t,t+1})^{\psi} (N^1_{t,t+1})^{1-\psi} w_{t+2} + R_{t+2} X^1_{t,t+2} + T^2_{t,t+2}
\end{aligned} \tag{O.12}$$

To ensure that utility is bounded, we assume $\phi\beta\gamma^{1-\sigma} < 1$.

O.2.2.3 First-order conditions for generation t household

Let $\lambda^a_{t,j}$ be the Lagrange multiplier for the budget constraint in period j, relevant for the generation of age a from the point of view of generation t (born in $t-1$).

$$
\begin{aligned}
c^1_t: &\quad \lambda^1_{t,t} = u'(c^1_t) \\
c^2_{t+1}: &\quad \lambda^2_{t,t+1} = \beta\gamma^{1-\sigma} u'(c^2_{t+1}) \\
c^3_{t+2}: &\quad \lambda^3_{t,t+2} = (\beta\gamma^{1-\sigma})^2 u'(c^3_{t+2}) \\
x^1_{t+1}: &\quad \gamma\lambda^1_{t,t} = E_t[R_{t+1}\lambda^2_{t,t+1}] \\
x^2_{t+2}: &\quad \gamma\lambda^2_{t,t+1} = E_{t+1}[R_{t+2}\lambda^3_{t,t+2}] \\
T^2_{t,t+2}: &\quad \lambda^3_{t,t+2} = \phi(\beta\gamma^{1-\sigma})^2 (N^2_{t,t+2})^{\eta+\sigma-1}(C^2_{t,t+2})^{-\sigma} \\
N^1_{t,t+1}: &\quad \lambda^1_{t,t}(\theta + bw^1_t(h_{t-1})) \\
&\qquad = \phi\beta\left(\tfrac{\eta+\sigma-1}{1-\sigma}\right)(N^1_{t,t+1})^{\eta+\sigma-2} E_t\left[\textstyle\sum_{a=1}^{T}(\beta\gamma^{1-\sigma})^{a-1}u(C^a_{t,t+a-1})\right] \\
&\qquad\quad +\phi E_t[(\beta\gamma^{1-\sigma})(N^1_{t,t+1})^{\eta+\sigma-1}(C^1_{t,t+1})^{-\sigma} \\
&\qquad\qquad\qquad *A^1(H^1_{t+1})^{\psi}(N^1_{t+1})^{-\psi}w_{t+1}(1-\psi(1-[bN^1_{t,t+2}+\zeta H^1_{t,t+2}]/N^1_{t,t+1}))] \\
&\qquad +\phi E_t[(\beta\gamma^{1-\sigma})^2(N^1_{t,t+1})^{\eta+\sigma-1}(C^2_{t,t+2})^{-\sigma} * A^2(H^1_{t,t+1})^{\psi}(N^1_{t+1})^{-\psi}w_{t+2}(1-\psi)] \\
H^1_{t,t+1}: &\quad \lambda^1_{t,t}(\nu + \zeta w^1_t(h_{t-1})) \\
&\qquad = \phi E_t[(\beta\gamma^{1-\sigma})(N^1_{t,t+1})^{\eta+\sigma-1}(C^1_{t,t+1})^{-\sigma} \\
&\qquad\qquad\qquad *A^1(H^1_{t,t+1})^{\psi-1}(N^1_{t,t+1})^{1-\psi}w_{t+1}\psi(1-[bN^1_{t,t+2}+\zeta H^1_{t,t+2}]/N^1_{t,t+1})] \\
&\qquad\quad +\phi E_t[(\beta\gamma^{1-\sigma})^2(N^1_{t,t+1})^{\eta+\sigma-1}(C^2_{t,t+2})^{-\sigma} * A^2(H^1_{t,t+1})^{\psi-1}(N^1_{t,t+1})^{1-\psi}w_{t+2}\psi]
\end{aligned}
$$

together with the budget constraints. Next, we make the following simplifying assumption: $\eta = 1-\sigma$. Using the first three equations to substitute out λ's and reverting back to per capita variables, we get:

$$
\begin{aligned}
&(c^1_t)^{-\sigma} = E_t[R_{t+1}\beta(\gamma c^2_{t+1})^{-\sigma}] \\
&(c^2_{t+1})^{-\sigma} = E_{t+1}[R_{t+2}\beta(\gamma_{t+1}c^3_{t+2})^{-\sigma}] \\
&(c^3_{t+2})^{-\sigma} = \phi(n_t c^2_{t+2})^{-\sigma} \\
&(c^1_t)^{-\sigma}(\theta + bw^1_t(h_{t-1})) \\
&= E_t[\phi(\beta\gamma^{1-\sigma})(n_t c^1_{t+1})^{-\sigma}A^1 h^{\psi}_t w_{t+1}(1-\psi(1-[b+\zeta h_{t+1}]n_{t+1})) + \phi(\beta\gamma^{1-\sigma})^2(n_t c^2_{t+2})^{-\sigma}A^2 h^{\psi}_t w_{t+2}(1-\psi)] \\
&(c^1_t)^{-\sigma}(\nu + \zeta w^1_t(h_{t-1})) \\
&= E_t[\phi(\beta\gamma^{1-\sigma})(n_t c^1_{t+1})^{-\sigma}A^1 h^{\psi-1}_t w_{t+1}\psi(1-[b+\zeta h_{t+1}]n_{t+1}) + \phi(\beta\gamma^{1-\sigma})^2(n_t c^2_{t+2})^{-\sigma}A^2 h^{\psi-1}_t w_{t+2}\psi]
\end{aligned}
\tag{O.13}
$$

These correspond to equations (16) in the main text.

O.2.3 Computing steady state and transition paths

First, we substitute out c_1 and c_3 to save on equations in Dynare++. Note that the first 3 equations in (O.13) together imply that

$$n_{t-1}n_{t-2}c_t^1 = n_{t-2}c_t^2\phi^{1/\sigma} = c_t^3\phi^{2/\sigma}. \tag{O.14}$$

The second equality in equation (O.14) follows directly from the FOC for $T_{t,t+2}^2$. Using the FOC for T^2 in the FOC for x^2, we get $(c_{t+1}^2)^{-\sigma} = \phi E_{t+1}[R_{t+2}\beta(\gamma n_t c_{t+2}^2)^{-\sigma}] = \phi E_{t+1}[R_{t+2}\beta(\gamma c_{t+2}^2)^{-\sigma}](n_t)^{-\sigma}$ and from the FOC for x^1, we get $(c_{t+1}^1)^{-\sigma} = E_{t+1}[R_{t+2}\beta(\gamma c_{t+2}^2)^{-\sigma}]$. Hence, $(c_{t+1}^2)^{-\sigma} = \phi(c_{t+1}^1 n_t)^{-\sigma}$ or $\phi^{1/\sigma}c_{t+1}^2 = n_t c_{t+1}^1$. This is equivalent to the first equality in equation (O.14). Using these equalities to substitute out c_t^1 and c_{t+2}^3, we get:

$$\begin{aligned}
x_{t+1}^1 \quad &: \quad (c_t^2)^{-\sigma} = \phi(n_{t-1})^{-\sigma}E_t[R_{t+1}\beta(\gamma c_{t+1}^2)^{-\sigma}] \\
N_{t,t+1}^1 \quad &: \quad (c_t^2)^{-\sigma}(\theta + bw_t^1(h_{t-1})) \\
&\quad = (n_{t-1})^{-\sigma}E_t[(\phi\beta\gamma^{1-\sigma})(c_{t+1}^2)^{-\sigma}A^1h_t^\psi w_{t+1}(1-\psi(1-[b+\zeta h_{t+1}]n_{t+1})) \\
&\quad\quad +(\phi\beta\gamma^{1-\sigma})^2(n_tc_{t+2}^2)^{-\sigma}A^2h_t^\psi w_{t+2}(1-\psi) \\
H_{t,t+1}^1 \quad &: \quad (c_t^2)^{-\sigma}(\nu + \zeta w_t^1(h_{t-1})) \\
&\quad = (n_{t-1})^{-\sigma}E_t[(\phi\beta\gamma^{1-\sigma})(c_{t+1}^2)^{-\sigma}A^1h_t^{\psi-1}w_{t+1}\psi(1-[b+\zeta h_{t+1}]n_{t+1}) \\
&\quad\quad +(\phi\beta\gamma^{1-\sigma})^2(n_tc_{t+2}^2)^{-\sigma}A^2h_t^{\psi-1}w_{t+2}\psi
\end{aligned}$$

Finally, let's get to the budget constraints. Combining age $a = 2$ and $a = 3$ budget constraints for generations t and $t-1$, respectively, by substituting out t_{t+1}^2, we get:

$$c_{t+1}^3 + n_{t-1}c_{t+1}^2 + n_{t-1}\gamma x_{t+2}^2 = n_{t-1}(A^2h_{t-1}^\psi w_{t+1} + R_{t+1}x_{t+1}^1) + R_{t+1}x_{t+1}^2.$$

Adding in the fourth budget constraint multiplied by n_{t-1} in the problem above, we get:

$$\begin{aligned}
&c_{t+1}^3 + n_{t-1}c_{t+1}^2 + n_{t-1}n_tc_{t+1}^1 + \gamma n_{t-1}x_{t+2}^2 + \gamma n_{t-1}n_tx_{t+2}^1 + n_{t-1}n_tn_{t+1}(\theta+\nu h_{t+1}) \\
&\quad = n_{t-1}[A^1h_t^\psi n_t(1-(b+\zeta h_{t+1})n_{t+1}) + A^2h_{t-1}^\psi]w_{t+1} + n_{t-1}R_{t+1}x_{t+1}^1 + R_{t+1}x_{t+1}^2.
\end{aligned}$$

Again, substituting out c_{t+1}^1 and c_{t+1}^3 and using capital market clearing (i.e. $n_tx_{t+2}^1 + x_{t+2}^2 = n_tk_{t+1}$), we get:

$$\begin{aligned}
&n_{t-1}c_{t+1}^2/\phi^{1/\sigma} + n_{t-1}c_{t+1}^2 + n_{t-1}c_{t+1}^2\phi^{1/\sigma} + \gamma_{t+1}n_{t-1}n_tk_{t+1} + n_{t-1}n_tn_{t+1}(\theta+\nu h_{t+1}) \\
&\quad = n_{t-1}[A^1h_t^\psi n_t(1-(b+\zeta h_{t+1})n_{t+1}) + A^2h_{t-1}^\psi]w_{t+1} + R_{t+1}n_{t-1}k_t.
\end{aligned}$$

Finally, divide by n_{t-1} to get:

$$\begin{aligned}
&\left(\tfrac{1+\phi^{1/\sigma}+\phi^{1/\sigma}}{\phi^{1/\sigma}}\right)c_{t+1}^2 + \gamma_{t+1}n_tk_{t+1} + n_tn_{t+1}(\theta+\nu h_{t+1}) \\
&\quad = [A^1h_t^\psi(n_t)(1-(b+\zeta h_{t+1})n_{t+1}) + A^2h_{t-1}^\psi]w_{t+1} + R_{t+1}k_t.
\end{aligned}$$

O.2.3.1 Computing c^2, n, k, h, w and r

The optimality equations above can be summarized as follows:

$$
\begin{aligned}
&(c_t^2)^{-\sigma} = (n_{t-1})^{-\sigma} E_t[(r_{t+1} + 1 - \delta)\phi\beta(\gamma c_{t+1}^2)^{-\sigma}] \\
&(c_t^2)^{-\sigma}(\theta + bA^1 h_{t-1}^{\psi} w_t) = (n_{t-1})^{-\sigma} E_t[(\phi\beta\gamma^{1-\sigma})(c_{t+1}^2)^{-\sigma} A^1 h_t^{\psi} w_{t+1}(1 - \psi(1 - [b + \zeta h_{t+1}]n_{t+1})) \\
&\qquad\qquad + (\phi\beta)^2(\gamma\gamma_{t+1})^{1-\sigma}(n_t c_{t+2}^2)^{-\sigma} A^2 h_t^{\psi} w_{t+2}(1 - \psi) \\
&(c_t^2)^{-\sigma}(\nu + \zeta A^1 h_{t-1}^{\psi} w_t) = (n_{t-1})^{-\sigma} E_t[(\phi\beta\gamma^{1-\sigma})(c_{t+1}^2)^{-\sigma} A^1 h_t^{\psi-1} w_{t+1}\psi(1 - [b + \zeta h_{t+1}]n_{t+1}) \\
&\qquad\qquad + (\phi\beta\gamma^{1-\sigma})^2(n_t c_{t+2}^2)^{-\sigma} A^2 h_t^{\psi-1} w_{t+2}\psi \\
&\left(\frac{1 + \phi^{1/\sigma} + \phi^{1/\sigma}}{\phi^{1/\sigma}}\right) c_t^2 + \gamma n_{t-1} k_t + n_{t-1} n_t(\theta + \nu h_t) \\
&\qquad\qquad = [A^1 h_{t-1}^{\psi} n_{t-1}(1 - (b + \zeta h_t)n_t) + A^2 h_{t-2}^{\psi}]w_t + R_t k_{t-1} \\
&w_t = s_t(1 - \alpha)k_{t-1}^{\alpha}\left[A^1 h_{t-1}^{\psi} n_{t-1}(1 - (b + \zeta h_t)n_t) + A^2 h_{t-2}^{\psi}\right]^{-\alpha} \\
&r_t = s_t \alpha k_{t-1}^{\alpha-1}\left[A^1 h_{t-1}^{\psi} n_{t-1}(1 - (b + \zeta h_t)n_t) + A^2 h_{t-2}^{\psi}\right]^{1-\alpha} \\
&\text{with } s_t = e^{\sigma_s \epsilon_t - \sigma_s^2/2}
\end{aligned}
$$

In a steady state, this boils down to a system of 6 equations in 6 unknowns, namely c^2, n, k, h, w and r, which is all we need to solve and simulate the model with Dynare.

Without uncertainty under some conditions, this system converges to a steady state where c^2, n, k, h are constant. Recall that the variables c^2, k are "detrended" and that the true per capita variables grow at rate γ. That is, $c_t^2 = \gamma^t \widehat{c}^2$ and $k_t = \gamma^t \widehat{k}_t$.

With uncertainty, the only state variables are n, the population growth rate, k, the capital stock per old worker, h human capital per worker and ϵ, the productivity shock.

Note that in the Dynare program, variables known at the beginning of period t must have "the subscript $t-1$" (i.e. subscript (-1) since all subscripts are relative to t). For fertility, this is quite natural: n_t is chosen in t, but we have to be careful with physical and human capital: k_t is the capital stock per old worker chosen in t and used in $t+1$, same for h_t.

O.2.3.2 Computing c^1, c^3, x^1, x^2, and t^2

Once, we have the steady state and transition paths of these 6 variables, we can use equation (O.14), to compute consumption at ages $a = 1$ and $a = 3$.

Next, given the 6 variables coming out of Dynare++, we can use the budget constraints and credit market clearing to back out the steady state values and paths of savings , x_t^1 and x_t^2, and transfers, t_t^2 as follows.

- Credit market clearing gives $N_{t-1}^1 x_t^1 + N_{t-1}^2 x_t^2 = K_t^d$. Dividing by N_{t-1}^2, we get $n_{t-2} x_t^1 + x_t^2 = n_{t-2} k_{t-1}$. Hence, we get an expression for savings at age $a = 1$ as a function of capital-to-old-worker ratio and savings at age $a = 2$:

$$x_t^1 = k_{t-1} - \frac{1}{n_{t-2}} x_t^2. \tag{O.15}$$

- From the budget constraint at age $a = 2$ and substituting in from equation (O.17), we have $c_t^2 + \gamma x_{t+1}^2 = w_t^2(h_{t-2}) + (r_t + 1 - \delta) x_t^1 - \frac{c_t^2}{\phi^{1/\sigma}} + \frac{r_t+1-\delta}{n_{t-2}} x_t^2$. Using credit market clearing, we get $c_t^2 + \gamma x_{t+1}^2 = w_t^2(h_{t-2}) - \frac{c_t^2}{\phi^{1/\sigma}} + (r_t + 1 - \delta) k_t$. Hence, we get an expression for savings at age $a = 2$ as a function of pensions, consumption and the capital-to-old-worker ratio:

$$x_{t+1}^2 = \frac{1}{\gamma} \left[w_t^2(h_{t-2}) - \frac{1 + \phi^{1/\sigma}}{\phi^{1/\sigma}} c_t^2 + (r_t + 1 - \delta) k_t \right] \tag{O.16}$$

- From the budget constraint at age $a = 3$, we have $c_t^3 + t_t^2 n_{t-2} = R_t x_t^2$. From above, we have $c_t^3 = \frac{c_t^2}{\phi^{1/\sigma}} n_{t-2}$. Substituting in gives $\frac{c_t^2}{\phi^{1/\sigma}} + t_t^2 = \frac{R_t}{n_{t-2}} x_t^2$. Hence, we get an expression for transfers as a function of pensions, consumption and savings at age $a = 2$ and fertility two periods ago:

$$t_t^2 = \frac{r_t + 1 - \delta}{n_{t-2}} x_t^2 - \frac{c_t^2}{\phi^{1/\sigma}}. \tag{O.17}$$

The 6 equations in (O.15) are all we need in the model section of the Dynare program. Once we have the steady state and transition paths of these variables, we can compute savings at age $a = 2$, x_{t+1}^2 from equation (O.16). Using the latter, we can compute transfers, t_t^2, from equation (O.17), and savings at age $a = 1$, x_{t+1}^1, from equation (O.15).

O.3 Sensitivity

O.3.1 Varying the age range of productivity shocks

One issue with the assumption that the only relevant shock for the dynasty is when their youngest cohort is age 20 to 30 is that it implicitly assumes that the shock lasts for the entire period of 20 years. In this section, we describe model results when the shock is set to the average productivity shock over the period where the cohort is age 20 to 40.

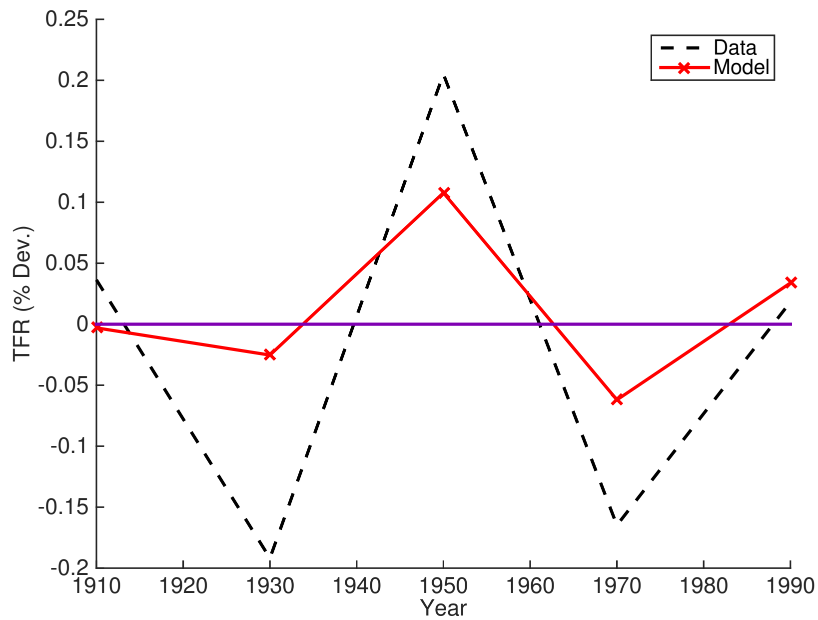
Table O.1 shows productivity shocks for this experiment compared to the baseline. Overall shocks are smoother and smaller because some busts and booms are smoothed out due to the longer time period. Therefore, as seen in Figure O.1, the baby bust is smaller than in the baseline experiment. Because of the smaller baby bust in the 1930s and smaller shocks, the Baby Boom is also smaller mainly because of the small baby bust. However, the productivity shock for the 1950s and 1960s is larger than for the 1950s alone, which is why the Baby Boom is still sizable. The problem with this alternative is that it implicitly assumes that fertility is uniform between age 20 and 40 when in reality most of them had completed their fertility by age 30.

Since this alternative is at the opposite extreme of the baseline, the two experiments together should give us reasonable bounds on how much of the fluctuations in fertility over the 20th century in the U.S. the model can account for.

Table O.1: Productivity shocks (in percent), Age Range 20-40

Decade	1910s(-20s)	1930s(-40s)	1950s(-60s)	1970s(-80s)	1990s(-2000s)
Experiment	-0.58	-3.30	10.24	1.85	-5.56
Baseline	-1.82	-13.09	7.49	5.68	-5.51

Figure O.1: Percent Deviations in Total Fertility Rate, Age Range 20-40



O.3.2 Varying the length of the period

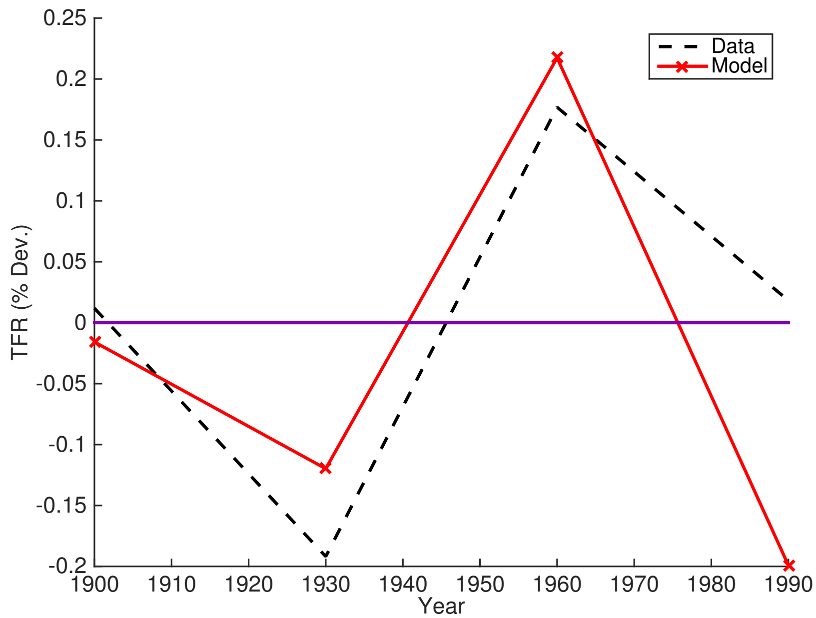
The length of a period also has an effect on the quantitative results of the exercise. To try to get an initial assessment of the size of these effects we solved a version of the model in which a period was 30 years which means that we changed only the parameters involved in the effective discount factor, namely β (discount factor), ϕ (altruism factor) and γ (productivity growth rate). We then recalibrated ϕ, θ and ψ to match the population growth, capital output ratio and goods costs as a fraction of income (as we did in the baseline). We also adjusted the productivity shocks the dynasty is faced with. In particular, we used the following productivity shocks for the periods 1900-1910, 1930-1940, 1960-1970 and 1990-2000 (analogous to Table 4 in the main text)

As can be seen in Figure O.2, the model response went from 58 percent to 62 percent of the baby bust in the 1930s accounted for and from 78 percent of the boom in the 1950s to 123 percent of the boom observed in the 1960s (which is smaller than in the 1950s). However, in the 1990s the model predicts that fertility should have been much lower than it actually was in the data.

Table O.2: Productivity shocks (in percent, 30-year period)

Decade	1900s	1930s	1960s	1990s
Productivity Shock	-1.88	-13.09	12.54	-4.51

Figure O.2: Percent Deviations in Total Fertility Rate, 30-year period

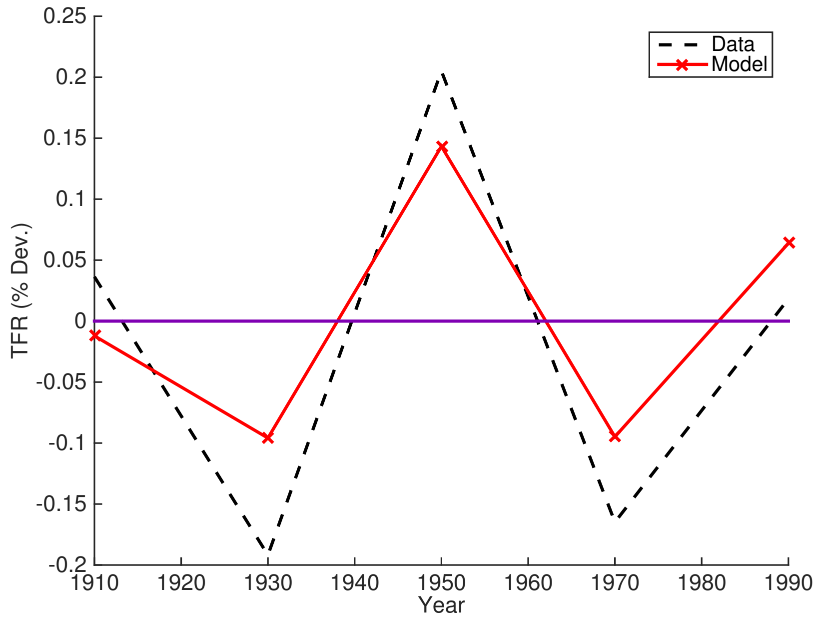


O.3.3 Varying intertemporal elasticity of substitution

Whether or not fertility is pro- or counter-cyclical depends on a combination of the curvature of utility and the breakdown of child costs between time and goods. According to our calculations from the PSID-CDS, during typical work hours parents spend 12.2 hours in childcare per child. Thus, out of a potential 100 hour workweek for both parents, about 12 percent are spent with each child while good costs are 16 percent of household income per child from USDA. With these particular calibration targets, our results are still quantitatively important when the intertemporal elasticity of substitution is above one (i.e., $\sigma < 1$). In particular, given $\sigma = 0.5$ we recalibrate ϕ, θ and ψ to match the population growth, capital-output ratio and goods costs as a fraction of income (as we did in the baseline).

As shown in Figure O.3, the model still captures 50 percent of the baby bust in the 1930s and 70 percent of the Baby Boom in the 1950s which is reassuring (compared to 58 percent and 77 percent in the baseline).

Figure O.3: Percent Deviations in Total Fertility Rate, $\sigma = 0.5$



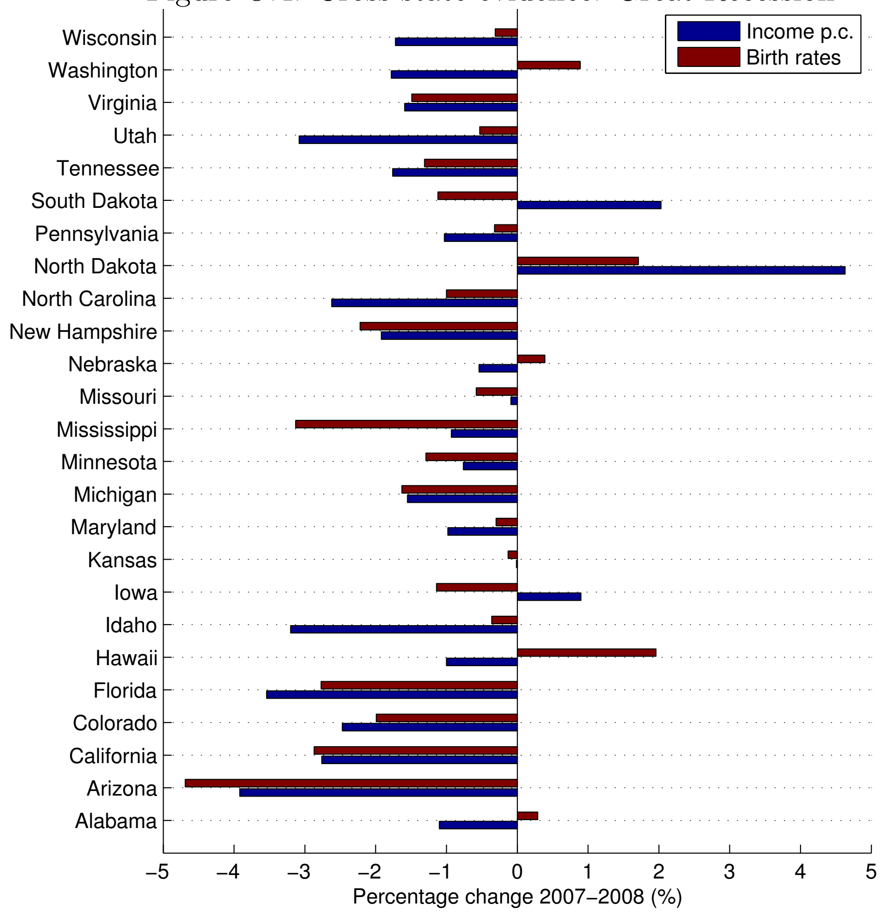
O.4 The Great Recession across 25 U.S. states

The most recent recession, starting in 2007, also seems to have had an impact on fertility in the United States. According to the National Center for Health Statistics, the birth rate (the number of births per 1000 women age 15 to 44) fell from 69.5 in 2007 to 66.7 in 2009. This corresponds to a 4 percent drop overall with most of it concentrated in younger age-groups. Preliminary data for the twelve-months period through June 2010 show a further decrease to a birth rate of 65.5, an additional 3 percent drop.

Data from a Pew Study in April 2010 across 25 U.S. states show that those states that were most affected by the recession between experienced a larger drop in birth rates from 2007 to 2008 (see Figure O.4). The correlation coefficient between percent changes in birth rates between 2007 and 2008 and percent changes in income per capita, between 2006 and 2007 and between 2007 and 2008, are 0.56 and 0.52, respectively.

Of course, this recession is still ongoing in the sense that, while GDP per capita may have started to increase again, growth rates have not yet attained their long term

Figure O.4: Cross-state evidence: Great Recession



levels. Because of this, it is yet to be seen how large the (completed) fertility response to current events will be and whether this generates a Baby Boom in the 2020s.

Compared to the U.S., the fertility response to the recent economic downturn among European countries is less evident if it exists at all. For example, using preliminary data from Eurostat, we find that between 2008 and 2009 many countries only saw their fertility level decrease slightly (and some not at all). The largest drop in fertility was in Spain with 4.6 percent, followed by Portugal with 3.7 percent—two countries whose economy was affected early on in the crisis.

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