

How do financial frictions affect the spending multiplier during a liquidity trap?

TECHNICAL APPENDIX

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January 30, 2013

This appendix of our paper, “How do financial frictions affect the spending multiplier during a liquidity trap?”, contains further material that could not be included in the paper due to space limitations. Section 1 describes the full derivation of the model. Section 2 details the computations for a model with taxation. Section 3 explains how we solve the model with a zero lower bound constraint. Section 4 provides robustness exercises.

1 The Model

1.1 Optimization Programs

The model builds on Bernanke, Gertler and Gilchrist (1999, BGG hereafter). It consists of six agents: the final goods producer, the intermediate goods producers, the capital good producer, the lender, the entrepreneur, and the household. A hatted variable denotes its deviation from the deterministic steady state. A variable without neither a hat nor a time subscript denotes its steady-state level.

1.1.1 Households

The economy is inhabited by a continuum of identical households. A typical household selects a sequence of consumption, labor supply and bond holdings so as to maximize his discounted lifetime utility. The objective of the representative household is thus

$$\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ \mathbb{U}(c_t, \ell_t^h) \right\}, \quad (1.1)$$

subject to the budget constraint

$$c_t + d_t \leq w_t \ell_t^h + R_{t-1}^d \frac{d_{t-1}(1 + \pi_t)^{\gamma_b}}{(1 + \pi_t)} - \frac{\Upsilon_t}{P_t} + \frac{\mathcal{A}_t}{P_t} + \text{div}_t, \quad (1.2)$$

where $\mathbb{E}_t$ is the expectation operator conditional to the information available in period t . $\beta \in (0, 1)$ is the subjective discount factor; c_t denotes real consumption; P_t is the price of final goods; $w_t \equiv W_t/P_t$ is the real wage rate; ℓ_t^h denotes labor supply; $1 + \pi_t = P_t/P_{t-1}$ represents the gross inflation rate; $d_t \equiv D_t/P_t$ denote the holding of net deposit in real terms; $(1 + \pi_{t+1})^{\gamma_b}$ is a term that adjusts nominal debt to the realized inflation from t to $t + 1$, and $\gamma_b \in [0, 1]$ is the coefficient of debt indexation. We choose a partial an inflation-indexation device to moderate the degree at which we deviate from the full Fisherian *debt-deflation* channel. $R_t^D = \exp(\varepsilon_t) R_t$ is the gross

nominal interest rate associated with one-period-maturity nominal deposits and R_t is the central bank's riskfree gross nominal interest rate. The disturbance term ε_t is a premium shock in the spirit of Smets and Wouters (2007), which creates an exogenous spread between the return of private assets and government bonds.¹ Finally, div_t , Υ_t , $\mathcal{A}_t$ denote real profits from monopolistic firms, nominal taxes, and nominal transfers from entrepreneurs, respectively, each one redistributed as a lump sum.

The disturbance term ε_t follows an AR(1) of the form

$$\varepsilon_t = \rho_\varepsilon \varepsilon_{t-1} + \epsilon_{\varepsilon,t},$$

where $\rho_\varepsilon \in (0, 1)$ and $\epsilon_{\varepsilon,t} \sim \text{i.i.d } (0, \sigma_\varepsilon)$.

We consider two types of preferences in the analysis. One in which consumption and labor enter separately in utility - where wealth effects are stronger-, and another in which there is non-separability - where wealth effects are weaker. The latter is discussed first.

Non-separable preferences Following Christiano, Eichenbaum, and Rebelo (2011), we consider non-separable preferences between consumption and labor. The preferences relation is

$$\mathbb{U}(c_t, \ell_t^h) \equiv \frac{\left(c_t^v (1 - \ell_t^h)^{1-v}\right)^{1-\sigma} - 1}{1 - \sigma}, \quad (1.3)$$

where $\sigma > 0$ is the coefficient of relative risk aversion and $v \in \{0, 1\}$ is a scale parameter. The representative household maximizes the discounted flow of utilities (1.1) where the utility function is defined by (1.3) subject to (1.2). The dynamic Lagrangian reads

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\left(c_t^v (1 - \ell_t^h)^{1-v}\right)^{1-\sigma} - 1}{1 - \sigma} - \lambda_t \left[c_t + d_t - w_t \ell_t^h - R_{t-1} \exp(\varepsilon_{t-1}) \frac{d_{t-1}(1 + \pi_t)^{\gamma_b}}{1 + \pi_t} + \frac{\Upsilon_t}{P_t} - \frac{\mathcal{A}_t}{P_t} - \text{div}_t \right] \right\},$$

where λ_t is the Lagrangian multiplier associated to the budget constraint - or the shadow marginal utility of wealth.

The first order conditions (FOCs) with respect to d_t , c_t and ℓ_t^h , are respectively

$$\lambda_t = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1} R_t \exp(\varepsilon_t) (1 + \pi_{t+1})^{\gamma_b}}{1 + \pi_{t+1}} \right\}, \quad (1.4)$$

$$c_t^{v(1-\sigma)} (1 - v) \left(1 - \ell_t^h\right)^{(1-v)(1-\sigma)-1} = \lambda_t w_t, \quad (1.5)$$

$$\left(1 - \ell_t^h\right)^{(1-v)(1-\sigma)} v c_t^{v(1-\sigma)-1} = \lambda_t. \quad (1.6)$$

Notice that using Equations (1.5) and (1.6), the labor supply can be simply stated as

$$\frac{1 - v}{v} \frac{c_t}{1 - \ell_t^h} = w_t. \quad (1.7)$$

¹ ε_t is a shock that is not generated within the typical BGG's financial accelerator mechanism. This credit spread shock happens in the relationship between the household and the lender. This characteristic would allow us to compare a standard NK model with a NK model with financial frictions *à la* BGG.

Separable preferences For robustness exercises, we also assume a separable utility function as

$$\mathbb{U}(c_t, \ell_t^h) \equiv \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \psi \frac{(\ell_t^h)^{1+\omega_w}}{1+\omega_w}, \quad (1.8)$$

where ω_w^{-1} is the Frisch elasticity of labor supply, and ψ is a normalizing constant. This time, the Lagrangian is

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \left\{ \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} - \psi \frac{(\ell_t^h)^{1+\omega_w}}{1+\omega_w} \right] - \lambda_t \left[c_t + d_t - w_t \ell_t^h - R_{t-1} \varepsilon_{t-1} \frac{d_{t-1}(1+\pi_t)^{\gamma_b}}{1+\pi_t} + \frac{\Upsilon_t}{P_t} - \frac{\mathcal{A}_t}{P_t} - \text{div}_t \right] \right\}.$$

The FOC with respect to d_t is still given by Equation (1.4), while the FOCs with respect to c_t and ℓ_t^h are

$$c_t^{-\sigma} = \lambda_t, \quad (1.9)$$

$$\psi \left(\ell_t^h \right)^{\omega_w} = \lambda_t w_t. \quad (1.10)$$

1.1.2 Entrepreneurs

There are a continuum of risk neutral entrepreneurs, indexed by $e \in [0, 1]$. At time t , type- e entrepreneur purchases at price Q_t the stock of capital $k_{e,t}$. She then rents out capital services to intermediate firms in $t+1$. She finances her capital expenditures with her own internal resources and debt. There is asymmetric information between the lender and the entrepreneur regarding the realized returns of capital.

Optimal financial contract Let $N_{e,t}$ be the available nominal net worth of type- e entrepreneur at the end of period t and $B_{e,t}$ the amount of nominal debt asked to the financial intermediary (lender). Accordingly,

$$Q_t k_{e,t} = B_{e,t} + N_{e,t}, \quad (1.11)$$

Notice that both capital and net worth are state variables, while the level of debt is a control variable. The lender obtains funds from the household with an opportunity cost of funds between t and $t+1$ equal to the riskless nominal rate of return R_t .

Following BGG, we assume that the ex-post gross nominal return on capital for type- e entrepreneur, $\tilde{R}_{t+1}^k$, is affected by an idiosyncratic disturbance, denoted by $\omega_{e,t+1}$, which is an i.i.d. random variable across time and types, with a continuous and once-differentiable c.d.f., $F(\omega)$, over a non-negative support. Also, we consider that ω is unknown to both the entrepreneur and the lender prior to the investment decision, with $E(\omega) = 1$ and $V(\omega) = \sigma_{\omega,t}$. We assume that the financial intermediary perfectly diversifies the idiosyncratic risk involved in lending. Thus, since the entrepreneur is risk neutral and the household is risk averse, the former will absorb all the risk associate with the lending contract.

As BGG, we follow the *costly state verification* environment of Townsend (1979), in which lenders bear a fixed monitoring cost in order to observe an individual borrower's realized return, while borrowers observe it for free. For convenience, BGG assume that the monitoring cost is a proportion $\mu \in [0, 1]$ of the realized gross payoff to the entrepreneur's capital, i.e., $\mu \omega_{e,t+1} \tilde{R}_{t+1}^k Q_t k_{e,t}$.

The entrepreneur chooses the value of his project's capital, $Q_t k_{e,t}$, and the associated level of borrowing $B_{e,t}$, prior to the realization of $\omega_{e,t+1}$. We assume that entrepreneurs loan contracts can be indexed to inflation. This assumption allows us to study the importance of the Fisher's debt-deflation channel on the transmission of fiscal shocks. If $\omega_{e,t+1}$ is higher than a threshold value $\bar{\omega}_{e,t+1}$, the entrepreneur promises to repay in period $t+1$ the inflation-indexed debt principal plus interests, or $R_{e,t+1}^L B_{e,t} (1+\pi_{t+1})^{\gamma_b}$, where R_t^L is the gross nominal non-default rate. The threshold value $\bar{\omega}_{e,t+1}$ is defined as:

$$E_t \left\{ \bar{\omega}_{e,t+1} \tilde{R}_{t+1}^k Q_t k_{e,t} \right\} = E_t \left\{ R_{e,t+1}^L B_{e,t} (1+\pi_{t+1})^{\gamma_b} \right\}, \quad (1.12)$$

If $\omega_{e,t+1}$ is lower than $\bar{\omega}_{e,t+1}$, the entrepreneur declares bankruptcy and gets nothing, while the lender audits the entrepreneur, pays the monitoring cost, and gets to keep any positive income of the entrepreneur. This mechanism proves to be truth-telling, since the entrepreneur is discouraged from pretending a bankruptcy.

Time and type sub-indexes are drop and it is assumed there is not aggregate uncertainty for briefness. We will use the fact that $R^L B(1 + \pi)^{\gamma_b} = \bar{\omega} \tilde{R}^k Qk$. In addition, define $\Gamma(\bar{\omega})$ and $\mu G(\bar{\omega})$ as follows:

$$\begin{aligned}\Gamma(\bar{\omega}) &= \bar{\omega} \int_{\bar{\omega}}^{\infty} f(\omega) d\omega + \int_0^{\bar{\omega}} \omega f(\omega) d\omega, \quad \text{and} \\ \mu G(\bar{\omega}) &= \mu \int_0^{\bar{\omega}} \omega f(\omega) d\omega.\end{aligned}$$

Entrepreneur returns: If $\omega \leq \bar{\omega}$, her earnings are zero. If $\omega > \bar{\omega}$, her earnings are $\omega \tilde{R}^k Qk - R^L \tilde{B} = (\omega - \bar{\omega}) \tilde{R}^k Qk$.

$$\begin{aligned}\text{returns}^e &= \int_{\bar{\omega}}^{\infty} (\omega - \bar{\omega}) \tilde{R}^k Qk f(\omega) d\omega \\ &= \left(\int_{\bar{\omega}}^{\infty} \omega f(\omega) d\omega - \bar{\omega} \int_{\bar{\omega}}^{\infty} f(\omega) d\omega \right) \tilde{R}^k Qk \\ &= [1 - \Gamma(\bar{\omega})] R^k Qk\end{aligned}$$

Lender returns: If $\omega \leq \bar{\omega}$, her earnings are $(1 - \mu) \omega \tilde{R}^k Qk$ (the lender collects realized gross returns of bankrupted entrepreneurs and pays monitoring costs). If $\omega > \bar{\omega}$, her earnings are $R^L B(1 + \pi)^{\gamma_b} = \bar{\omega} \tilde{R}^k Qk$ (fix, irrespective of the level of ω , and collected from non-defaulting entrepreneurs).

$$\begin{aligned}\text{returns}^L &= (1 - \mu) \int_0^{\bar{\omega}} \tilde{R}^k Qk f(\omega) d\omega + \int_{\bar{\omega}}^{\infty} \bar{\omega} \tilde{R}^k Qk f(\omega) d\omega \\ &= [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] \tilde{R}^k Qk\end{aligned}$$

$\Rightarrow$ It worth mentioning that the nature of the contract, either in nominal terms or in inflation-indexed terms, does not change the expressions for expected returns expressed above. But it will change the participation constraint of the lender (PCL). The latter states that lender expected returns on the loan should be greater or equal to the opportunity costs of her funds.

In the case of **inflation-indexed debt contracts**, the PCL is:

$$\text{E}_t \left\{ [\Gamma(\bar{\omega}_{e,t+1}) - \mu G(\bar{\omega}_{e,t+1})] \tilde{R}_{t+1}^k Q_{t+1} k_{e,t} \right\} \geq R_t (Q_{t+1} k_{e,t} - N_{e,t}) \text{E}_t \left\{ (1 + \pi_{t+1})^{\gamma_b} \right\},$$

In the last two expressions, we used the fact that $Qk = B + N$. *The PCL under inflation-indexed debt assumes implicitly that government bonds are also indexed to inflation.* Otherwise, there will be no changes in the optimal debt contract, nor in the law of motion of net worth.² Next, the design of the optimal contract for **inflation-indexed debt** is analyzed; the case of **nominal debt contracts** is contained in the latter by setting the term $\gamma_b = 0$.

Let $x_{e,t} \equiv Q_{t+1} k_{e,t} / N_{e,t}$ denote entrepreneur e 's leverage ratio and $\tilde{r}_t \equiv \text{E}_t \left\{ \tilde{R}_{t+1}^k / R_t \right\}$ represent the expected discounted return of capital. The optimal contract consists thus in choosing x and $\bar{\omega}$ in order to maximize type- e

²The financial intermediary alternative is to buy riskless government bonds. Assuming that government bonds are signed only in nominal terms leave the PCL completely identical. Differences will be observed, however, in the non-defaulting rate R^L .

entrepreneur's expected returns with respect to the participation constraint of the lender. For brevity, we drop type subindexes. The problem is

$$\max_{x_t, \bar{\omega}_{t+1}} E_t \{ [1 - \Gamma(\bar{\omega}_{t+1})] \tilde{r}_t x_t \}$$

subject to

$$E_t \{ [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \tilde{r}_t x_t \} \geq (x_t - 1) E_t \{ (1 + \pi_{t+1})^{\gamma_b} \}$$

The FOCs are:

$$/x : E_t \{ [1 - \Gamma(\bar{\omega}_{t+1}) + \Lambda_t [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})]] \tilde{r}_t \} = E_t \{ \Lambda_t (1 + \pi_{t+1})^{\gamma_b} \} \quad (1.13)$$

$$/\bar{\omega} : E_t \{ \tilde{r}_t x_t [-\Gamma_{\omega}(\bar{\omega}_{t+1}) + \Lambda_t (\Gamma_{\omega}(\bar{\omega}_{t+1}) - \mu G_{\omega}(\bar{\omega}_{t+1}))] \} = 0 \quad (1.14)$$

$$/\Lambda : E_t \{ [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \tilde{r}_t x_t \} = (x_t - 1) E_t \{ (1 + \pi_{t+1})^{\gamma_b} \} \quad (1.15)$$

where Λ is the lagrangian multiplier and $\Gamma_{\omega}(\cdot)$ and $G_{\omega}(\cdot)$ denote the derivative of $\Gamma(\cdot)$ and $G(\cdot)$ w.r.t. $\bar{\omega}$. BGG show that combining the last three equations and aggregating across entrepreneurs allows to express the equilibrium in the credit market as follows:

$$E_t \left\{ \frac{\tilde{R}_{t+1}^k}{R_t} \right\} = x \left(\frac{Q_t k_t}{N_t}, \bar{\omega}_{t+1}, E_t \{ \pi_{t+1} \} \right),$$

where $x(\cdot)$ a function with $\frac{\partial x(\cdot)}{\partial x_t} > 0$ for $N_{t+1} < Q_t k_t$; N_t is nominal aggregate net worth and k_t is the aggregate stock of capital bought in period t . This relationship represents the key feature of the accelerator mechanism: entrepreneurs with a high self-financing ratio (or bigger net worth) are less likely to default, implying a lower premium on external funds, which reduces the cost of credit.

Capital returns and entrepreneurs in general equilibrium At the end of period t , type- e entrepreneur buys the stock of capital $k_{e,t}$ at price Q_t . In time $t + 1$, and after observing all the shocks, she rents her capital stock to intermediate firms at a real rental rate z_{t+1} . She then sells the un-depreciated capital to the capital producer, who also discounts a proportional fee per unit of capital, cap_t , due to an adjustment cost which must be paid out to produce new capital goods (see the capital producer problem in Section ?? for further details).

The *undistorted* gross nominal rate of capital returns, R_{t+1}^k , equals

$$R_{t+1}^k \equiv (1 + \pi_{t+1}) \frac{z_{t+1} + (1 - \delta - \text{cap}_{t+1}) q_{t+1}}{q_t}, \quad \text{or} \quad (1.16)$$

$$\equiv (1 + \pi_{t+1})(1 + r_{t+1}^k), \quad (1.17)$$

where δ is the capital depreciation rate. We assume that the capital returns perceived by entrepreneurs are distorted by a *premium* shock, ε_t , so $\tilde{R}_{t+1}^k = R_{t+1}^k \exp(-\varepsilon_t)$. Notice that a positive innovation of ε_t reduces the value of capital and investment, while it increases the required return on bonds (reducing consumption, see Section 1.1.1). Thus, both consumption and investment decrease. As explained by Smets and Wouters (2007), this *premium* shock helps to explain the co-movement of investment and consumption observed in the data.

Remark 1 In a standard New Keynesian (NK) model à la Smets and Wouters (2007), a $\varepsilon_t > 0$ naturally implies a spread between R^k and R , since there is arbitrage between capital and bonds. The frictionless financial market of that model makes a separation between consumers and entrepreneurs irrelevant, since there is not a role for an endogenous credit spread. In order to make a premium shock in the financial frictions model comparable to said shock in the standard NK model, we need to include ε_t in both household and entrepreneur problems. The derivation in a typical NK model would be as follows:

$$\max E_t \sum_{T=t}^{\infty} (\beta)^{T-t} \left\{ \frac{(c_T^v (1 - \ell_T^h)^{1-v})^{1-\sigma} - 1}{1 - \sigma} \right\}, \text{ subject to}$$

$$c_T + i_T + \frac{d_T}{R_T \exp(\varepsilon_T)} \leq w_T \ell_T^h + \frac{d_{T-1}(1 + \pi_T)^{\gamma_b}}{1 + \pi_T} + z_T k_{T-1} - \frac{\Upsilon_T}{P_T} + \frac{\mathcal{A}_T}{P_T} + \text{div}_T, \text{ and} \quad ((\Lambda))$$

$$k_T \leq (1 - \delta)k_{T-1} + i_T - \frac{\vartheta}{2} \left(\frac{i_T}{k_{T-1}} - \delta \right)^2 k_{T-1} \quad ((\mu))$$

$$\begin{aligned} /i_t &: q_t^{-1} = 1 - \vartheta \left(\frac{i_T}{k_{T-1}} - \delta \right), \text{ where } q_t \equiv \frac{\mu_t}{\lambda_t} \\ /k_t &: \mu_t = \beta E_t \{ (\lambda_{t+1} z_{t+1} + \mu_{t+1} [1 - \delta - \text{cap}_{t+1}]) \} \\ &: \text{ where } \text{cap}_{t+1} = \frac{\vartheta}{2} \left(\frac{i_{t+1}}{k_t} - \delta \right)^2 - \vartheta \left(\frac{i_{t+1}}{k_t} - \delta \right) \frac{i_{t+1}}{k_t}. \end{aligned}$$

From the household perspective, it should be the case that, in equilibrium, the interest rate perceived in bonds, $\exp(\varepsilon_t) R_t$, equals the rate of returns on capital R_{t+1}^k , which is implicitly defined in the F.O.C. of capital. This condition ($/k_t$:) can be rewritten as (assuming certainty equivalence):

$$\begin{aligned} \left[\beta E_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} \right\} \right]^{-1} &= \frac{E_t \{ z_{t+1} + q_{t+1} (1 - \delta - \text{cap}_t) \}}{q_t}, \text{ or} \\ \exp(\varepsilon_t) R_t &= \frac{E_t \{ P_{t+1} z_{t+1} + P_{t+1} q_{t+1} (1 - \delta - \text{cap}_t) \}}{P_t q_t} \equiv E_t \{ R_{t+1}^k \}. \end{aligned}$$

And so, there is an spread between the risk free rate and the rate of return of capital:

$$E_t \{ \hat{R}_{t+1}^k \} - \hat{R}_t = \varepsilon_t.$$

■

We follow BGG by considering that entrepreneurs live for finite horizons. Thus, we assume that each entrepreneur has a probability of exit the economy of $1 - \gamma$. Assuming that entrepreneurs will not live forever ensures that they will not accumulate enough wealth to be fully self-financed. The nominal aggregate net worth at the end of period t , N_t , is given by

$$N_t = \gamma V_t + W_t^e \quad (1.18)$$

where V_t is aggregate equity from capital holdings in period t . If debt contracts are signed with inflation-indexed instruments, then aggregate equity is defined as:

$$\begin{aligned} V_t &= [1 - \Gamma(\bar{\omega}_t)] \tilde{R}_t^k Q_{t-1} k_{t-1}, \quad \text{or} \\ &= \tilde{R}_t^k Q_{t-1} k_{t-1} (1 - \mu G(\bar{\omega}_t)) - R_{t-1} B_{t-1} (1 + \pi_t)^{\gamma_b} \end{aligned} \quad (1.19)$$

since from the PCL for period $t - 1$ we can see: $\Gamma(\bar{\omega}_t) \tilde{R}_t^k Q_{t-1} k_{t-1} = \mu G(\bar{\omega}_t) \tilde{R}_t^k Q_{t-1} k_{t-1} + R_{t-1} B_{t-1} (1 + \pi_{t+1})^{\gamma_b}$. When debt contracts are denominated in nominal terms ($\gamma_b = 0$), an unexpected positive change in inflation in period t largely improves the value of equity since debt is predetermined and $\tilde{R}_t^k = \exp(-\varepsilon_t) (1 + r_t^k) (1 + \pi_t)$. Fisher's debt-deflation channel (Fisher, 1933) appears naturally in this environment, and acts as an amplification mechanism on the investment decision of entrepreneurs.

Following BGG and Carlstrom and Fuerst (1997), we assume that entrepreneurs participate in the general labor market.³ Also, we assume that entrepreneurs supply one unit of labor at each and every period and earn the nominal wage W_t^e . In definition (1.18), γV_t is the equity held by entrepreneurs who survive in t . Those who die in t , transfer their wage to new entrepreneurs entering the economy, consume part of their equity, such as $c_t^e = (1 - \gamma) \varrho \frac{V_t}{P_t}$, while the rest, $\mathcal{A}_t = (1 - \gamma) (1 - \varrho) \frac{V_t}{P_t}$, is lump-sum transferred to households.

³This is a technical issue, since it is necessary to start entrepreneurs off with some net worth in order to allow them to begin operations.

1.1.3 Capital Producer

Capital producers operate in a perfectly competitive market. At the end of period $t-1$, entrepreneurs buy capital stock to be used in period t from the capital producers. Once intermediate goods have been sold and capital rents have been paid, entrepreneurs sell back to the capital producer the remaining, un-depreciated stock of capital. The representative capital producer then builds new capital stock, k_t , by combining investment goods, i_t , and un-depreciated capital, $(1-\delta)k_{t-1}$, where δ is the depreciation rate of capital. In general terms, the capital producer problem is

$$\max_{i_t} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \frac{\lambda_{t+1}}{\lambda_t} \{Q_t [k_t - (1-\delta - \text{cap}_t)k_{t-1}] - P_t i_t - Q_t \text{cap}_t k_{t-1}\}, \quad \text{subject to}$$

$$k_t = (1-\delta)k_{t-1} + K(i_t, i_{t-1}, k_{t-1}),$$

where $\beta^t \frac{\lambda_{t+1}}{\lambda_t}$ defines the appropriate discount factor for this problem. The function $K(i_t, i_{t-1}, k_{t-1})$ denotes adjustment costs in capital formation. We consider two different types of adjustment costs. The type we use for the benchmark exercise is the *capital* adjustment cost where, in order to produce new capital goods, the capital producer needs to use a combination of old capital and new investment (see Christiano, Eichenbaum and Rebelo, 2011). She then charges a fee cap_t to the entrepreneur to discount the old capital she needs to use to produce new units. The second type we use for a robustness check is the *investment* adjustment cost, where the capital producer uses a combination of old investment goods with new investment goods to produce new capital units (see Christiano, Eichenbaum and Evans, 2005). In this case, she does not charge an extra fee to entrepreneurs, so $\text{cap}_t = 0$. The characterization of the capital producer problem for each adjustment cost type is given as follows:

Capital adjustment cost

$$\begin{aligned} K(i_t, i_{t-1}, k_{t-1}) &= i_t - \frac{\vartheta}{2} \left(\frac{i_t}{k_{t-1}} - \delta \right)^2 k_{t-1}, \quad \text{with} \\ \text{cap}_t &= \frac{\partial K(\cdot)}{\partial k_{t-1}} = \frac{\vartheta}{2} \left(\frac{i_t}{k_{t-1}} - \delta \right)^2 - \vartheta \left(\frac{i_t}{k_{t-1}} - \delta \right) \quad (\text{see Remark 1}). \end{aligned}$$

In equilibrium, the relative price of capital, $q_t \equiv Q_t/P_t$, is given by

$$/i_t : q_t^{-1} = 1 - \vartheta \left(\frac{i_t}{k_{t-1}} - \delta \right). \quad (1.20)$$

Investment adjustment cost

$$\begin{aligned} K(i_t, i_{t-1}, k_{t-1}) &= \left[1 - \Phi \left(\frac{i_t}{i_{t-1}} \right) \right] i_t, \quad \text{with} \\ \text{cap}_t &= 0. \end{aligned}$$

In equilibrium, the relative price of capital, $q_t \equiv Q_t/P_t$, is given by

$$\begin{aligned} /i_t : q_t &= \left[\phi_{1,t} + \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1} q_{t+1}}{\lambda_t q_t} \phi_{2,t} \right\} \right]^{-1} \quad \text{where} \\ : \phi_{1,t} &= 1 - \Phi \left(\frac{i_t}{i_{t-1}} \right) - \Phi' \left(\frac{i_t}{i_{t-1}} \right) \frac{i_t}{i_{t-1}} \quad \text{and} \quad \phi_{2,t} = \left(\frac{i_{t+1}}{i_t} \right)^2 \Phi' \left(\frac{i_{t+1}}{i_t} \right) \end{aligned} \quad (1.21)$$

1.1.4 Retailer

The final good y_t , used for consumption and investment, is produced in a competitive market by combining a continuum of intermediate goods indexed by $j \in [0, 1]$, via the CES production function

$$y_t = \left(\int_0^1 y_{j,t}^{\frac{\theta_p-1}{\theta_p}} dj \right)^{\frac{\theta_p}{\theta_p-1}}, \quad (1.22)$$

where $y_{j,t}$ denotes the overall demand addressed to the producer of intermediate good j and θ_p is the elasticity of demand for a producer of intermediate good. The maximization of profits yields typical demand functions

$$y_{j,t} = \left(\frac{P_{j,t}}{P_t} \right)^{-\theta_p} y_t, \quad (1.23)$$

with

$$P_t = \left(\int_0^1 P_{j,t}^{1-\theta_p} dj \right)^{\frac{1}{1-\theta_p}}, \quad (1.24)$$

where $P_{j,t}$ denotes the price of intermediate good produced by firm j .

1.1.5 Intermediate Goods

Intermediate firms produce differentiated goods by assembling services of labor and capital, namely $\ell_{j,t}$ and $k_{j,t-1}$, respectively. Capital services are rented from the entrepreneur which owns the capital stock. Type- j firm's total labor input, $\ell_{j,t}$ is composed by household labor, $\ell_{j,t}^h$, and entrepreneurial labor, $\ell_{j,t}^e$, according to

$$\ell_{j,t} = [\ell_{j,t}^h]^\Omega [\ell_{j,t}^e]^{1-\Omega}$$

Type- j intermediate good is produced with the following constant return to scale technology

$$y_{j,t} = \ell_{j,t}^{1-\alpha} k_{j,t-1}^\alpha. \quad (1.25)$$

Let $\mathbb{S}(y_{j,t})$ denote the total real cost of producing $y_{j,t}$ units of good j

$$\mathbb{S}(y_{j,t}) = w_t \ell_{j,t}^h + w_t^e \ell_{j,t}^e + z_t k_{j,t-1}, \quad (1.26)$$

where $w_t^e \equiv W_t^e/P_t$ is the real wage of entrepreneur's labor and $z_t \equiv Z_t/P_t$ is the real rental rate of capital. Each monopolistic firm chooses capital and labor services in order to minimize $\mathbb{S}(y_{j,t})$ subject to the production function (1.25), taking w_t , w_t^e and z_t as given. Accordingly, labor and capital demands satisfy

$$w_t = s_t \Omega (1 - \alpha) \frac{y_t}{\ell_{j,t}^h}, \quad (1.27)$$

$$w_t^e = s_t (1 - \Omega) (1 - \alpha) \frac{y_t}{\ell_{j,t}^e}, \quad (1.28)$$

$$z_t = s_t \alpha \frac{y_t}{k_{j,t-1}}. \quad (1.29)$$

where $s_t \equiv \partial \mathbb{S}(\cdot) / \partial y_{j,t}$ is the the real marginal cost.

Price Setting We assume that in each period of time, type- j monopolistic firm's price setting decision is modelled through the Calvo's (1983) staggering mechanism. Let $y_{j,t}$ denote the quantity of output sold by the representative entrepreneur to the type- j retailer measured in wholesale goods: one unit of wholesale goods is transformed in one unit of retail goods. In each period, the retailer faces a constant probability, $1 - \alpha_p$, of being able to re-optimize its price. Firm j takes the demand function (1.23) into account when setting its price. Additionally, it takes into account the fact that this price rate will presumably hold for more than one period.

Let denote $P_{j,t}^*$, the nominal price chosen in time t and $y_{j,t,T}^*$, the demand for good j in period T if firm j last reoptimized its price in period t . Therefore, firm j selects $P_{j,t}^*$ so as to maximize the present discounted sum of profit streams, taking as given the demand curve

$$P_{j,t}^* = \left\{ \begin{array}{l} \max_{P_{j,t} \geq 0} \quad \mathbb{E}_t \sum_{T=t}^{\infty} (\beta \alpha_p)^{T-t} \lambda_T \left\{ \left(\frac{P_{j,t}}{P_T} \right) y_{j,t} - \mathbb{S}(y_{j,t}) \right\} \\ \text{subject to } y_{j,t} = \left(\frac{P_{j,t}}{P_t} \right)^{-\theta_p} y_t \end{array} \right\}$$

The FOC is given by

$$\mathbb{E}_t \sum_{T=t}^{\infty} (\beta \alpha_p)^{T-t} \lambda_T \frac{y_{j,t,T}^*}{P_{j,t}^*} \left\{ \frac{P_{j,t}^*}{1 + \pi_{t,T}} - \mu_p s_t \right\} = 0, \quad (1.30)$$

where $p_{j,t}^* \equiv P_{j,t}^*/P_t$, $1 + \pi_{t,T} \equiv P_T/P_t$, $\mu_p \equiv \theta_p/(\theta_p - 1)$ denotes the mark-up of the monopolistic firm, and $s_t \equiv \partial \mathbb{S}(\cdot) / \partial y_{j,t}$.

1.1.6 Resource Constraint

The resource constraint is given by

$$y_t = c_t + i_t + c_t^e + g_t + \mu G(\bar{\omega}_{e,t}) R_t^k q_{t-1} k_{t-1}, \quad (1.31)$$

where g_t denotes government expenditures.

1.1.7 Monetary and Fiscal Policy

We assume that the gross nominal interest rate R_t is thus chosen according to

$$R_t = \max(1, R_t^{not}), \quad (1.32)$$

where R_t^{not} is the desired (or notional) interest rate by the central bank in response to the inflation. As such, R_t^{not} follows a simple rule of the form

$$\hat{R}_t^{not} = \rho_R \hat{R}_{t-1}^{not} + (1 - \rho_R) [a_\pi \hat{\pi}_t + a_{\Delta_y} \Delta \hat{y}_t], \quad (1.33)$$

$$\frac{R_t^{not}}{R} = \left(\frac{R_{t-1}^{not}}{R} \right)^{\rho_R} \left[\left(\frac{1 + \pi_t}{1 + \pi} \right)^{a_\pi} \left(\frac{y_t}{y_{t-1}} \right)^{a_{\Delta_y}} \right]^{1 - \rho_R}, \quad (1.34)$$

where $\rho_R \in (0, 1)$ is a smoothing parameter, a_π is the elasticity of R_t^{not} with respect to inflation deviations, R is the steady-state gross nominal interest rate, and π is the central bank's steady-state inflation target. The central bank sets R_t equal to R_t^{not} if and only if its policy rule recommendation implies a non-negative level for the nominal interest rate. If this is not the case, the central bank simply fixes its target rate equal to zero.

The fiscal authority purchases consumption goods, g_t which is financed by lump-sum taxes. The government budget constraint is given by

$$g_t = \frac{\Upsilon_t}{P_t}, \quad (1.35)$$

where g_t follows a AR(1) process.

In the symmetric equilibrium, all entrepreneurs, households, and firms are identical and make the same decisions. In addition, equilibrium on the labor market yields $\int_0^1 \ell_{j,t} dj = \ell_t^h$. The symmetric equilibrium is characterized by an allocation $\{y_t, c_t, c_t^e, i_t, \ell_t^h, k_t, N_t, V_t\}$ and a sequence of prices, wages, and co-state variables $\{P_t, W_t, W_t^e, R_t, R_t^k, Q_t, z_t, \lambda_t\}$ such that, for every realization of stochastic shocks, the optimization conditions in each sector, the monetary and fiscal rules, and the aggregate shocks' law of motions are satisfied.

1.2 Non Linear Model: Summary

1.2.1 Household's Preferences

Euler equation on consumption Non-separable preferences

$$\left(1 - \ell_t^h\right)^{(1-v)(1-\sigma)} v c_t^{v(1-\sigma)-1} = \lambda_t \quad (1.36)$$

Separable preferences

$$c_t^{-\sigma} = \lambda_t. \quad (1.37)$$

Risk free bond equation

$$\frac{\lambda_t}{\exp(\varepsilon_t) R_t} = \beta E_t \left\{ \frac{\lambda_{t+1} (1 + \pi_{t+1})^{\gamma_b}}{1 + \pi_{t+1}} \right\}. \quad (1.38)$$

$$r_t \equiv \frac{R_t}{E_t \{1 + \pi_{t+1}\}}. \quad (1.39)$$

Labor supply Non-separable preferences

$$\frac{1-v}{v} \frac{c_t}{1 - \ell_t^h} = w_t \quad (1.40)$$

Separable preferences

$$\psi \left(\ell_t^h \right)^{\omega_w} = \lambda_t w_t, \quad (1.41)$$

1.2.2 Intermediate Good Sector

Production function

$$y_t = \ell_t^{1-\alpha} k_{t-1}^\alpha. \quad (1.42)$$

$$\ell_t = \left[\ell_t^h \right]^\Omega \left[\ell_t^e \right]^{1-\Omega}. \quad (1.43)$$

$$\ell_t^e = 1. \quad (1.44)$$

Cost minimization

$$w_t = s_t \Omega (1 - \alpha) \frac{y_t}{\ell_t^h}. \quad (1.45)$$

$$w_t^e = s_t (1 - \Omega) (1 - \alpha) \frac{y_t}{\ell_t^e}. \quad (1.46)$$

$$z_t = s_t \alpha \frac{y_t}{k_{t-1}}. \quad (1.47)$$

Phillips curve

$$\mathbb{E}_t \sum_{T=t}^{\infty} (\beta \alpha_p)^{T-t} \lambda_T \frac{y_{j,t,T}^*}{P_{j,t}^*} \left\{ \frac{\delta_{t,T}^p p_{j,t}^*}{1 + \pi_{t,T}} - \mu_p s_t \right\} = 0. \quad (1.48)$$

Inflation definition

$$1 + \pi_t = \frac{P_t}{P_{t-1}}. \quad (1.49)$$

1.2.3 Entrepreneur

Definitions

$$N_t + B_t = Q_t k_t \quad (1.50)$$

$$\Gamma(\bar{\omega}_{t+1}) = \bar{\omega}_{t+1} \int_0^{\bar{\omega}_{t+1}} f(\omega) d\omega + \int_0^{\bar{\omega}_{t+1}} \omega f(\omega) d\omega. \quad (1.51)$$

$$\mu G(\bar{\omega}_{t+1}) = \mu \int_0^{\bar{\omega}_{t+1}} \omega f(\omega) d\omega. \quad (1.52)$$

$$x_t = Q_t \frac{k_t}{N_t}. \quad (1.53)$$

$$\check{r}_t = \mathbb{E}_t \left\{ \frac{\tilde{R}_{t+1}^k}{R_t} \right\}. \quad (1.54)$$

In real terms: $n_t = N_t/P_t$; $q_t = Q_t/P_t$; $b_t = B_t/P_t$;

$$n_t = q_t k_t - b_t \quad (1.55)$$

$$x_t = q_t \frac{k_t}{n_t}. \quad (1.56)$$

Optimal contract equations

$$\mathbb{E}_t \left\{ \check{r}_t [1 - \Gamma(\bar{\omega}_{t+1}) + \Lambda_t [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})]] \right\} = \mathbb{E}_t \{ \Lambda_t (1 + \pi_{t+1})^{\gamma_b} \}, \quad (1.57)$$

$$\mathbb{E}_t \{ \check{r}_t x_t \Gamma_{\omega}(\bar{\omega}_{t+1}) \} = \mathbb{E}_t \{ \check{r}_t x_t \Lambda_t [\Gamma_{\omega}(\bar{\omega}_{t+1}) - \mu G_{\omega}(\bar{\omega}_{t+1})] \}, \quad (1.58)$$

$$\mathbb{E}_t \{ [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \check{r}_t x_t \} = [x_t - 1] \mathbb{E}_t \{ (1 + \pi_{t+1})^{\gamma_b} \}. \quad (1.59)$$

Return of capital

$$R_t^k = (1 + \pi_t) \frac{z_t + (1 - \delta - \text{cap}_t) q_t}{q_{t-1}}. \quad (1.60)$$

$$\tilde{R}_t^k = \exp(-\varepsilon_{t-1}) R_t^k \quad (1.61)$$

Net worth

$$\begin{aligned}
N_t &= \gamma V_t + W_t^e \\
V_t &= \tilde{R}_t^k Q_{t-1} k_{t-1} (1 - \mu G(\bar{\omega}_t)) - R_{t-1} B_{t-1} (1 + \pi_t)^{\gamma_b}. \\
c_t^e &= (1 - \gamma) \varrho \frac{V_t}{P_t}.
\end{aligned}$$

In real terms: $n_t = N_t/P_t$; $q_t = Q_t/P_t$; $v_t = V_t/P_t$; $b_t = B_t/P_t$; $w_t^e = W_t^e/P_t$,

$$n_t = \gamma v_t + w_t^e \quad (1.62)$$

$$v_t = \frac{1}{1 + \pi_t} \left[\tilde{R}_t^k q_{t-1} k_{t-1} (1 - \mu G(\bar{\omega}_t)) - R_{t-1} b_{t-1} (1 + \pi_t)^{\gamma_b} \right]. \quad (1.63)$$

$$c_t^e = (1 - \gamma) \varrho v_t. \quad (1.64)$$

1.2.4 Capital Producer

Investment adjustment cost

$$k_t = (1 - \delta) k_{t-1} + \left[1 - \Phi \left(\frac{i_t}{i_{t-1}} \right) \right] i_t. \quad (1.65)$$

$$\begin{aligned}
q_t &= \left[\phi_{1,t} + \beta E_t \left\{ \frac{\lambda_{t+1} q_{t+1}}{\lambda_t q_t} \phi_{2,t} \right\} \right]^{-1} \text{ where} \\
\phi_{1,t} &= 1 - \Phi \left(\frac{i_t}{i_{t-1}} \right) - \Phi' \left(\frac{i_t}{i_{t-1}} \right) \frac{i_t}{i_{t-1}} \quad \text{and} \quad \phi_{2,t} = \left(\frac{i_{t+1}}{i_t} \right)^2 \Phi' \left(\frac{i_{t+1}}{i_t} \right)
\end{aligned} \quad (1.66)$$

$$\text{cap}_t = 0$$

Capital adjustment cost

$$k_t = (1 - \delta) k_{t-1} + i_t - \frac{\vartheta}{2} \left(\frac{i_t}{k_{t-1}} - \delta \right)^2 k_{t-1}. \quad (1.67)$$

$$q_t^{-1} = 1 - \vartheta \left(\frac{i_t}{k_{t-1}} - \delta \right). \quad (1.68)$$

$$\begin{aligned}
\text{cap}_t &= \frac{\vartheta}{2} \left(\frac{i_t}{k_{t-1}} - \delta \right)^2 - \vartheta \left(\frac{i_t}{k_{t-1}} - \delta \right) \\
&= -\frac{\vartheta}{2} \left(\left[\frac{i_t}{k_{t-1}} \right]^2 - \delta^2 \right)
\end{aligned}$$

Resource constraint

$$y_t = c_t + i_t + g_t + c_t^e + \mu G(\bar{\omega}_t) r_t^k q_{t-1} k_{t-1}. \quad (1.69)$$

1.2.5 Monetary and Fiscal Policy

$$R_t = \max(1, R_t^{\text{not}}), \quad (1.70)$$

$$\frac{R_t^{\text{not}}}{R} = \left(\frac{R_{t-1}^{\text{not}}}{R} \right)^{\rho_R} \left[\left(\frac{1 + \pi_t}{1 + \pi} \right)^{a_\pi} \left(\frac{y_t}{y_{t-1}} \right)^{a_{\Delta y}} \right]^{1 - \rho_R}, \quad (1.71)$$

$$G_t = \Upsilon_t. \quad (1.72)$$

1.3 Model's Steady State

In this sub-section, we compute the model's steady-state of real and financial variables when this is relevant for log-linearization.

1.3.1 Nominal Variables

- Steady state inflation is zero, which implies (for gross rates)

$$1 + \pi = 1,$$

$$R = r,$$

$$R^k = r^k.$$

1.3.2 Real Variables

- From Equation (1.38)

$$r = \frac{1}{\beta}. \quad (1.73)$$

- From Equation (1.54)

$$r^k = (\tilde{r})(r) = \frac{\tilde{r}}{\beta}. \quad (1.74)$$

- From Equation (1.48)

$$\mu_p s = 1. \quad (1.75)$$

- From Equation (1.60)

$$z = q \left[r^k - (1 - \delta) \right]. \quad (1.76)$$

- From Equation (1.65) or (1.67)

$$\delta = \frac{i}{k}. \quad (1.77)$$

- From Equation (1.66) or (1.68)

$$q = 1. \quad (1.78)$$

since in the s.s. $\phi_1 = 1$ and $\phi_2 = 0$.

- From Equations (1.45)-(1.47)

$$w = s\Omega(1 - \alpha) \frac{y}{\ell^h}, \quad (1.79)$$

$$w^e = s(1 - \Omega)(1 - \alpha) \frac{y}{\ell^e}, \quad (1.80)$$

$$\frac{y}{k} = \frac{z}{s\alpha}. \quad (1.81)$$

- From Equations (1.55) and (1.56)

$$x = \frac{qk}{n}, \quad (1.82)$$

$$\frac{b}{n} = x - 1. \quad (1.83)$$

- From Equation (1.19)

$$v = [1 - \Gamma(\bar{\omega})] q r^k k,$$

which implies, using (1.64)

$$\frac{c^e}{k} = (1 - \gamma) \varrho [1 - \Gamma(\bar{\omega})] r^k.$$

- From Equation (1.69)

$$\frac{c}{y} = 1 - \frac{i}{y} - \frac{c^e}{y} - \frac{g}{y} - \mu G(\bar{\omega}) q r^k \frac{k}{y}. \quad (1.84)$$

- Non-separable preferences: how to set v ? Set v compatible with $\ell^h = \frac{1}{3}$. Depart from the labor market equilibrium condition, and solve for v .

$$\begin{aligned} (l^d) \quad s\Omega(1 - \alpha) \frac{y}{\ell^h} &= \frac{1 - v}{v} \frac{c}{1 - \ell^h} \quad (l^s), \\ v &= \frac{1}{(A + 1)} \quad \text{where } A = s\Omega(1 - \alpha) \frac{1 - \ell^h}{\ell^h} \frac{y}{c}. \end{aligned}$$

1.3.3 Financial Frictions

- Steady state of optimal contract equations:

$$\check{r} (1 - \Gamma(\bar{\omega}) + \Lambda [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})]) = \Lambda, \quad (1.85)$$

$$\frac{\Gamma_\omega(\bar{\omega})}{\Gamma_\omega(\bar{\omega}) - \mu G_\omega(\bar{\omega})} = \Lambda, \quad (1.86)$$

$$\Gamma(\bar{\omega}) - \mu G(\bar{\omega}) = \frac{x - 1}{\check{r}x}. \quad (1.87)$$

BGG choose $\bar{\omega}$, σ_ω , γ , and μ in order to match the following moments

- An annually rate of business failure of 3 per cent, or $F(\bar{\omega}) = 0.03/4$ in quarterly terms,
- A risk premium of 200 basis points, or $\check{r} = 1.02^{\frac{1}{4}}$ in quarterly terms, and
- A leverage ratio of 50 per cent, or $(x - 1)/x = .5$, with $x \equiv Qk/N = k/n = 2$.

In “funx3.m”, we use the Matlab command `fsolve` to pick up the value of $\bar{\omega}$, σ_ω , γ and μ which satisfy the following system of steady-state equations

$$F(\bar{\omega}) = 0.03/4, \quad (1.88)$$

$$\Gamma(\bar{\omega}) - \mu G(\bar{\omega}) = \frac{x - 1}{x\check{r}}, \quad (1.89)$$

$$\check{r} = \left\{ 1 - \Gamma(\bar{\omega}) + \frac{\Gamma_\omega(\bar{\omega}) (\Gamma(\bar{\omega}) - \mu G(\bar{\omega}))}{[\Gamma_\omega(\bar{\omega}) - \mu G_\omega(\bar{\omega})]} \right\}^{-1} \frac{\Gamma_\omega(\bar{\omega})}{[\Gamma_\omega(\bar{\omega}) - \mu G_\omega(\bar{\omega})]}, \text{ and} \quad (1.90)$$

$$n = \gamma v + w^e \quad (1.91)$$

Using the fact that $v = [1 - \Gamma(\bar{\omega})] r^k k$ and $x = k/n$, equation (1.91) can be rewritten as

$$\frac{1}{r^k x} = \gamma [1 - \Gamma(\bar{\omega})] + \frac{w^e}{r^k k}.$$

Using (1.80) and (1.81), and the assumption that $\ell^e = 1$, than $w^e k^{-1} = \alpha^{-1}(1 - \Omega)(1 - \alpha)z$, and the last equation can be finally stated as

$$\frac{1}{r^k x} = \gamma [1 - \Gamma(\bar{\omega})] + \frac{(1 - \Omega)(1 - \alpha)z}{\alpha r^k}. \quad (1.92)$$

In Matlab, we will use equations (1.88), (1.89), (1.90), and (1.92).

Remark 2 *Partial expectation formula*

$$\int_{\bar{\omega}_t}^{\infty} \omega f(\omega) d\omega = \exp\left(\frac{\mu_{\omega} + \sigma_{\omega}^2}{2}\right) F_N(v + \sigma_{\omega}),$$

where F_N is the cumulative distribution function of the standard normal, $v \equiv (\mu_{\omega} - \ln(\bar{\omega}))/\sigma_{\omega}$ being the standard version of $\ln(\bar{\omega})$, and

$$F(\bar{\omega}_t) = \text{prob}(\omega_t \leq \bar{\omega}_t) = \int_0^{\bar{\omega}_t} f(\omega) d\omega, .$$

■

Therefore, notice that:

$$\Gamma(\bar{\omega}) = \bar{\omega} [1 - F(\bar{\omega})] + E(\omega) - F_N((\mu_{\omega} - \ln(\bar{\omega}))/\sigma_{\omega} + \sigma_{\omega}).$$

$$\Gamma_{\omega}(\bar{\omega}) = [1 - F(\bar{\omega})].$$

$$\Gamma_{\omega\omega}(\bar{\omega}) = -f(\bar{\omega}).$$

$$\mu G(\bar{\omega}) = \mu [E(\omega) - F_N((\mu_{\omega} - \ln(\bar{\omega}))/\sigma_{\omega} + \sigma_{\omega})].$$

$$\mu G_{\omega}(\bar{\omega}) = \mu \bar{\omega} f(\bar{\omega}).$$

$$\mu G_{\omega\omega}(\bar{\omega}) = \mu [f(\bar{\omega}) + \bar{\omega} f_{\omega}(\bar{\omega})],$$

with

$$f_{\omega}(\bar{\omega}) = -\frac{f(\bar{\omega})}{\bar{\omega}} \left[1 - \frac{(\mu_{\omega} - \ln(\bar{\omega}))}{\sigma_{\omega}^2} \right].$$

In addition, the elasticity of the external finance premium to the leverage ratio is defined by the model's steady state:

$$\chi \equiv \left(\frac{1}{x - 1} \right) \left[1 + \frac{f_2}{f_0 f_1} \right]^{-1},$$

where

$$\begin{aligned} f_0 &\equiv 1 - \check{r} [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})], \\ f_1 &\equiv \bar{\omega} \left[\frac{\Gamma_{\omega\omega}(\bar{\omega})}{\Gamma_{\omega}(\bar{\omega})} - \frac{\Gamma_{\omega\omega}(\bar{\omega}) - \mu G_{\omega\omega}(\bar{\omega})}{\Gamma_{\omega}(\bar{\omega}) - \mu G_{\omega}(\bar{\omega})} \right], \\ f_2 &\equiv \bar{\omega} \left[\frac{\Gamma_{\omega}(\bar{\omega}) - \mu G_{\omega}(\bar{\omega})}{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})} \right]. \end{aligned}$$

1.4 Model's Log-Linearization

In this sub-section, the model's equations are log-linearized around the non-stochastic steady state.

1.4.1 Household's Preferences

Euler equation

- Non-separable preference. From Equation (1.36)

$$-\frac{\ell^h}{1-\ell^h}\hat{\ell}_t^h(1-v)(1-\sigma)+\hat{c}_t(v(1-\sigma)-1)=\hat{\lambda}_t. \quad (1.93)$$

- Separable preferences. From Equation (1.37)

$$-\sigma\hat{c}_t=\hat{\lambda}_t. \quad (1.94)$$

Risk free bond equation

- Equation (1.38)

$$\hat{\lambda}_t-\hat{R}_t=E_t\left\{\hat{\lambda}_{t+1}-(1-\gamma_b)\hat{\pi}_{t+1}\right\}+\hat{\varepsilon}_t. \quad (1.95)$$

- Equation (1.39)

$$\hat{r}_t=\hat{R}_t-E_t\hat{\pi}_{t+1}. \quad (1.96)$$

Labor Supply Equation

- Non-separable preference. From Equation (1.40)

$$\left(\frac{\ell^h}{1-\ell^h}\right)\hat{\ell}_t^h=\hat{w}_t-\hat{c}_t. \quad (1.97)$$

- Separable preferences. From Equation (1.41)

$$\omega_w\hat{\ell}_t^h=\hat{\lambda}_t+\hat{w}_t. \quad (1.98)$$

1.4.2 Intermediate Good Sector

Production function

- Equations (1.42), (1.43) and (1.44)

$$\hat{y}_t=(1-\alpha)\Omega\hat{\ell}_t^h+\alpha\hat{k}_{t-1}, \quad (1.99)$$

$$\hat{\ell}_t=\Omega\hat{\ell}_t^h. \quad (1.100)$$

Cost minimization

- Equation (1.45)

$$\hat{w}_t=\hat{s}_t+\hat{y}_t-\hat{\ell}_t^h. \quad (1.101)$$

- Equation (1.46)

$$\hat{w}_t^e=\hat{s}_t+\hat{y}_t. \quad (1.102)$$

- Equation (1.47)

$$\hat{z}_t=\hat{s}_t+\hat{y}_t-\hat{k}_{t-1}. \quad (1.103)$$

Phillips curve

- Equation (1.48)

$$\hat{\pi}_t = \kappa_p \hat{s}_t + \beta \mathbb{E}_t \{ \hat{\pi}_{t+1} \}, \quad (1.104)$$

where

$$\kappa_p = \frac{(1 - \alpha_p)(1 - \beta \alpha_p)}{\alpha_p}.$$

1.4.3 Entrepreneur

Definitions

- Return of capital: Equation (1.60)

$$\hat{R}_t^k = \hat{\pi}_t + \hat{z}_t \left[\frac{z}{r^k} \right] + \hat{q}_t \left[\frac{1 - \delta}{r^k} \right] - \hat{q}_{t-1} + (1 - F) \times (\hat{i}_t - \hat{k}_{t-1}) \left[\frac{\vartheta \delta^2}{r^k} \right], \quad (1.105)$$

where F is a dummy that take 1 for investment adjustment cost and 0 for capital adjustment cost.

- Equation (1.61)

$$\widehat{\tilde{R}}_t^k = \hat{R}_t^k - \varepsilon_{t-1}$$

- Equation (1.54)

$$\widehat{\tilde{r}}_t = \mathbb{E}_t \{ \widehat{\tilde{R}}_{t+1}^k \} - \hat{R}_t = \mathbb{E}_t \{ R_{t+1}^k \} - \varepsilon_t - \hat{R}_t. \quad (1.106)$$

- Equation (1.55)

$$\hat{n}_t = x \hat{q}_t + x \hat{k}_t - (x - 1) \hat{b}_t \quad (1.107)$$

- Equation (1.56)

$$\hat{x}_t = \hat{q}_t + \hat{k}_t - \hat{n}_t. \quad (1.108)$$

Optimal contract equations

- Equation (1.57)

$$\widehat{\tilde{r}}_{t-1} = f_0 \hat{\Lambda}_t + \gamma_b \hat{\pi}_t, \quad (1.109)$$

- Equation (1.58)

$$\hat{\Lambda}_t = f_1 \widehat{\omega}_t, \quad (1.110)$$

- Equation (1.59)

$$\mathbb{E}_t \widehat{\tilde{r}}_t = \hat{x}_t \left(\frac{1}{x - 1} \right) - f_2 \mathbb{E}_t \widehat{\omega}_{t+1} + \gamma_b \mathbb{E}_t \hat{\pi}_{t+1} \quad (1.111)$$

where

$$\begin{aligned} f_0 &\equiv 1 - \check{r} [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] \\ f_1 &\equiv \bar{\omega} \left[\frac{\Gamma_{\omega\omega}(\bar{\omega})}{\Gamma_{\omega}(\bar{\omega})} - \frac{\Gamma_{\omega\omega}(\bar{\omega}) - \mu G_{\omega\omega}(\bar{\omega})}{\Gamma_{\omega}(\bar{\omega}) - \mu G_{\omega}(\bar{\omega})} \right] \\ f_2 &\equiv \bar{\omega} \left[\frac{\Gamma_{\omega}(\bar{\omega}) - \mu G_{\omega}(\bar{\omega})}{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})} \right] \end{aligned}$$

By combining (1.109), (1.110), and (1.111), we can summarize the relationship between the external finance premium and leverage as following

$$E_t \hat{r}_t = \chi \hat{x}_t + \gamma_b E_t \hat{\pi}_{t+1} \quad (1.112)$$

$$E_t \{\hat{R}_{t+1}^k\} - \varepsilon_t - \hat{R}_t = \chi \hat{x}_t + \gamma_b E_t \hat{\pi}_{t+1} \quad (1.113)$$

$$E_t \{\hat{R}_{t+1}^k\} - \hat{R}_t = \chi \hat{x}_t + \gamma_b E_t \hat{\pi}_{t+1} + \varepsilon_t \quad (1.114)$$

where

$$\chi \equiv \left(\frac{1}{x-1} \right) \left[1 + \frac{f_2}{f_0 f_1} \right]^{-1}$$

In AIM, we use the combination of (1.109) and (1.110) and (1.112).

Net worth

- Using the fact that $v = [1 - \Gamma(\bar{\omega})] r^k k$, $n = \gamma v + w^e$, and $c^e = (1 - \gamma) \varrho [1 - \Gamma(\bar{\omega})] r^k k$, Equations (1.62), (1.63), and (1.64) become

$$\hat{n}_t = n_0 \hat{v}_t + [1 - n_0] \hat{w}_t^e, \quad (1.115)$$

$$\hat{v}_t + \hat{\pi}_t = v_0 [\hat{R}_t^k + \hat{q}_{t-1} + \hat{k}_{t-1}] - [v_0 - 1] [\hat{R}_{t-1} + \hat{b}_{t-1} + \gamma_b \hat{\pi}_t] - v_1 \hat{\omega}_t, \quad (1.116)$$

$$\frac{c^e}{k} \hat{c}_t^e = \frac{c^e}{k} \hat{v}_t. \quad (1.117)$$

$$\begin{aligned} \text{where} \quad : \quad v_0 &\equiv \frac{1 - \mu G(\bar{\omega})}{1 - \Gamma(\bar{\omega})}, \quad v_1 \equiv \frac{\bar{\omega} \mu G(\bar{\omega})}{1 - \Gamma(\bar{\omega})}, \\ : \quad n_0 &\equiv \gamma [1 - \Gamma(\bar{\omega})] r^k x. \end{aligned}$$

1.4.4 Capital Producer

Law of motion of capital

- Equation (1.65) or (1.67)

$$\hat{k}_t = \hat{k}_{t-1} (1 - \delta) + \delta \hat{i}_t, \quad (1.118)$$

Price of capital

- Investment adjustment cost. From Equation (1.66)

$$\frac{1}{\varkappa} \hat{q}_t = (\hat{i}_t - \hat{i}_{t-1}) - \beta (E_t \{\hat{i}_{t+1}\} - \hat{i}_t) \quad (1.119)$$

where $\varkappa \equiv \Phi''(1)$.

- Capital adjustment cost. From Equation (1.68)

$$\hat{q}_t = \vartheta \delta [\hat{i}_t - \hat{k}_{t-1}]. \quad (1.120)$$

Merging the two condition yields:

$$(1 - F) \hat{q}_t + F \frac{1}{\varkappa} \hat{q}_t = (1 - F) \vartheta \delta \hat{i}_t - (1 - F) \vartheta \delta \hat{k}_{t-1} + F (\hat{i}_t - \hat{i}_{t-1}) - \beta F (E_t \{\hat{i}_{t+1}\} - \hat{i}_t). \quad (1.121)$$

where F is a dummy that take 1 for investment adjustment cost and 0 for capital adjustment cost.

1.4.5 Resource Constraint

- Equation (1.69)

$$\hat{y}_t = \hat{c}_t \frac{c}{y} + \hat{i}_t \frac{i}{y} + \hat{g}_t \frac{g}{y} + \hat{c}_t^e \frac{c^e}{y} + \left[\hat{r}_t^k + \hat{q}_{t-1} + \hat{k}_{t-1} \right] \left[\mu G(\bar{\omega}) r^k \frac{k}{y} \right] + \hat{\bar{\omega}}_t \bar{\omega} \mu G_\omega(\bar{\omega}) r^k \frac{k}{y}. \quad (1.122)$$

1.4.6 Monetary Policy

$$\hat{R}_t^{not} = \rho_R \hat{R}_{t-1}^{not} + (1 - \rho_R) a_\pi \hat{\pi}_t + (1 - \rho_R) a_{\Delta y} (\hat{y}_t - \hat{y}_{t-1}). \quad (1.123)$$

Shocks

- Risk premium (recessionary) shock

$$\varepsilon_t = \rho_\varepsilon \varepsilon_{t-1} + \epsilon_{\varepsilon,t}.$$

- Government spending shock

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \epsilon_{g,t}.$$

2 Model's Extensions: Taxation

In this section, we consider three types of alternative fiscal policies: a labor-income tax cut, a capital-income tax cut, and partial expensing allowance on investment.

2.1 Labor-Income Tax

We assume that households and entrepreneurs pay an income tax that affects disposable labor-income, which equals $(1 - \tau_{\ell,t}) w_t \ell_t^h$ for the former and $(1 - \tau_{\ell,t}) w_t^e$ for the latter.

Household labor supply, defined by the FOC of the lifetime utility maximization, is modified to

$$\frac{1-v}{v} \frac{c_t}{1-\ell_t^h} = (1 - \tau_{\ell,t}) w_t. \quad (2.1)$$

The log-linear version is

$$\hat{c}_t + \frac{\ell^h}{1-\ell^h} \hat{\ell}_t^h = \hat{w}_t - \frac{\tau_\ell}{1-\tau_\ell} \hat{\tau}_{\ell,t}.$$

The formula for v is also affected by taxes:

$$v = \frac{1}{(A' + 1)} \quad \text{where } A' = s\Omega(1-\alpha)(1-\tau_\ell) \frac{1-\ell^h}{\ell^h} \frac{y}{c}.$$

Entrepreneurs net worth now changes to

$$N_t = \gamma V_t + (1 - \tau_{\ell,t}) W_t^e, \quad (2.2)$$

which log-linearized form is

$$\hat{n}_t = n_0 \hat{v}_t + (1 - n_0) \hat{w}_t^e - (1 - n_0) \frac{\tau_\ell}{1 - \tau_\ell} \hat{\tau}_{\ell,t}.$$

The labor-income tax rate follows $\log(\tau_{\ell,t}) = (1 - \rho_\ell) \log(\tau_\ell) + \rho_\ell \log(\tau_{\ell,t-1}) + \epsilon_{\ell,t}$. The steady-state level of taxes, τ_ℓ , equals 28 percent, ρ_ℓ is set to 0.8, and $\epsilon_{\ell,t}$ is the tax innovation.

2.2 Capital-Income Tax

We assume that entrepreneurs pay taxes on their capital earnings. The nominal rate of capital returns, given by Equation (1.16), becomes

$$R_{t+1}^k = (1 + \pi_{t+1}) \frac{(1 - \tau_{k,t+1}) z_{t+1} + (1 - \delta - \text{cap}_{t+1}) q_{t+1}}{q_t}. \quad (2.3)$$

The steady-state rental rate of capital, given by Equation (1.76) becomes

$$z = \frac{r^k - (1 - \delta)}{1 - \tau_k}. \quad (2.4)$$

The log-linear model is similar than previously except for Equation (1.105) which is

$$\hat{R}_t^k = \hat{\pi}_t + \hat{z}_t (1 - \tau_k) \frac{z}{r^k} - \hat{\tau}_{k,t} \tau_k \frac{z}{r^k} + \hat{q}_t \left[\frac{1 - \delta}{r^k} \right] - \hat{q}_{t-1} + F \times \left(\hat{i}_t - \hat{k}_{t-1} \right) \left[\frac{\partial \delta^2}{r^k} \right]. \quad (2.5)$$

The capital-income tax rate follows $\log(\tau_{k,t}) = (1 - \rho_k) \log(\tau_k) + \rho_k \log(\tau_{k,t-1}) + \epsilon_{k,t}$. The steady-state level of taxes, τ_k , equals 36 percent, ρ_k is set to 0.8, and $\epsilon_{k,t}$ is the tax innovation.

2.3 Partial Expensing Allowances on Investment

Following Edge and Rudd (2011), we assume two types of capital deductions: expensing allowances on investment and depreciation allowances. The former corresponds to a partial rebate of the purchase price of capital while the later compensates the firm for capital depreciation. The capital producer benefits from these investment-tax incentives and solves the following problem

$$\max_{i_t} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \frac{\lambda_{t+1}}{\lambda_t} \left\{ \begin{aligned} & Q_t(1 - \tau_q) [k_t - (1 - \delta - \text{cap}_t)k_{t-1}] - (1 - \tau_q X_t) P_t i_t \\ & + \tau_q \sum_{v=1}^{\infty} \delta(1 - \delta)^{v-1} (1 - X_{t-v}) P_{t-v} i_{t-v} - (1 - \tau_q) Q_t \text{cap}_t k_{t-1} \end{aligned} \right\}, \quad (2.6)$$

subject to

$$k_t = (1 - \delta) k_{t-1} + i_t - \frac{\vartheta}{2} \left(\frac{i_t}{k_{t-1}} - \delta \right)^2 k_{t-1},$$

with $\text{cap}_t = \frac{\partial K(\cdot)}{\partial k_{t-1}} = \frac{\vartheta}{2} \left(\frac{i_t}{k_{t-1}} - \delta \right)^2 - \vartheta \left(\frac{i_t}{k_{t-1}} - \delta \right)$.

The expensing allowance, defined by X_t , corresponds to a subsidy on the purchasing price of capital and follows an AR(1) process. The depreciation of capital is also compensated by the depreciation allowance, defined by the term $\sum_{v=1}^{\infty} \delta(1 - \delta)^{v-1} P_{t-v} i_{t-v}$. As noticed by Edge and Rudd (2011), expensed capital cannot benefit from the depreciation allowance, such that each term in sum in Equation (2.6) is multiplied by $(1 - X_{t-v})$.

The FOC with respect to i_t is given by

$$/i_t : q_t(1 - \tau_q) \left[1 - \vartheta \left(\frac{i_t}{k_{t-1}} - \delta \right) \right] = 1 - \tau_q X_t - (1 - X_t) \sum_{v=1}^{\infty} \delta \tau^q (1 - \delta)^{v-1} \beta^v \frac{\lambda_{t+v}}{\lambda_t} \prod_{h=1}^v (1 + \pi_{t+h})^{-1}. \quad (2.7)$$

Using the Euler equation given by (1.4),

$$R_t \exp(\varepsilon_t) = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1} (1 + \pi_{t+1})^{\gamma_b}}{\lambda_t (1 + \pi_{t+1})} \right\},$$

we can rewrite the sum in Equation (2.7) as

$$\begin{aligned} \sum_{v=1}^{\infty} \frac{\beta^v \frac{\lambda_{t+v}}{\lambda_t}}{\prod_{h=1}^v (1 + \pi_{t+h})} \delta \tau^q (1 - \delta)^v &= \sum_{v=1}^{\infty} \frac{\prod_{h=1}^v (1 + \pi_{t+h+1})^{\gamma_b}}{\prod_{h=1}^v R_{t+h} \exp(\varepsilon_{t+h})} \delta \tau^q (1 - \delta)^{v-1} \\ &= V_t^{\delta}, \end{aligned}$$

where V_t^{δ} can be written in a recursive form as

$$V_t^{\delta} = \frac{(1 + \pi_{t+1})^{\gamma_b}}{R_t \exp(\varepsilon_t)} \delta \tau^q + \frac{(1 + \pi_{t+1})^{\gamma_b}}{R_t \exp(\varepsilon_t)} (1 - \delta) \mathbb{E}_t \left\{ V_{t+1}^{\delta} \right\}.$$

Therefore, Equation (2.7) is

$$q_t(1 - \tau_q) \left[1 - \vartheta \left(\frac{i_t}{k_{t-1}} - \delta \right) \right] = 1 - \tau_q X_t - (1 - X_t) V_t^{\delta}. \quad (2.8)$$

At the steady-state, the previous equation yields

$$q = \frac{1 - V^\delta}{1 - \tau_q},$$

$$\text{where } V^\delta = \delta \tau^q \left[\frac{1}{\beta} - (1 - \delta) \right]^{-1} \text{ and } X = 0.$$

Log-linearizing Equation (2.8) yields

$$\hat{q}_t = \left(\hat{i}_t - \hat{k}_{t-1} \right) \delta \vartheta - \left[\frac{\tau_q - V^\delta}{1 - V^\delta} \right] X_t - \left[\frac{V^\delta}{1 - V^\delta} \right] \hat{V}_t^\delta,$$

with

$$\hat{V}_t^\delta = \beta (1 - \delta) E_t \left\{ \hat{V}_{t+1}^\delta \right\} - \left(\hat{R}_t + \hat{\varepsilon}_t - \gamma_b \hat{\pi}_{t+1} \right).$$

The expensing allowance proportion is given by $X_t = \rho_X X_t + \epsilon_{X,t}$, with $\rho_X = 0.8$ and $\epsilon_{X,t} \sim \text{i.i.d } (0, \sigma_X)$.

3 Solving a RE model with a ZLB constraint

In this section, we present two methodologies to solve a linear model with the zero lower bound constraint. It is worth noticing that these two methods are identical when the recessionary shock follows an autoregressive process. In our codes, we use the extended path solution (see `IRF_computation_Extended.m`) for general purposes.

3.1 Piecewise-linear à la Bodenstein *et al.* (2009).

The first solution method mixes the equilibrium conditions (structural-form model) with the equilibrium dynamics (reduced-form model) in a set of piecewise-linear recursive equations. We borrow heavily from Bodenstein *et al.* (2009). All the equations of the model are linearized outside the ZLB period, while during the ZLB we impose $\hat{R}_t = 1 - \frac{1}{\beta}$ (which is the maximum negative deviation allowed for the nominal interest rate from its s.s level, $\frac{1}{\beta} - 1$). The equilibrium conditions are written in the matrix form

$$\bar{\mathbf{A}} E_t \mathbf{s}_{t+1} + \bar{\mathbf{B}} \mathbf{s}_t + \bar{\mathbf{C}} \mathbf{s}_{t-1} + \bar{\mathbf{D}} \boldsymbol{\varepsilon}_t = 0, \quad (3.1)$$

where E_t is the expectation operator conditional on the information available on period t , $\mathbf{s}_t$ is a $(n \times 1)$ -vector of variables where n denotes the total number of state and control variables in the economy, $\mathbf{X} \in \{\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}\}$ are $(n \times n)$ -matrices containing structural coefficients, $\boldsymbol{\varepsilon}_t$ is a $(k \times 1)$ -vector of innovations, and $\bar{\mathbf{D}}$ is a $(n \times k)$ -matrix. The linearized model is solved using the AIM algorithm (see Anderson and Moore, 1985), which yields the following reduced-form solution

$$\mathbf{s}_t = \mathbf{P} \mathbf{s}_{t-1} + \mathbf{Q} \boldsymbol{\varepsilon}_t. \quad (3.2)$$

This system does not hold when the nominal interest rate hits the zero lower bound. The solution method thus involve a piecewise-linear approach that is solved through backward-induction during the ZLB period.⁴ Assume that the ZLB period lasts for the interval $[\underline{t}, \bar{t}]$, such that $0 \leq \underline{t} \leq \bar{t} \leq T$, where T is the maximum length of the simulation. Let $\hat{R}_t^{not} = f(\mathbf{s}_{t+1}, \mathbf{s}_t, \mathbf{s}_{t-1}, \boldsymbol{\varepsilon}_t)$ denote a “notional” interest rate which is determined according to the central bank’s (linear) monetary rule. Further, for convenience, let the monetary rule f be the first-listed equation in the system. Outside the ZLB period, when $\hat{R}_t^{not} \geq 1 - \frac{1}{\beta}$, we have that $\hat{R}_t = \hat{R}_t^{not}$. Assume this last expression is the second-listed equation in the system; further, assume that the nominal interest rate $\hat{R}_t$ is the second-listed variable in vector $\mathbf{s}_t$ while $\hat{R}_t^{not}$ is the first-listed variable in vector $\mathbf{s}_t$. Thus, outside the ZLB it must be the case

⁴During the ZLB period, the solution method is deterministic, i.e. it is valid for a *known* sequence of shocks.

that $\bar{\mathbf{B}}(2, 2) = -1$. In contrast, during the ZLB period, $\hat{R}_t^{not} < 1 - \frac{1}{\beta}$ while $\hat{R}_t = 1 - \frac{1}{\beta}$, and thus $\hat{R}_t \neq \hat{R}_t^{not}$. To accommodate these facts into the aggregate dynamics, we make use of the following auxiliary system (we drop the expectation operator as we assume there is no uncertainty about the future sequence of shocks)

$$\bar{\mathbf{A}}\mathbf{s}_{t+1} + \bar{\mathbf{B}}^* \mathbf{s}_t + \bar{\mathbf{C}}\mathbf{s}_{t-1} + \bar{\mathbf{D}}\boldsymbol{\varepsilon}_t + \bar{\mathbf{d}} = 0, \quad (3.3)$$

where $\bar{\mathbf{B}}^*$ is almost identical to $\bar{\mathbf{B}}$, except that $\bar{\mathbf{B}}^*(2, 2) = 0$, and $\bar{\mathbf{d}}$ is a $(n \times 1)$ -vector filled with zeros except that $\bar{\mathbf{d}}(2) = -\left(1 - \frac{1}{\beta}\right)$. Next, consider the model solution when the economy have just left the ZLB period; the idea is to use this solution in a sequence of recursive equations that will trace back the economy up to the moment the recessionary shock hit.

In time $\bar{t} + 1$, the dynamics of $\mathbf{s}_{\bar{t}+1}$ follow the reduced-form 3.2, thus:

$$\mathbf{s}_{\bar{t}+1} = \mathbf{P}\mathbf{s}_{\bar{t}} + \mathbf{Q}\boldsymbol{\varepsilon}_{\bar{t}+1}.$$

Placing this solution into 3.3 and solving for $\mathbf{s}_{\bar{t}}$ yields

$$\mathbf{s}_{\bar{t}} = \mathbf{G}^{(1)}\mathbf{s}_{\bar{t}-1} + \mathbf{h}^{(1)} + \mathbf{J}^{(1)}, \quad (3.4)$$

where

$$\begin{aligned} \mathbf{G}^{(1)} &= -(\bar{\mathbf{A}}\mathbf{P} + \bar{\mathbf{B}}^*)^{-1} \bar{\mathbf{C}}, \\ \mathbf{h}^{(1)} &= -(\bar{\mathbf{A}}\mathbf{P} + \bar{\mathbf{B}}^*)^{-1} \bar{\mathbf{d}}, \\ \mathbf{J}^{(1)} &= -(\bar{\mathbf{A}}\mathbf{P} + \bar{\mathbf{B}}^*)^{-1} (\bar{\mathbf{A}}\mathbf{Q}\boldsymbol{\varepsilon}_{\bar{t}+1} + \bar{\mathbf{D}}\boldsymbol{\varepsilon}_{\bar{t}}). \end{aligned}$$

Using the solution of $\mathbf{s}_{\bar{t}}$, we can infer from system 3.3 the solution for $\mathbf{s}_{\bar{t}-1}$, which yields

$$\mathbf{s}_{\bar{t}-1} = \mathbf{G}^{(2)}\mathbf{s}_{\bar{t}-2} + \mathbf{h}^{(2)} + \mathbf{J}^{(2)}, \quad (3.5)$$

where

$$\begin{aligned} \mathbf{G}^{(2)} &= -(\bar{\mathbf{A}}\mathbf{G}^{(1)} + \bar{\mathbf{B}}^*)^{-1} \bar{\mathbf{C}}, \\ \mathbf{h}^{(2)} &= -(\bar{\mathbf{A}}\mathbf{G}^{(1)} + \bar{\mathbf{B}}^*)^{-1} (\bar{\mathbf{A}}\mathbf{h}^{(1)} + \bar{\mathbf{d}}), \\ \mathbf{J}^{(2)} &= -(\bar{\mathbf{A}}\mathbf{G}^{(1)} + \bar{\mathbf{B}}^*)^{-1} (\bar{\mathbf{A}}\mathbf{J}^{(1)} + \bar{\mathbf{D}}\boldsymbol{\varepsilon}_{\bar{t}-1}). \end{aligned}$$

Recursively, we have that, for $k = \bar{t}, \bar{t} - 1, \dots, \underline{t}$,

$$\mathbf{s}_k = \mathbf{G}^{(\bar{t}-k+1)}\mathbf{s}_{k-1} + \mathbf{h}^{(\bar{t}-k+1)} + \mathbf{J}^{(\bar{t}-k+1)},$$

where

$$\begin{aligned} \mathbf{G}^{(\bar{t}-k+1)} &= -(\bar{\mathbf{A}}\mathbf{G}^{(\bar{t}-k)} + \bar{\mathbf{B}}^*)^{-1} \bar{\mathbf{C}}, \\ \mathbf{h}^{(\bar{t}-k+1)} &= -(\bar{\mathbf{A}}\mathbf{G}^{(\bar{t}-k)} + \bar{\mathbf{B}}^*)^{-1} (\bar{\mathbf{A}}\mathbf{h}^{(\bar{t}-k)} + \bar{\mathbf{d}}), \\ \mathbf{J}^{(\bar{t}-k+1)} &= -(\bar{\mathbf{A}}\mathbf{G}^{(\bar{t}-k)} + \bar{\mathbf{B}}^*)^{-1} (\bar{\mathbf{A}}\mathbf{J}^{(\bar{t}-k)} + \bar{\mathbf{D}}\boldsymbol{\varepsilon}_k). \end{aligned}$$

and $\mathbf{G}^{(0)} = \mathbf{P}$, $\mathbf{h}^{(0)} = \mathbf{0}$, $\mathbf{J}^{(0)} = \mathbf{P}\mathbf{Q}\boldsymbol{\varepsilon}_{\bar{t}+1}$.

The dynamics of the system for $0 < t < \underline{t}$ are determined with the original system 3.1. Thus, for $h = \underline{t} - 1, \underline{t} - 2, \dots, 1$, we have that

$$\mathbf{s}_h = \mathbf{G}^{(\bar{t}-h+1)} \mathbf{s}_{h-1} + \mathbf{h}^{(\bar{t}-h+1)} + \mathbf{J}^{(\bar{t}-h+1)}, \quad (3.6)$$

where

$$\begin{aligned} \mathbf{G}^{(\bar{t}-h+1)} &= - \left(\bar{\mathbf{A}} \mathbf{G}^{(\bar{t}-h)} + \bar{\mathbf{B}} \right)^{-1} \bar{\mathbf{C}}, \\ \mathbf{h}^{(\bar{t}-h+1)} &= - \left(\bar{\mathbf{A}} \mathbf{G}^{(\bar{t}-h)} + \bar{\mathbf{B}} \right)^{-1} \bar{\mathbf{A}} \mathbf{h}^{(\bar{t}-h)}, \\ \mathbf{J}^{(\bar{t}-h+1)} &= - \left(\bar{\mathbf{A}} \mathbf{G}^{(\bar{t}-h)} + \bar{\mathbf{B}} \right)^{-1} \left(\bar{\mathbf{A}} \mathbf{J}^{(\bar{t}-h)} + \bar{\mathbf{D}} \boldsymbol{\varepsilon}_h \right), \end{aligned}$$

with $h = \underline{t} - 1, \underline{t} - 2, \dots, 1$. Finally, we assume that $\mathbf{s}_0 = \boldsymbol{\varepsilon}_0 = \mathbf{0}$.

3.2 Full-deterministic solution: Extended path solution of a linear model

The idea is to pick a sufficiently large simulation horizon T such that all variables in the economy have reached their steady state levels at that time. Knowing that solution will allow to form recursive equations to solve the model through backward-induction (equivalent to perfect foresight), using only the systems 3.1 and 3.3. Thus, since we know that $\mathbf{s}_{T+1} = \mathbf{0}$, we use 3.1 to obtain, for $i = T, T - 1, \dots, \bar{t} + 1$,

$$\mathbf{s}_i = \hat{\mathbf{G}}^{(T-i+1)} \mathbf{s}_{i-1} + \hat{\mathbf{J}}^{(T-i+1)}, \quad (3.7)$$

where

$$\begin{aligned} \hat{\mathbf{G}}^{(T-i+1)} &= - \left(\bar{\mathbf{A}} \hat{\mathbf{G}}^{(T-i)} + \bar{\mathbf{B}} \right)^{-1} \bar{\mathbf{C}}, \\ \hat{\mathbf{J}}^{(T-i+1)} &= - \left(\bar{\mathbf{A}} \hat{\mathbf{G}}^{(T-i)} + \bar{\mathbf{B}} \right)^{-1} \left(\bar{\mathbf{A}} \hat{\mathbf{J}}^{(T-i)} + \bar{\mathbf{D}} \boldsymbol{\varepsilon}_{T-1} \right), \end{aligned}$$

and $\hat{\mathbf{G}}^{(0)} = \hat{\mathbf{J}}^{(0)} = \mathbf{0}$. During the ZLB period, we use the system 3.3. Thus, for $k = \bar{t}, \bar{t} - 1, \dots, \underline{t}$ we have that

$$\mathbf{s}_k = \hat{\mathbf{G}}^{(T-k+1)} \mathbf{s}_{k-1} + \hat{\mathbf{h}}^{(T-k+1)} + \hat{\mathbf{J}}^{(T-k+1)} \quad (3.8)$$

where

$$\begin{aligned} \hat{\mathbf{G}}^{(T-k+1)} &= - \left(\bar{\mathbf{A}} \hat{\mathbf{G}}^{(T-k)} + \bar{\mathbf{B}}^* \right)^{-1} \bar{\mathbf{C}}, \\ \hat{\mathbf{h}}^{(T-k+1)} &= - \left(\bar{\mathbf{A}} \hat{\mathbf{G}}^{(T-k)} + \bar{\mathbf{B}}^* \right)^{-1} \left(\bar{\mathbf{A}} \hat{\mathbf{h}}^{(T-k)} + \bar{\mathbf{d}} \right), \\ \hat{\mathbf{J}}^{(T-k+1)} &= - \left(\bar{\mathbf{A}} \hat{\mathbf{G}}^{(T-k)} + \bar{\mathbf{B}}^* \right)^{-1} \left(\bar{\mathbf{A}} \hat{\mathbf{J}}^{(T-k)} + \bar{\mathbf{D}} \boldsymbol{\varepsilon}_k \right), \end{aligned}$$

and $\hat{\mathbf{h}}^{(z)} = \mathbf{0}$ for $z \leq T - \bar{t}$. The rest of the solution for $0 < t < \underline{t}$ is similar to the preceding section, we just need to replace $\mathbf{G}$, $\mathbf{h}$, and $\mathbf{J}$ with the corresponding hatted matrices.

4 Robustness Analysis

In this section, we report additional robustness exercises that are not presented in the paper. For every single alternative specification, our main results hold.

4.1 Separable Preferences

We assume separable preferences between consumption and labor, as defined by the utility function (1.8). We consider a log utility function such that $\sigma = 1$ and $\omega_w = 1$.

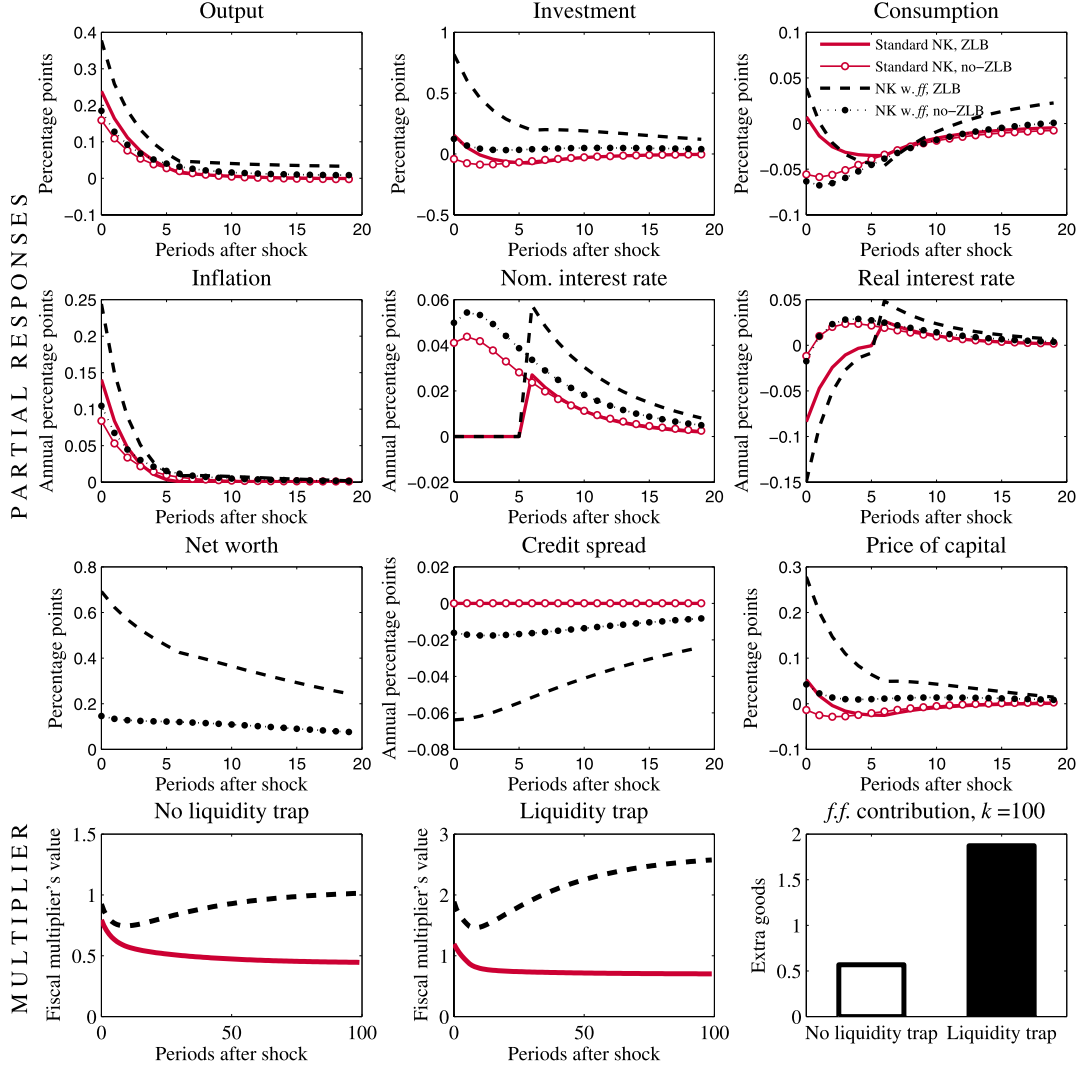


Figure 1: Separable preferences.

4.2 Price Rigidities

We assume instead a weaker degree of price rigidities ($\alpha_p = 0.5$).

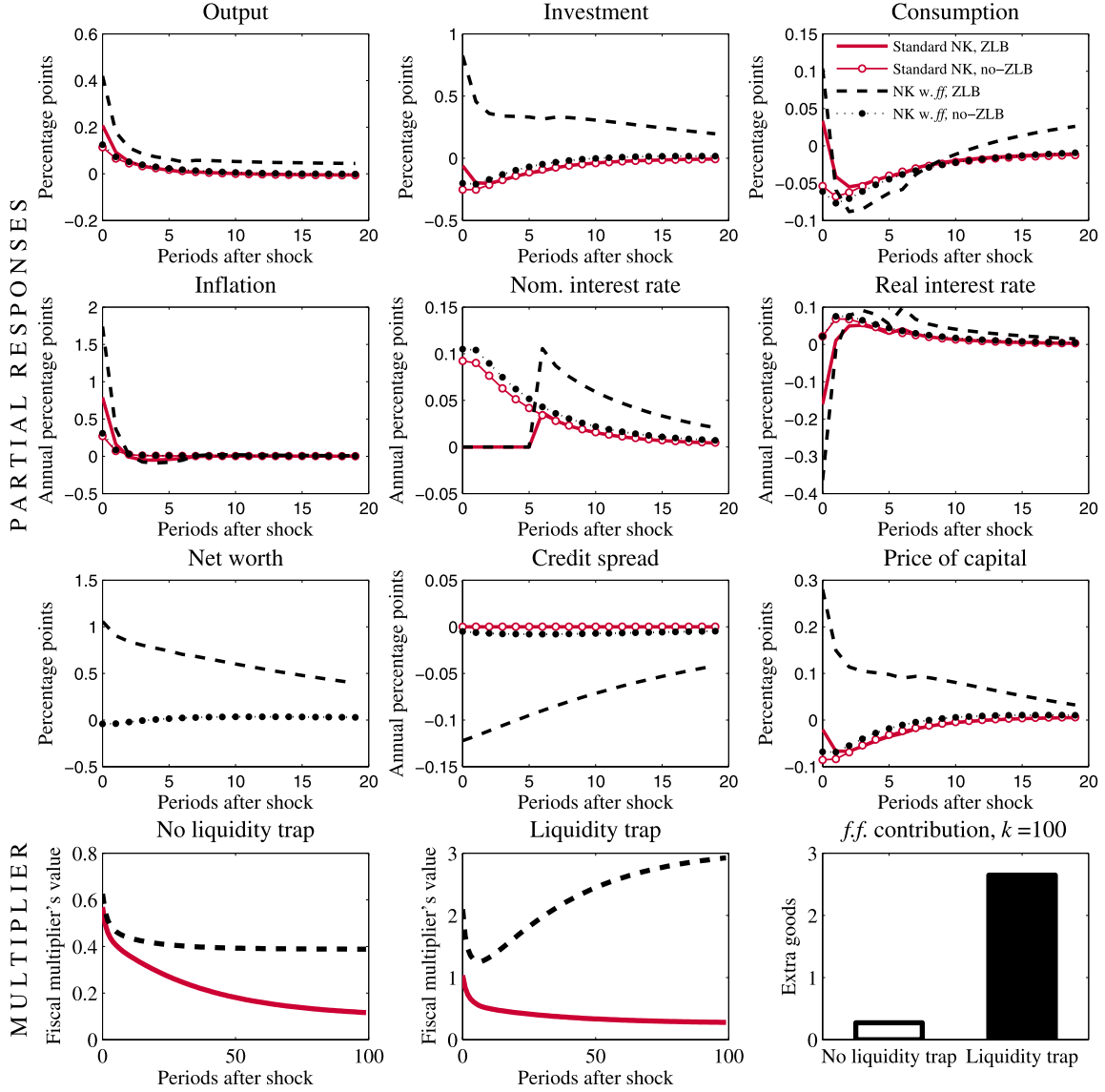


Figure 2: Weaker degree of price rigidities.

4.3 Variable-Elasticity Production Function

We assume that the final good y_t is produced via the variable-elasticity production function (Kimball, 1995) – instead of a CES production function given by Equation (1.22) – such that

$$\int_0^1 G\left(\frac{y_{j,t}}{y_t}\right) dj = 1, \quad (4.1)$$

where the function $G(\cdot)$ is increasing, strictly concave, and satisfies the normalization $G(1) = 1$. This assumption reinforces the degree of strategic complementarity in price-setting decisions. A final good firm chooses $\check{y}_{j,t}$ in order to maximize its profits

$$\max P_t y_t - \int_0^1 P_{j,t} y_{j,t} dj,$$

subject to the function production (4.1). Solving the optimization program yields the overall demand addressed to the producer of intermediate good who chooses $\check{y}_{j,t}$:

$$G'\left(\frac{\check{y}_{j,t}}{y_t}\right) = \frac{\check{P}_{j,t}}{P_t} \int_0^1 \frac{y_{j,t}}{y_t} G'\left(\frac{y_{j,t}}{y_t}\right) dj. \quad (4.2)$$

The aggregate price level is implicitly defined by

$$\int_0^1 P_{j,t} y_{j,t} dj = P_t y_t. \quad (4.3)$$

Let $\theta_p(\xi_t)$ denote the elasticity of demand for a producer of intermediate good facing the relative demand $\xi_t = y_{j,t}/y_t$. According to the implicit demand function (4.1), $\theta_p(\xi_t)$ obeys

$$\theta_p(\xi_t) = -\frac{G'(\xi_t)}{\xi_t G''(\xi_t)}.$$

This last equation illustrates that intermediate good firms face a varying elasticity of demand for their output, implying a time varying markup, which is denoted by $\mu_p(\xi_t)$, and obeys

$$\mu_p(\xi_t) = \frac{\theta_p(\xi_t)}{\theta_p(\xi_t) - 1}.$$

Further computations leads to a New Keynesian Philips Curve

$$\hat{\pi}_t = \kappa_p \hat{s}_t + \beta E_t \{\hat{\pi}_{t+1}\}, \quad (4.4)$$

where the slope κ_p is now defined by

$$\kappa_p = \frac{(1 - \alpha_p)(1 - \beta\alpha_p)}{\alpha_p(1 + \theta_p\epsilon_\mu)}, \quad \text{with } \epsilon_\mu \equiv \frac{\mu'_p(1)}{\mu_p(1)}.$$

We calibrate $\epsilon_\mu = 0.15$, meaning that a 2% increase in relative prices results in a 24% decline in demand. This value is consistent with Eichenbaum and Fisher (2004).

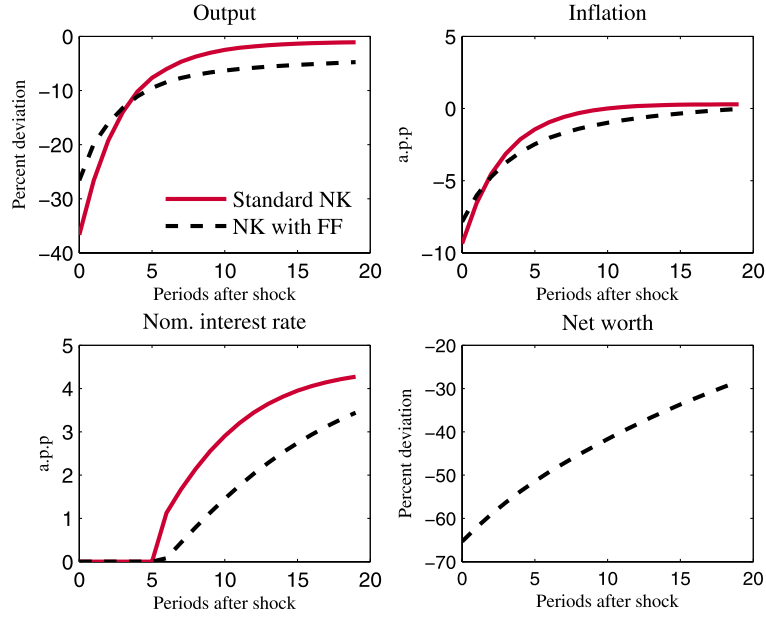


Figure 3: Variable-elasticity production function: recessionary shock.

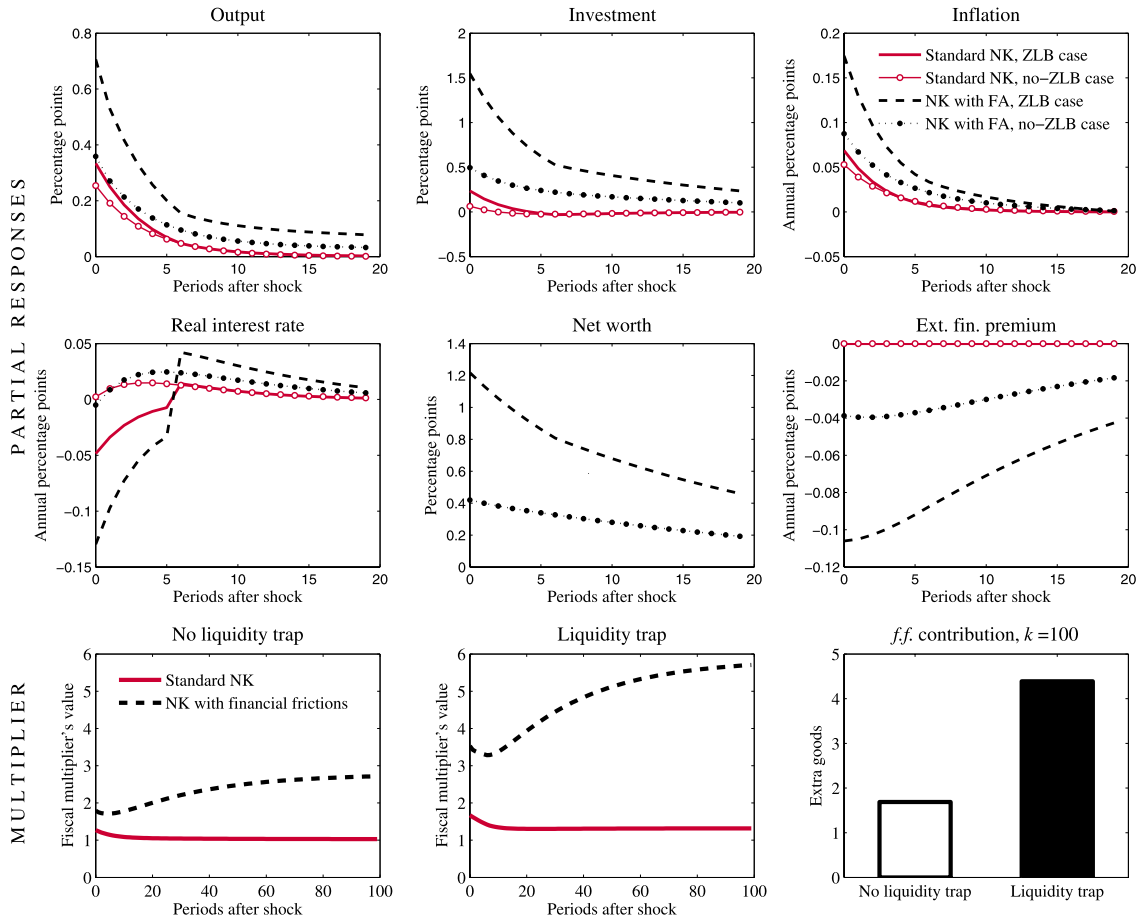


Figure 4: Variable-elasticity production function: Partial IRFs.

4.4 Investment Adjustment Costs

We adopt the investment-adjustment costs specification, as in Equations (1.65) and (1.66). Following CER, we set $\varkappa$ to 5. Since investment is more persistent, the liquidity trap lasts for 9 quarters. Notice that in the *NK with f.f* model, it starts at quarter 1 instead of quarter 0. The strong persistence of investment strengthens the capital accumulation channel, such that the *f.f* contribution is larger than in the benchmark case.

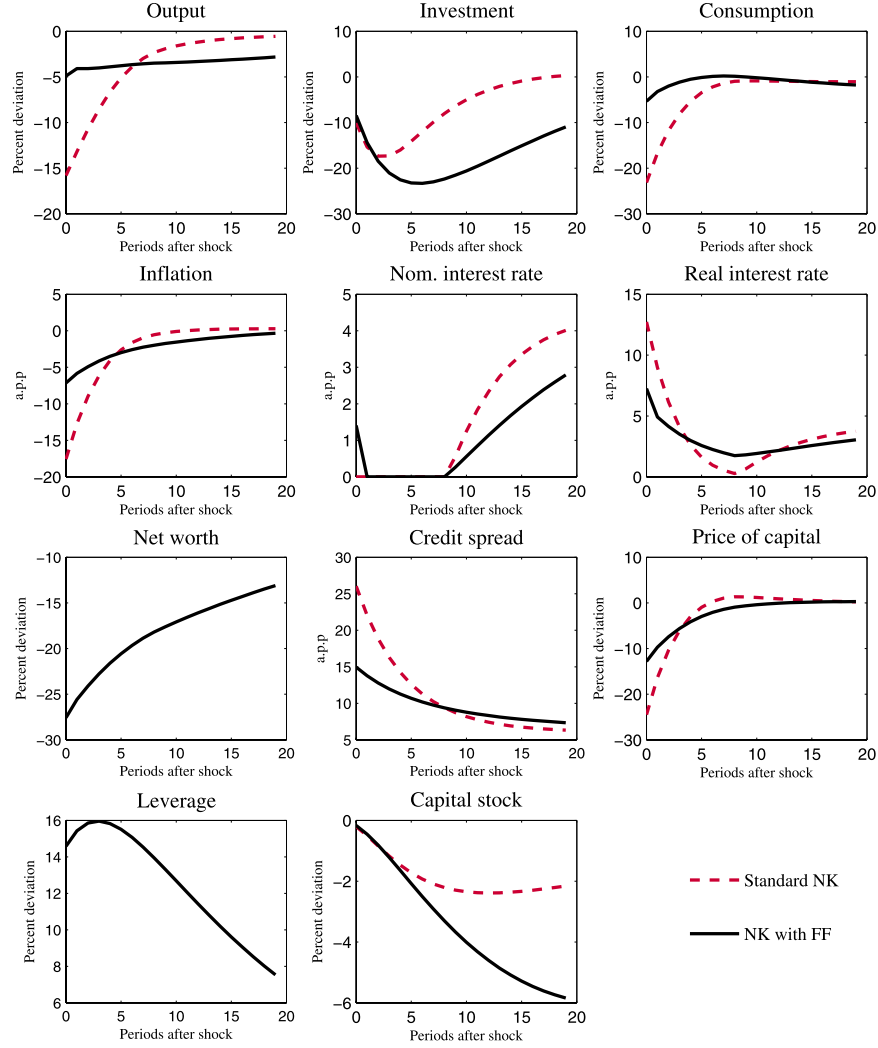


Figure 5: Investment adjustment cost: recessionary shock.

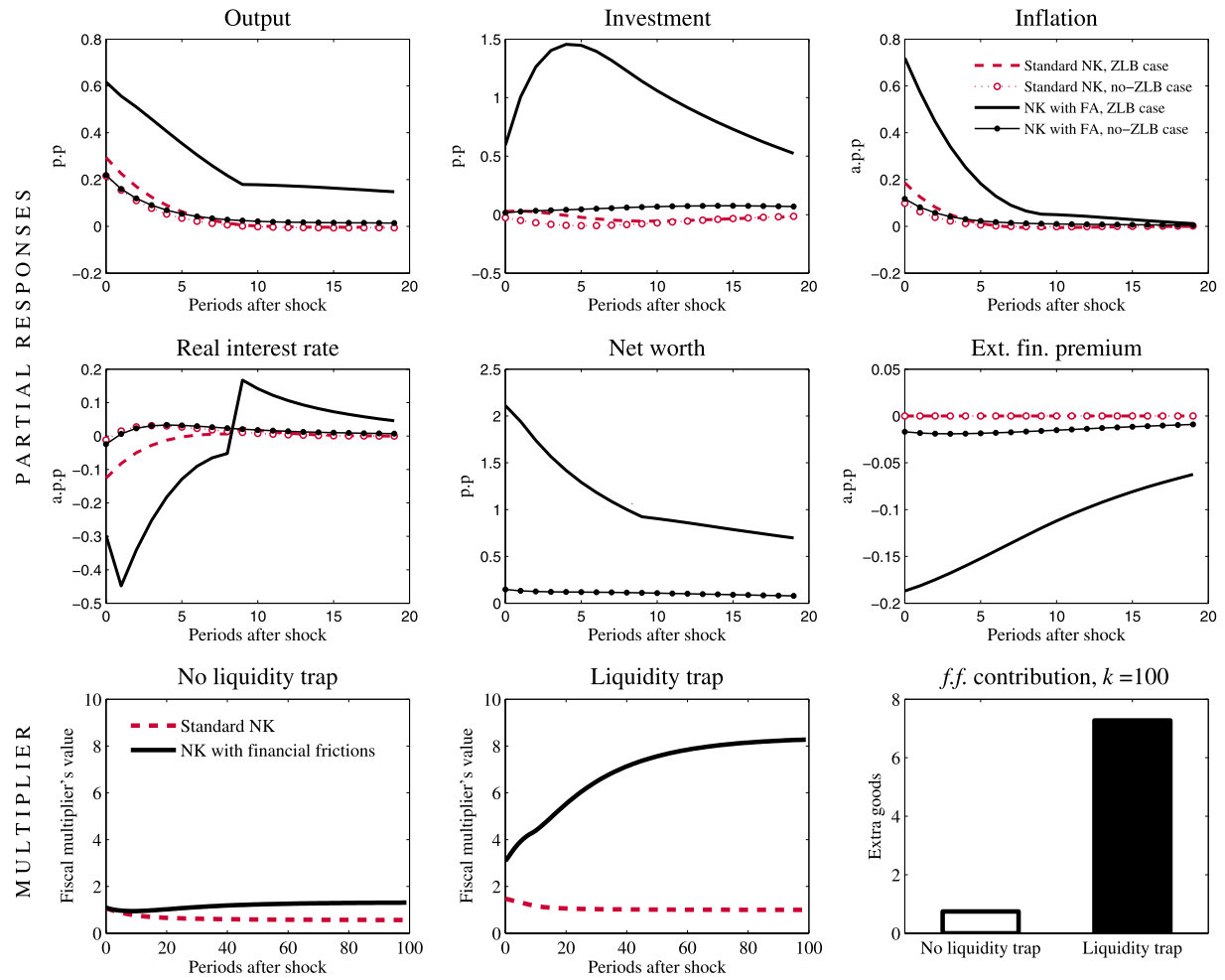


Figure 6: Investment adjustment cost: partial IRFs.

4.5 Alternative financial sector calibration

We assume a higher risk premium at the s.s., reaching 4 percent per year, instead than 2 percent. As a consequence, the leverage elasticity of the external finance, χ , doubles, reaching 0.08 (a value closed to the estimate of Gilchrist and Zakrajšek, 2011).

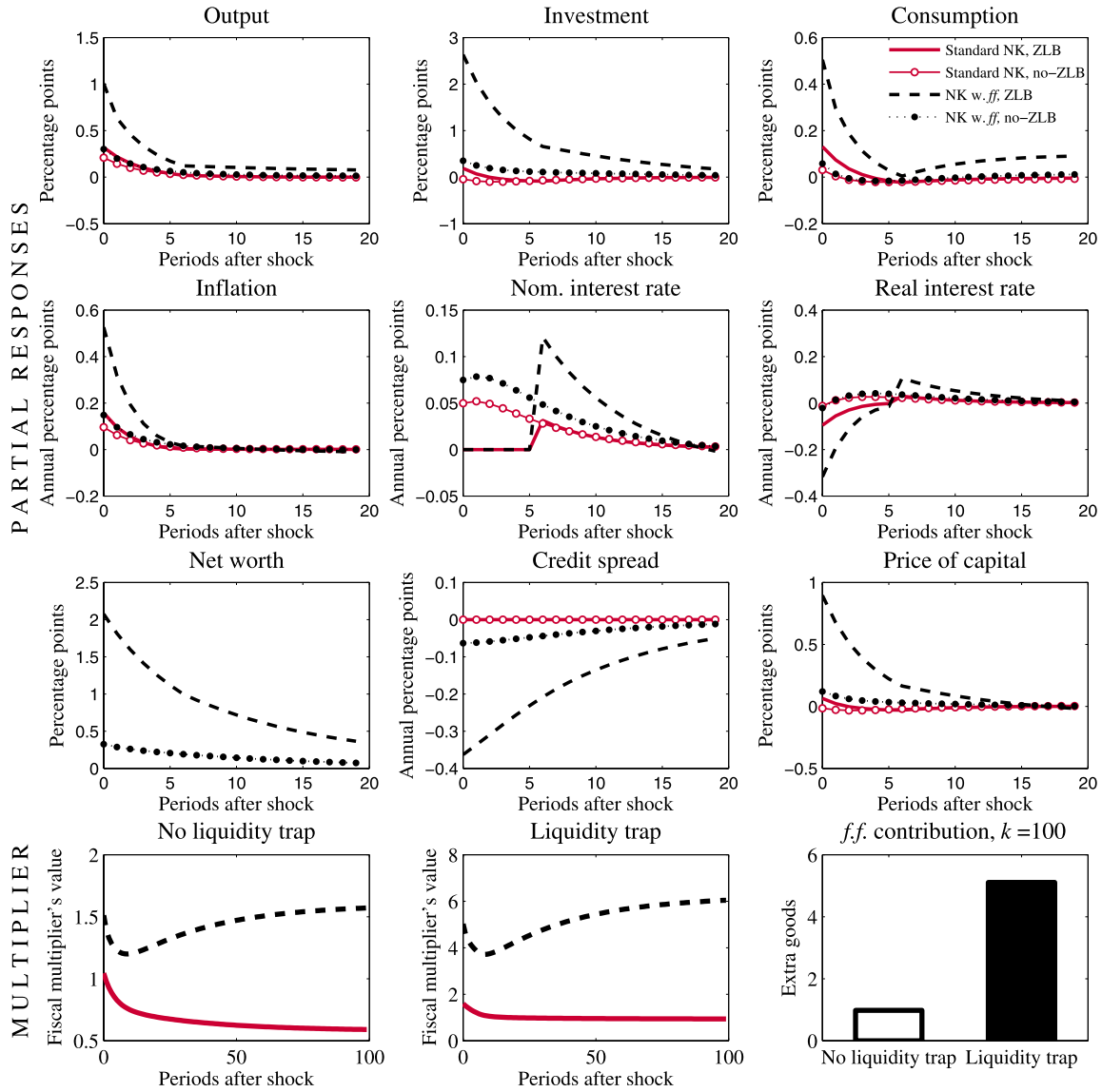


Figure 7: Alternative financial sector calibration.

4.6 Size of the Recessional Shock

We now assume that the two model variants are hit by a premium shock of the same size. Consequently, due to the presence of financial frictions, the duration of the liquidity trap will be different across the two models. We reduce the size of the shock to obtain 3 quarters under the ZLB regime in standard NK, and 11 quarters in NK with *f.f.* (a larger shock would make harder to find a determinate solution). The *f.f.* contribution during a ZLB is significantly larger.

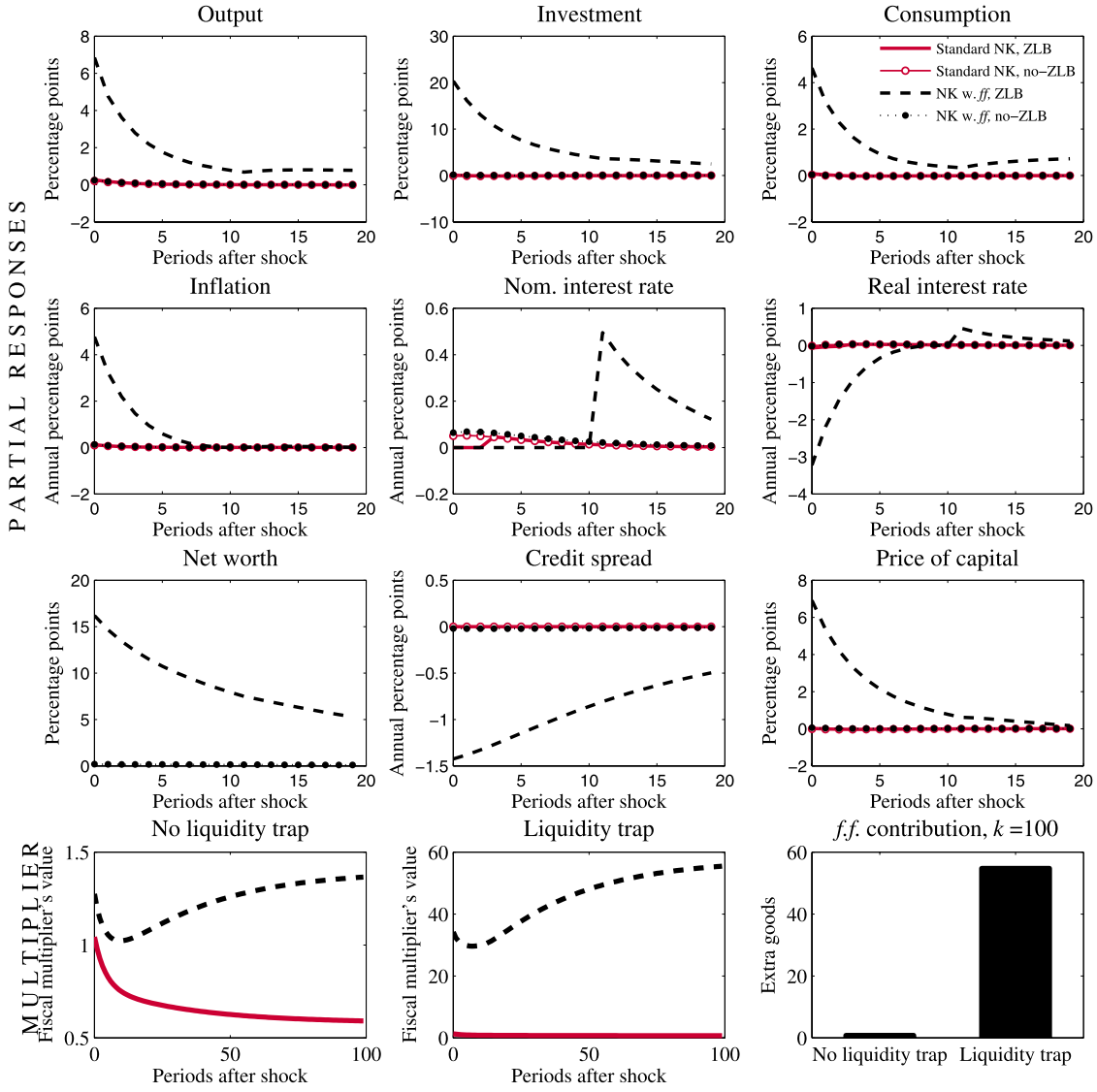


Figure 8: Same size of the recessional shock in both models.

4.7 Long-run multipliers for different model's specifications

This is the summary of long-run present-value multipliers and *f.f.* contribution for different exercises and modeling specifications.

Table 1. Long-run multipliers and output gains for different specifications

	No-liquidity trap regime			Liquidity trap regime		
	Present-value mult.			Present-value mult.		
	NK model	NK w. f.f.	<i>f.f.</i> contrib.	NK model	NK w. f.f.	<i>f.f.</i> contrib.
Benchmark	0.59	1.37	0.77	0.94	3.74	2.80
Full depreciation	1.30	1.74	0.44	1.46	2.01	0.54
No-Fisher effect	0.59	0.68	0.09	0.94	1.50	0.57
Monetary policy rule						
(1): Low inertia	0.63	1.21	0.56	1.65	17.96	16.31
(2): Inflation-targeting	0.10	-0.03	-0.13	0.99	3.05	2.07
(3): No response to output	0.57	1.38	0.81	0.85	3.70	2.85
Model structure						
(4): Separable preferences	0.44	1.01	0.57	0.70	2.57	1.87
(5): Weaker price rigidities	0.12	0.39	0.27	0.28	2.92	2.65
(6): Investment adjustment cost	0.57	1.31	0.74	0.78	3.71	2.93
(7): Different financial calibration	0.59	1.57	0.98	0.94	6.04	5.11
(8): Same size of the shock	0.59	1.37	0.77	0.70	55.47	54.76

Note: Long-run multipliers and *f.f.* contributions are computed at horizon 100.

4.8 Comparison with Christiano, Eichenbaum, and Rebelo (2011)

Christiano, Eichenbaum, and Rebelo (2011, hereafter CER) have a different strategy to generate a liquidity trap. They assume that the rate at which agents discount the future temporarily increases. Since agents value more future consumption, savings increase and current consumption abruptly falls. The nominal interest rate is thus pushed downwards, making the ZLB constraint binding. In their model with capital, which displays capital adjustment costs, the discount factor increases for 10 quarters, while the ZLB binds for 6. In their benchmark exercise, they assume that government spending rises only during the presence of the liquidity trap (likewise a brick-shaped pattern); then, they evaluate the impact spending multiplier for periods 1 to 6.

The standard NK model is equivalent to the model used by CER; we can thus easily compare our results to theirs. Two important clarifying points are worth mentioning. First, CER uses a simple monetary rule with no inertia nor response to output; their calibration is $\rho i = a_{\Delta y} = 0$, and $a_{\pi} = 1.5$. Second, the discount factor is qualitatively different than the premium shock: the former appears in the Euler equation of households (eq 1.4) but not in the definition of capital returns (so $\tilde{R}_t^k = R_t^k$), as the premium shock does. In sum, a discount factor shock does not generate a credit spread in the standard NK model, and does not guarantee a co-movement between investment and consumption. Figure 9 reproduces CER recession (figure 4 in their paper) and compares it with the NK with *f.f.* counterpart. The discount factor shock is smaller in the NK with *f.f.* in order to make the duration of ZLB comparable across models. The impact multipliers are displayed in the bottom row, center panel of the figure. While CER's multiplier is roughly 4, the NK with *f.f.*'s multiplier reaches 12, implying a significant contribution of credit frictions to the multiplier's size.

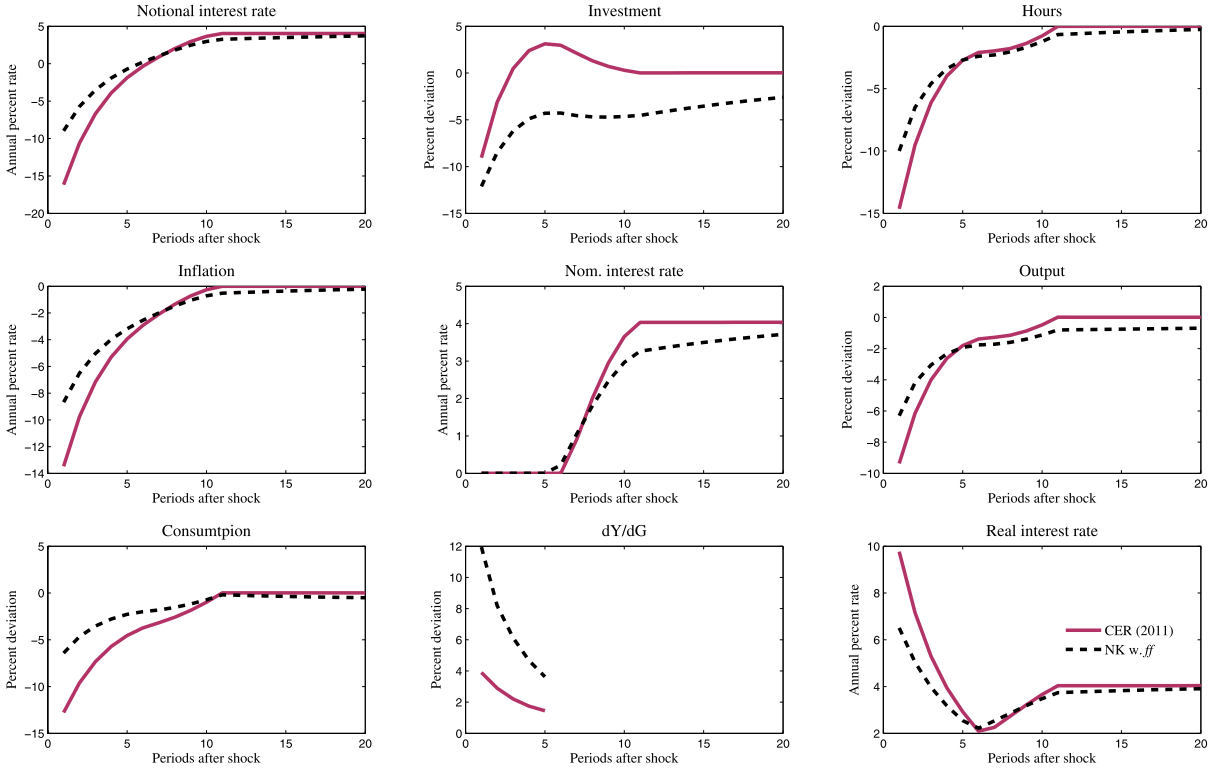


Figure 9: Partial IRFs to a government spending shock in CER paper: model with and without financial frictions.

Finally, we present the partial impulses along with the present-value multipliers of the previous exercise.

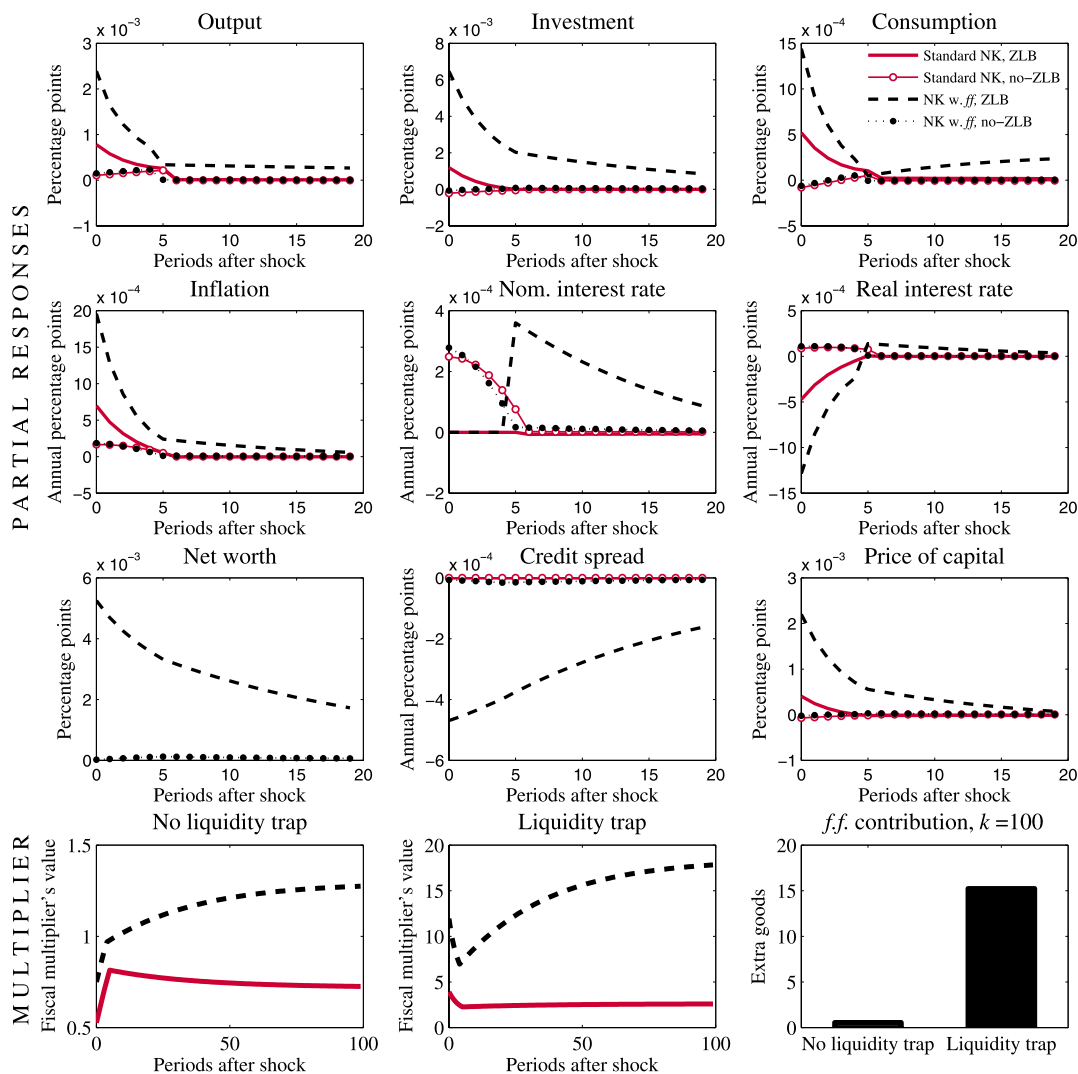


Figure 10: Comparison with CER