Bargaining with Commitment between Workers and Large Firms: Online Appendix

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This document provides supplementary material for the paper 'Bargaining with Commitment between Workers and Large Firms.'

Hyperlinks from this document to references (for example, equation numbers and statements of Propositions) which are found in the main paper will work properly only if the main paper file

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is placed in the same directory as this document.

This document has five sections.

Appendix A provides a more formal justification for the form of the Hamilton-Jacobi-Bellman equation for the surplus, which was formulated directly in recursive form in equation (3) in the main paper.

Appendix B provides an expanded discussion of the cross-sectional results described in Section 3 of the paper, particularly in Proposition 2 and the following discussion. It demonstrates that, if the production and vacancy-posting cost functions are quadratic, then it is possible that the distribution of firms by employment is Pareto and that Gibrat's law holds at least for large firms.

Appendix C gives more detail on the alternative contracting environments discussed at the end of Section 3 of the paper, following Proposition 3. As part of this discussion, Section C.3 describes the construction of an extensive-form microfoundation for the bargaining outcome so described.

Appendix D discusses welfare-enhancing tax and subsidy policies when the planner cannot observe firm productivity z directly, and so must condition hiring taxes and subsidies on easier-to-observe firm characteristics such as size and age.

Appendix E discusses a generalization of the bargaining equation SS, in which the bargaining power of a worker is allowed to depend on two key characteristics of the firm with which she is negotiating. The two characteristics are n, the number of incumbent employees, and z, the firm's productivity. If $\beta(n+1;z)$ depends on n and z in a very specific way, then constrained efficiency of the equilibrium can obtain; however, there is no reason *ex ante* to expect that this will occur. The condition for efficiency is a generalized version of the Hosios condition, but depends not only on the unemployment elasticity of the matching function but also on the production function, the vacancy-posting cost function, and other parameters of the model.

A The Hamilton-Jacobi-Bellman Equation for the Surplus

In the main text, it was stated without further justification that the surplus satisfies the HJB equation (3), repeated here for convenience:

$$\begin{aligned} rS(n;z) &= y(n;z) - nrV^u - c(v(n;z)) \\ &\qquad + qv(n;z)\left[S(n+1;z) - S(n;z) - (V(n+1;z) - V^u)\right] - \delta S(n;z). \end{aligned}$$

It is straightforward to derive this equation more formally. I give two possibilities here. The first argument involves stopping times.

First notice that the flow rate at which surplus is accrued in the absence of either a contact with an additional worker or the shutdown of the firm is equal to the output of the firm $y(n;z)$ less the sum of the incumbent workers' flow opportunity costs, nrV^u, and less the flow vacancy posting cost $c(v(n;z))$. Denote this by

$$\varphi(n;z) \equiv y(n;z) - nrV^u - c(v(n;z)).$$

Second, let T be the stopping time defined by the first arrival time either of a successful contact with a new worker by the firm, or the destruction of the firm. Since each of these events occur according to a Poisson process, respectively with parameters $qv(n;z)$ and δ, the arrival time of the first of the two to occur is exponentially distributed with parameter $qv(n;z)+\delta$; with probability $qv(n;z)/(qv(n;z)+\delta)$, the first event to occur is that a worker is contacted, while with complementary probability $\delta/(qv(n;z)+\delta)$, the firm is destroyed before a worker is contacted. In the first case, the new value of the surplus accruing to the firm and the n incumbent workers is $S(n+1;z)$, less the payment in excess of the outside option to the new hire, $V(n+1;z)-V^u$. Denote this value by

$$S'(n;z) = S(n+1;z) - [V(n+1;z) - V^u]$$

In the second case, the new value of the surplus is zero.

It follows that

$$S(n;z) = \int_0^\infty I(T)(qv(n;z)+\delta)e^{-(qv(n;z)+\delta)T}\,dT \tag{A.1}$$

where

$$\begin{aligned} I(T) &= \int_0^T e^{-rt}\varphi(n;z)\,dt + e^{-rT}\frac{qv(n;z)}{qv(n;z)+\delta}S'(n;z) \\ &= \frac{1}{r}\left[1-e^{-rT}\right]\varphi(n;z) + e^{-rT}\frac{qv(n;z)}{qv(n;z)+\delta}S'(n;z) \end{aligned}$$

is the expected value of the surplus conditional on the stopping time T. Substitute into (A.1) to see that

$$S(n;z) = \frac{1}{r}\left[1-\frac{qv(n;z)+\delta}{r+qv(n;z)+\delta}\right]\varphi(n;z) + \frac{qv(n;z)}{r+qv(n;z)+\delta}S'(n;z)$$

which can be rearranged to establish (3).

An alternative justification for (3) can be obtained by considering a discrete-time approximation to the HJB equation. Consider a small time interval $\Delta t > 0$. In this the time interval, the probability that the firm hires precisely one additional worker is $qv(n;z)\Delta t + o(\Delta t)$, where, as usual, $o(x)$ denotes terms which converge to zero faster than x as $x \to 0$. The form of this expression arises because the firm posts $v(n;z)$ vacancies if it has n workers and productivity z; this delivers a Poisson arrival rate of contacts with workers of $qv(n;z)$.) The continuation value of the firm in this event is $S'(n;z)$; this has present discounted value $e^{-r\Delta t}S'(n;z) = (1-r\Delta t)S'(n;z) + o(\Delta t)$.

Next, the probability that the firm is destroyed in this time interval is $\delta\Delta t + o(\Delta t)$, since destruction occurs at Poisson rate δ. In this case the continuation value is zero.

Third, notice that the probability of contacting two or more workers is $O(\Delta t^2)$ (since $v(n+1; z)$ is bounded above by $v(1, \bar{z}) < \infty$) and therefore $o(\Delta t)$. Similarly, the probability that the firm both contacts one or more new workers and also is hit by the job destruction shock is also $O(\Delta t^2)$.

It follows that the probability that the firm neither hires nor is destroyed during the time interval under consideration is $1 - (qv(n; z) + \delta)\Delta t + o(\Delta t)$. In this case it earns flow surplus $\varphi(n; z)$ for the duration of the interval, and this has present discounted value

$$\int_0^{\Delta t} \varphi(n; z) e^{-rt}\, dt = \frac{1}{r}\left[1 - e^{-r\Delta t}\right] \varphi(n; z) = \varphi(n; z)\Delta t + o(\Delta t).$$

The continuation value of the firm in this case is again $S(n; z)$ since its state is unchanged; this has present discounted value $e^{-r\Delta t} S(n; z) = (1 - r\Delta t)S(n; z) + o(\Delta t)$.

It follows from all of this that the surplus $S(n; z)$ satisfies

$$\begin{aligned} S(n; z) &= [1 - (qv(n; z) + \delta)\Delta t]\, [\varphi(n; z)\Delta t + (1 - r\Delta t)S(n; z)] \\ &\qquad\qquad\qquad + qv(n; z)\Delta t(1 - r\Delta t)S'(n; z) + o(\Delta t) \\ &= S(n; z) + \varphi(n; z)\Delta t - (r + qv(n; z) + \delta)S(n; z)\Delta t + qv(n; z)S'(n; z)\Delta t + o(\Delta t) \end{aligned}$$

Cancel $S(n; z)$ from both sides, divide by Δt, drop terms which are $o(\Delta t)/\Delta t = o(1)$, and rearrange to establish (3).

Similar justifications can be given for other HJB equations which are written directly in recursive form in the paper. In particular, this applies to equations (6), (7), and (10), which characterize respectively the values of an employed worker, a firm, and an unemployed worker.

B Proofs of Claims about Cross-Sectional Implications

In this section I characterize the cross-sectional implications of the model of bargaining with commitment. The results and arguments are closely analogous to those used by Acemoglu and Hawkins (in press) to prove similar results for the case of continuously bargained wages.

The two key equations of the model are the HJB equation for the surplus, equation (A.1) in the main paper, and the equation characterizing vacancy posting under PER and SS, that is, equation (A.2) in the main paper. These are repeated here for convenience.

$$(r + \delta)S(n; z) = y(n; z) - nrV^u - c(v(n; z)) + qv(n; z)(1 - \beta)[S(n + 1; z) - S(n; z)], \tag{B.1}$$

$$v(n; z) = \arg\max_{v \geq 0} \left\{-c(v) + qv(1 - \beta)\left[S(n + 1; z) - S(n; z)\right]\right\}. \tag{B.2}$$

Characterizing the cross-sectional properties is more straightforward in a continuous-employment version of the model., as discussed in footnote 8 of the main paper. In this setting, (B.1) and (B.2) become

$$(r+\delta)S(n;z) = y(n;z) - nrV^u - c(v(n;z)) + qv(n;z)(1-\beta)\frac{\partial}{\partial n}S(n;z), \tag{B.3}$$

$$v(n;z) = \arg\max_{v\geq 0}\left\{-c(v) + qv(1-\beta)\frac{\partial}{\partial n}S(n;z)\right\}. \tag{B.4}$$

It is easiest to proceed by specifying a parametric functional form for the production function and the cost of posting vacancies. Accordingly, in the remainder of this section, I assume that the production function is

$$y(n;z) = zn - \frac{1}{2}\sigma n^2, \tag{B.5}$$

where $\sigma > 0$ is constant across firms. I also assume that the vacancy-posting cost function takes the form

$$c(v) = \frac{1}{2}\gamma v^2. \tag{B.6}$$

This allows establishing closed-form solutions for the surplus $S(\cdot)$ and for vacancy-posting $v(\cdot)$. I guess and verify that there is a quadratic solution to (B.3) and (B.4) for $S(\cdot)$, of the form

$$S(n;z) = A(z) + B(z)n - \frac{1}{2}Cn^2,$$

and solve for the unknown coefficients. Note that C does not depend on z.

Proposition B.1. *Under the quadratic functional forms in (B.5) and (B.6), the maximum size of a firm with productivity z is*

$$n^*(z) = \frac{z - rV^u}{\sigma}. \tag{B.7}$$

The surplus is given for $0 \leq n \leq n^*(z)$ by

$$S(n;z) = A(z) + B(z)n - \frac{1}{2}Cn^2 \tag{B.8}$$

where

$$\begin{aligned}
C &= \frac{2\sigma}{r+\delta+\sqrt{(r+\delta)^2 + 4q^2(1-\beta)^2\frac{\sigma}{\gamma}}} \\
B(z) &= \frac{2(z-rV^u)}{r+\delta+\sqrt{(r+\delta)^2 + 4q^2(1-\beta)^2\frac{\sigma}{\gamma}}}, \\
A(z) &= \frac{q^2(1-\beta)^2 B(z)^2}{2\gamma(r+\delta)}.
\end{aligned}$$

Vacancy posting is

$$v(n;z) = \begin{cases} \lambda(n^*(z) - n) & 0 \leq n \leq n^*(z) \\ 0 & n > n^*(z), \end{cases} \tag{B.9}$$

where λ is the unique positive solution to

$$\lambda^2 + \frac{r+\delta}{q(1-\beta)}\lambda - \frac{\sigma}{\gamma} = 0.$$

Proof. Given the guessed functional form for $S(\cdot)$, it is immediate from the first-order condition in (B.6) that $v(n;z) = q(1-\beta)(B(z) - Cn)/\gamma$. Substitute into the HJB equation and equate coefficients on the constant, linear, and quadratic terms in n to verify that the claimed functional form is the unique solution for $S(\cdot)$. Substitute into the equation for $v(n;z)$ just given to verify the claims about $v(\cdot)$. A useful identity in verifying the form of λ is that

$$C = \frac{\gamma}{2q^2(1-\beta)^2}\left(-(r+\delta) + \sqrt{(r+\delta)^2 + 4q^2(1-\beta)^2\frac{\sigma}{\gamma}}\right).$$

□

Next, observe that conditional on survival the growth rate of an individual firm is $qv(n;z)$. The Poisson death rate of the firm is δ. Firms of productivity z are born (with zero workers) at rate $\phi(z)e$, where e is the aggregate rate of entry. Denote by $g(n;z)$ the density of firms with productivity z by firm size. The Liouville equation governing the density of firm size among firms with productivity z given the process just described is

$$0 = \frac{\partial}{\partial t} g\left(n;z\right) = -\frac{\partial}{\partial n}\left[qv(n;z)g(n;z)\right] - \delta g(n;z) + \delta j(n)$$

where $j(\cdot)$ is the indicator function taking the value 1 if $n=0$ and 0 otherwise, and the coefficient δ multiplying it arises from flow balance (since a fraction δ of firms exit per unit of time, the same fraction must enter). This is an ordinary differential equation for $g(n;z)$. After substituting for $v(n;z)$ using (B.9), this differential equation takes the form

$$\frac{\frac{\partial}{\partial n}g(n;z)}{g(n;z)} = \left[1 - \frac{\delta}{q\lambda}\right]\frac{1}{n^*(z) - n}.$$

Integrating once with respect to n establishes that

$$g(n;z) = D(z)(n^*(z) - n)^{\frac{\delta}{q\lambda}-1}, \tag{B.10}$$

where

$$D(z) = \frac{\delta}{q\lambda} n^*(z)^{-\frac{\delta}{q\lambda}}$$

is constant with respect to n. Integrating a second time shows that the fraction of firms with productivity z with employment exceeding n is

$$\bar{G}(n;z) = \begin{cases} \left[1 - \frac{n}{n^*(z)}\right]^{\frac{\delta}{q\lambda}} & n < n^*(z) \\ 0 & n \geq n^*(z). \end{cases}$$

This preliminary material allows me to show that, given enough heterogeneity in productivity z, the model of privately efficient bargaining can be consistent with Zipf's law.

Proposition B.2. *In steady-state equilibrium, if the productivity distribution $\Phi(\cdot)$ is Pareto, then the firm size distribution has a Pareto tail with the same shape parameter.*

The proof of this result, which makes use of the preceding expression for $\bar{G}(n;z)$, is identical to that of Proposition 3 in Acemoglu and Hawkins (in press), and is therefore omitted.

I can also establish the following result, showing that given enough heterogeneity in productivity, then the cross-sectional pattern of firm growth is consistent with Gibrat's law, at least for the largest firms.

Proposition B.3. *In steady-state equilibrium, if the productivity distribution $\Phi(\cdot)$ is Pareto, then Gibrat's law holds for large firms.*

Proof. The proof is very similar to—but algebraically simpler than—that of Proposition 4 in Acemoglu and Hawkins (in press).

Denote by $\zeta(n;z) = qv(n;z)/n$ the proportional growth rate of a firm with productivity z and n employees. Using (B.9), it follows that

$$\zeta(n;z) = \frac{qv(n;z)}{n} = \frac{q\lambda}{n}(n^*(z) - n).$$

To establish Gibrat's law, it is useful to change variables from z to ζ. This can be done because conditional on n, there is a bijective relationship between z and ζ. To see why, substitute from (B.7) into (B.9) to see that

$$\zeta = \frac{q\lambda}{n}\left[\frac{z - rV^u}{\sigma} - n\right],$$

so that

$$z = rV^u + \frac{\sigma}{q\lambda}(\zeta + q\lambda)n. \tag{B.11}$$

This implies that holding n constant, we have that the value of z consistent with growth rate ζ satisfies

$$\frac{\partial z}{\partial \zeta} = \frac{\sigma}{q\lambda}n. \tag{B.12}$$

Among firms with productivity z, the density over firm size is given by (B.10). Substitute to eliminate $n^*(z)$ in terms of n and ζ, to see that this density can be written

$$g(n;z) = \delta n^{-1}\zeta^{\frac{\delta}{q\lambda}-1}(\zeta+q\lambda)^{-\frac{\delta}{q\lambda}} \tag{B.13}$$

Finally, the density of z can be written under the Pareto distributional assumption as

$$\begin{aligned} f(z) &= \kappa z_m^\kappa z^{-\kappa-1} \\ &= \kappa z_m^\kappa \left(rV^u + \frac{\sigma}{q\lambda}(\zeta+q\lambda)n\right)^{-\kappa-1} \\ &= \kappa z_m^\kappa \left(\frac{q\lambda}{\sigma}\right)^{\theta+1} (\zeta+q\lambda)^{-(\kappa+1)}\, n^{-(\kappa+1)} \left(1+\frac{rV^u}{\frac{\sigma}{q\lambda}(\zeta+q\lambda)n}\right)^{-(\kappa+1)}. \end{aligned} \tag{B.14}$$

Combining (B.12), (B.13), and (B.14), we obtain that the density of firms with size n and growth rate ζ can be written

$$\begin{aligned} & g(n;z)\cdot f(z)\cdot\left(\frac{\partial z}{\partial \zeta}\right)^{-1} \\ &= \delta\kappa z_m^\kappa \left(\frac{q\lambda}{\sigma}\right)^{\kappa+2} n^{-(\kappa+3)}\zeta^{-1+\frac{\delta}{q\lambda}}\,(\zeta+q\lambda)^{-(\kappa+1)-\frac{\delta}{q\lambda}} \left(1+\frac{rV^u}{\frac{\sigma}{q\lambda}(\zeta+q\lambda)n}\right)^{-(\kappa+1)}. \end{aligned}$$

The right side of this equation is the product of a function of n, a function of ζ, and an expression that converges to 1 uniformly in ζ as $n \to +\infty$. It follows that Gibrat's law holds asymptotically.

□

In summary, the cross-sectional features of equilibrium under bargaining over privately efficient contracts are similar to those found by Acemoglu and Hawkins (in press) for bargaining over current wages alone.

C Alternative Contracting Environments

As discussed near the end of Section 3 of the paper, there are many possible timings of payments within an employment contract that are consistent with PER and SS. In the paper, I discussed in detail one particular arrangement, using contracts under which a worker is paid a wage which is constant within an employment relationship. I also briefly introduced two alternatives, one using signing bonuses and one using transfers made to incumbent workers when a new worker is hired. In this section I discuss both of these in more detail. Section C.1 discusses the signing bonus model, and Section C.2 the model using transfers on replacement hiring. Finally, in Section C.3, I discuss a non-cooperative microfoundation for the bargaining game.

C.1 Signing Bonuses

A particularly straightforward model of bargaining that is consistent with privately efficient recruiting can be constructed using what Shimer (1996) calls 'simple contracts.' Assume that when a worker is hired by a firm, the two of them sign a contract that promises the worker two types of payments: first, a one-time 'signing bonus,' and second, an ongoing flow wage. The signing bonus is a one-time payment $p(n+1;z)$ made immediately when the worker is hired. Its value depends on the firm's productivity z and on the size of its existing workforce. The flow wage is equal to the flow value of the worker's outside option rV^u, which makes the worker indifferent between remaining employed at the firm and quitting to unemployment.

In this environment, the HJB equation for the value of a firm with productivity z and with n incumbent employees (who received their signing bonuses in the past) takes the form

$$(r+\delta)J(n;z) = y(n;z) - nrV^u - c(v(n;z)) + qv(n;z)\left[J(n+1;z) - J(n;z) - p(n+1;z)\right]. \quad \text{(C.1)}$$

The intuition for this equation is similar to that for (3) in the main paper. The capital gain term in brackets incorporates the fact that when the firm hires, it must pay the signing bonus to the worker.

The firm chooses its vacancy posting intensity to maximize its expected profit, that is,

$$v(n;z) = \arg\max_v \left\{-c(v) + qv\left[J(n+1;z) - J(n;z) - p(n+1;z)\right]\right\}. \quad \text{(C.2)}$$

Completing the description of this bargaining model requires explaining how the signing bonus paid to a new hire is determined. Consistent with SS, assume that the worker's gain from the match is a fraction β of the (marginal) surplus associated with the match. Because the ongoing flow payment received by the worker when employed has the same value as her flow outside option, this gain is simply equal to the signing bonus $p(n+1;z)$.[1] I can therefore write the condition that a newly hired worker and her new employer share the marginal gain associated with their match in the ratio $\beta : (1-\beta)$ as

$$(1-\beta)p(n+1;z) = \beta\left[J(n+1;z) - J(n;z) - p(n+1;z)\right],$$

or equivalently

$$p(n+1;z) = V(n+1;z) - V^u = \beta\left[J(n+1;z) - J(n;z)\right]. \quad \text{(C.3)}$$

In this model, the firm receives all of the surplus arising from an ongoing employment relation-

[1] To see this more formally, write the HJB equation defining $E(n;z)$, the value of an incumbent worker who already received her signing bonus in the past, as

$$rE(n;z) = rV^u + qv(n;z)\left[E(n+1;z) - E(n;z)\right] + \delta\left[V^u - E(n;z)\right].$$

It is clear that the only solution to this equation is such that $E(n;z) = V^u$ for all $(n;z)$. Thus, the value of a new employment relationship with a firm with productivity z and n incumbent employees is simply $p(n+1;z) + V^u$, a gain of $p(n+1;z)$ relative to the outside option of unemployment.

ship, since it compensates incumbent workers for the opportunity cost of their time, and pays for vacancy posting and for the signing bonuses earned by new hires. Thus the firm's value $J(\cdot)$ coincides with the surplus $S(\cdot)$ defined by (3). It is straightforward to verify this using (C.1), (C.2), and (C.3). Because the firm chooses $v(\cdot)$ to maximize its own profits and because these coincide with the joint surplus belonging to the firm and its workers, (C.2) coincides with (4). Since (C.3) also coincides with (5), it is clear that the signing bonus model satisfies PER and SS, thus establishing existence.

Like the constant wage model, the signing bonus model does have drawbacks. The implications for wage dynamics within an employment relationship are counterfactual, since earnings are negatively correlated with an employee's tenure at the firm. A deeper issue arises because after she has been paid her signing bonus, a worker is indifferent between continuing working or quitting to unemployment. While this is consistent with employment-at-will in the sense that she cannot do better by walking away from the contract, it requires making the strong assumption that a worker who has just been matched with a vacancy has the bargaining power to extract compensation greater than the value of unemployment from the firm, but incumbent employees have none. This would apply if the worker can commit not to try to renegotiate her contract or if the firm can commit not to negotiate with incumbent workers, but these are not obviously attractive assumptions.

C.2 Bilaterally Renegotiation-Proof Contracts

Although the constant wage contracts studied in Section 3 of the paper do not give workers any reason to want to alter an existing contract, they still have the feature that the firm has an incentive to renegotiate with incumbent workers if it can bargain with them as if newly-hired, since this would reduce the wages of high-seniority workers. Since the firm is assumed to be able to commit to long-term contracts, this is not necessarily a problem. However, it is interesting that a stronger form of renegotiation-proofness can be achieved using a different contractual structure for earnings. In this section I show briefly how this is possible.

Assume that a firm make two kinds of payments to its employees. First, the firm pays a flow wage $w(n;z)$ to each worker. Second, it also makes a lump-sum transfer $t(n;z)$ to each incumbent worker when it makes a new hire. $w(n;z)$ and $t(n;z)$ depend only on the firm's productivity and its current employment level. The HJB equation defining the value of a firm with productivity z and n incumbent workers is

$$(r+\delta)J(n;z) = y(n;z) - nw(n;z) - c(v(n;z)) + qv(n;z)\left[-nt(n;z) + J(n+1;z) - J(n;z)\right], \quad \text{(C.4)}$$

while the value of an incumbent employee at such a firm is[2]

$$rV(n;z) = w(n;z) + \delta\left[V^u - V(n;z)\right] + qv(n;z)\left[t(n;z) + V(n+1;z) - V(n;z)\right]. \quad \text{(C.5)}$$

[2]Here $V(m;z)$ denotes the value of an incumbent worker who is currently one of m employees, and $w(m;z)$ the wage paid to such an employee; in the constant wage model $V(m;z)$ denoted the value of an incumbent worker who was the mth hired among a pool of at least m employees, and $w(m;z)$ the corresponding wage.

The firm chooses the intensity of vacancy posting to maximize profits, so that

$$v(n;z) = \arg\max_{v} \left[-c(v) + qv\left[-nt(n;z) + J(n+1;z) - J(n;z)\right]\right] \tag{C.6}$$

In deciding its intensity of vacancy posting, the firm takes into account the cost of the transfers it will need to make to its incumbent workers when it hires an $(n+1)$st worker.

As in the constant wage model, the firm does not earn the full surplus since workers are paid wages in excess of their outside option. The surplus in this model satisfies[3]

$$S(n;z) \equiv \left[J(n;z) - J(0;z)\right] + n\left[V(n;z) - V^u\right]. \tag{C.7}$$

Combining (C.4) and (C.5) shows that the surplus $S(\cdot)$ satisfies the familiar HJB equation (3).

It remains to specify how wages and transfers are determined. To determine the overall level of a worker's earnings, assume as before that a newly-hired worker enjoys a fraction β of the marginal surplus arising from the match, that is,

$$(1-\beta)\left[V(n+1;z) - V^u\right] = \beta\big[J(n+1;z) - J(n;z) + n[V(n+1;z) - V(n;z)]\big]. \tag{C.8}$$

It is immediate from the characterization of the surplus in (C.7) that SS holds.

Finally, to determine how a worker's lifetime earnings are split between wages and transfers and in order to ensure privately efficient recruiting, an additional assumption is needed. To motivate this, note that because of decreasing returns to labor, a worker's value $V(n;z)$ will be decreasing in n for given z. (This follows from Proposition 2 in Section 3 of the paper.) Thus, the worker experiences a capital loss $V(n+1;z) - V(n;z)$ when the firm hires an $(n+1)$st worker. I assume that transfers are determined in order to compensate incumbent workers for this loss, that is,

$$t(n;z) = V(n;z) - V(n+1;z). \tag{C.9}$$

This implies using (C.5) and SS that

$$w(n;z) = rV^u + (r+\delta)\left[V(n;z) - V^u\right] = rV^u + (r+\delta)\beta\left[S(n;z) - S(n-1;z)\right]. \tag{C.10}$$

The marginal value of hiring to the firm can then be written

$$\begin{aligned} -nt(n;z) + J(n+1;z) - J(n;z) &= J(n+1;z) - J(n;z) + n\left[V(n+1;z) - V(n;z)\right] \\ &= S(n+1;z) - S(n;z) - \left[V(n+1;z) - V^u\right] \end{aligned}$$

where the second equality follows from the definition of $S(\cdot)$ in (C.7). It is then immediate from (C.6) that PER is satisfied.

[3] The form of (C.7) differs from that of (3) because in the current model all n incumbent workers have the same value. The cost of payments to a newly hired worker is not accounted for at the time of hiring, but rather on a flow basis.

It is straightforward to see that the wage determination protocol just described is proof against worker-induced renegotiations in a stronger sense than the constant wage model described in the paper. Suppose that a worker is unexpectedly granted a one-time opportunity to tear up her contract with the firm and negotiate again as if newly matched. Then the contract she will be able to negotiate will deliver her exactly the same future wages and transfers as were guaranteed by her existing contract. This is because at any time after incumbent workers have received the lump-sum transfer promised in their contract at the moment an $(n+1)$st worker is hired, all the n incumbent workers receive thereafter the same value as does the new hire. (This value is given, according to SS, by $V(n+1;z) = V^u + \beta\left[S(n+1;z) - S(n;z)\right]$.) Thus, the worker has no incentive to initiate such a renegotiation. Further, because any new hire receives a value in excess of the value of unemployment, the contracts of this section are consistent with employment-at-will.[4]

More interestingly, the contracts studied in this section are in fact also proof against renegotiations induced by the firm. Suppose that the firm is unexpectedly granted a one-time opportunity to tear up its contract with one of its employees and negotiate again as if newly matched. Again, this contract will deliver the same payoffs to firm and worker as the original contract, so the firm has no incentive to initiate the renegotiation.[5]

The bargaining model introduced in this section assumes that firms can commit to make transfers to incumbent workers in order to compensate them for the capital loss they experience when an additional worker is hired, in order to remove the firm's incentive for excessive hiring. It is not clear whether such transfers are observed in practice. However, because the firm has commitment power, the transfers need not be delivered literally at the time the new worker is hired, but could instead be paid as higher future wages.[6] This is similar to how the constant wage model delivers newly-hired workers the same values as in the signing bonus model of Section C.1, by spreading out the signing bonus over the worker's entire employment spell at the firm.

C.3 Non-cooperative Extensive-Form Microfoundation

In this section I give details of the construction of a noncooperative microfoundation for bargaining over long-term contracts. The construction extends that of Binmore et al. (1986, hereafter BRW) to the multi-worker setting. For the sake of concreteness, I give the construction in the setting where firms and workers bargain over bilaterally renegotiation-proof contracts, as introduced in Section C.2 above.

The reason why it is useful to provide this construction is that one attractive feature of the Nash bargaining solution in the one-worker per firm benchmark and of the work of SZ is that for both models it is known that a non-cooperative extensive-form microfoundation can be constructed. This

[4] The proof of this statement is similar to that for the constant wage model.

[5] The concept of renegotiation-proofness specified here only applies to renegotiation of a single contract. Specifying a concept of multilateral bargaining which allows for multiple workers' contracts to be renegotiated at the same time, while interesting, is beyond the scope of this paper. Intuitively, allowing for this moves us towards the 'extreme employment-at-will' world of SZ where no commitment to contracts is feasible.

[6] The distinction between flow wages and lump-sum transfers is exaggerated in the environment studied in which time is continuous but employment is discrete.

makes the reduced-form surplus-sharing rule which is standard in the random search literature, following Diamond (1982) and Pissarides (1985), more attractive. In the case of one-to-one matching, BRW show how the Nash surplus-splitting rule can arise as the unique subgame perfect equilibrium of a non-cooperative sequential offer-counteroffer game in which there is a vanishingly small possibility after each rejected offer of negotiations breaking down. Similarly, in the multi-worker firm setting, SZ show how their model of bargaining without commitment can be microfounded using an extensive-form non-cooperative game involving the firm and all its incumbent workers which builds on BRW. For the sake of setting the model of bargaining with commitment introduced in this paper on an equal footing with that of bargaining only over current wages, it is therefore helpful to show that this model also has a game-theoretic underpinning.

In fact it is more straightforward to construct a microfoundation for the model of bargaining with commitment than it was for SZ, since only bilateral negotiations need to be considered. Consider first the situation where a firm negotiates with a newly hired worker. The simplest case arises if this is the last worker the firm will ever hire.[7] Because the firm has committed to a long-term contract with each of its incumbent workers, these contracts are taken as exogenous in the new bargain. Allowing for this, the firm and the new worker can simply play a bilateral bargaining game as in BRW. Next, if the firm has a smaller workforce, so that it anticipates the possibility of additional hiring in the future, the marginal surplus to be split with the current worker incorporates this possibility. Once this is taken into account, the bargaining game is still bilateral, so again the construction of BRW applies in the obvious way.

I now explain this more formally. To reduce the notational burden, and because I wish to describe a bargaining game between a single, fixed firm and its employees, I omit notation for that single firm's productivity z in this section.

Consider a firm with $n \geq 0$ incumbent employees, with each incumbent employee $k = 1, \ldots, n$ employed under a long-term contract which promises sequences of flow wages $(w^k(\cdot))$ and lump-sum transfers $(t^k(\cdot))$, so that the flow wage paid when employment at the firm is m (for $m \geq k$) is $w^k(m)$, and so that the transfer paid to the worker when the firm hires a $(m+1)$st worker is $t^k(m)$. As long as the firm does not meet a new worker, at each time τ it chooses an intensity of vacancy posting v_τ, and pays the wage $w^k(n)$ to each incumbent worker.

Next suppose that the firm's vacancy posting is successful and it meets a new worker (as occurs at Poisson rate qv_τ). In this case, the firm and the new worker engage in an offer-counteroffer game as in BRW. The game occurs in fictitious time (that is, it is completed instantaneously, no matter how many offers are rejected). There is a small exogenous chance of breakdown after each rejected counteroffer, the probability being d_w after a rejected offer made by the worker and d_f after a rejected offer made by the firm. If breakdown occurs, or if no offer is ever accepted, the worker returns to unemployment and the firm continues matched only to its incumbent employees. I assume that $d_w = \beta d$ and $d_f = (1-\beta)d$ for some $d > 0$, and consider the limit as $d \to 0$. This follows the approach described in Section 4 of BRW.

[7] Such a worker exists because of decreasing returns in production.

An offer in the BRW game consists of a long-term contract. This contract promises a sequence of flow wages $w^k(n)$ for $k \leq n \leq n^*$ to be paid by the firm to the worker if employment at the firm is n, together with a sequence of lump-sum transfers $t^k(n)$ for $k \leq n \leq n^* - 1$ to be paid by the firm to the worker when it hires a $(n+1)$st worker. The firm is able to commit to whatever contract is agreed, so that it governs the employment relationship with this worker until the firm is exogenously destroyed. Upon agreement, the newly-matched worker is employed by the firm, which begins to pay her the wage $w^{n+1}(n+1)$ as specified in the new contract. The firm makes lump-sum transfers $t^k(n)$ to each of its incumbent workers, and changes their flow wage from $w^k(n)$ to $w^k(n+1)$ as specified in their existing contracts.

The appropriate noncooperative equilibrium concept in this environment is a version of Markov perfect equilibrium. To emphasize that the aggregate variables V^u and q are taken as given in the bargaining game between a single firm and its workers, I call this equilibrium concept *Markov perfect partial equilibrium*, or MPPE. An MPPE of the game just described consists of (a) a maximum employment level n^*; (b) a sequence of contracts indexed by $k = 1, \ldots, n^*$ where for each k the contract consists of a sequence $(w^k(n))_{n=k}^{n^*}$ of nonnegative wages and a sequence $(t^k(n))_{n=k}^{n^*-1}$ of nonnegative transfers to be made by the firm to the kth worker hired when total employment at the firm is equal to $n \geq k$; and (c) a vacancy posting strategy $v(\cdot)$ for the firm as a function of its current employment level (with $v(n) = 0$ for $n \geq n^*$), such that (1) the firm chooses vacancy posting to maximize its profit and (2) each contract $\left((w^k(n))_{n=k}^{n^*}, (t^k(n))_{n=k}^{n^*-1}\right)$ is the unique equilibrium of the BRW offer-counteroffer game played at the time of hiring between the firm and the kth worker to be hired. The equilibrium is Markov perfect in the sense that wages, transfers, and vacancy-posting depend only on the number of workers at the firm. It is partial in the sense that, as noted, the value of an unemployed worker, V^u, and the vacancy-filling rate, q, are exogenous.

To state this definition more precisely requires introducing some notation. The value function of a firm with n workers, taking as given the set of contracts $\mathbf{w}^n = (w^i(\cdot))_{i=1}^n$ and $\mathbf{t}^n = (t^i(\cdot))_{i=1}^n$ already signed with incumbent workers, satisfies the HJB equation

$$
\begin{aligned}
(r+\delta)J(n;\mathbf{w}^n,\mathbf{t}^n) = y(n) - \sum_{k=1}^{n} w^k(n) - c(v(n;\mathbf{w}^n,\mathbf{t}^n)) \\
+ qv(n;\mathbf{w}^n,\mathbf{t}^n)\big[-\sum_{k=1}^{n} t^k(n) + J(n;\mathbf{w}^n,\mathbf{t}^n) - J(n;\mathbf{w}^n,\mathbf{t}^n)\big].
\end{aligned}
$$

The HJB equation for the kth worker hired when total employment at the firm is n takes the form

$$
\begin{aligned}
(r+\delta)\left[V^k(n;\mathbf{w}^n,\mathbf{t}^n) - V^u\right] = w^k(n) - rV^u \\
+ qv(n)\left[t^k(n) + V^k(n+1;\mathbf{w}^{n+1},\mathbf{t}^{n+1}) - V^k(n;\mathbf{w}^n,\mathbf{t}^n)\right].
\end{aligned}
$$

Vacancy posting is optimal if $v(n) = v(n; \mathbf{w}^n, \mathbf{t}^n)$ solves

$$v(n; \mathbf{w}^n, \mathbf{t}^n) = \arg\max_{v \geq 0} \left\{ -c(v) + qv\big[-\sum_{k=1}^{n} t^k(n) + J(n+1; \mathbf{w}^{n+1}, \mathbf{t}^{n+1}) - J(n; \mathbf{w}^n, \mathbf{t}^n)\big] \right\}.$$

The surplus shared between the firm and the $(n+1)$st-hired worker at matching, to be divided in the BRW game, is

$$\big[V(n+1; \mathbf{w}^{n+1}, \mathbf{t}^{n+1}) - V^u\big] + \left[-\sum_{i=k}^{n} t^k(n) + J(n+1; \mathbf{w}^{n+1}, \mathbf{t}^{n+1}) - J(n; \mathbf{w}^n, \mathbf{t}^n)\right].$$

Under the assumptions made above on the breakdown probabilities following an offer rejection, the unique equilibrium of the BRW game (in the limit as $d \to 0$) is such that the worker and firm contract so as to maximize their joint surplus, taking as given the contracts the firm has signed with incumbent workers and anticipating that the firm will in future choose vacancy posting to maximize its own profits and will negotiate in the same way with the resulting new hires. This implies that the surplus will be divided in ratio $\beta : (1-\beta)$ between the worker and the firm, that is

$$(1-\beta)\big[V(n+1; \mathbf{w}^{n+1}, \mathbf{t}^{n+1}) - V^u\big] = \beta\left[-\sum_{k=1}^{n} t^k(n) + J(n+1; \mathbf{w}^{n+1}, \mathbf{t}^{n+1}) - J(n; \mathbf{w}^n, \mathbf{t}^n)\right].$$

The main result of this section is that there is an MPPE in which the values of firms and workers and the vacancy-posting strategies of firms satisfy PER and SS as described in the main paper. The equilibrium allocation therefore decentralizes the allocation described in Propositions 1 and 2 in Section 3 of the paper.

To construct this equilibrium, let n^*, $S(\cdot)$, $V(\cdot)$, and $v(\cdot)$ be the unique maximum firm size, surplus, worker value, and vacancy posting strategy consistent with PER and SS. The existence and uniqueness of these objects is established by Proposition 1 in the main paper. In addition, let $t(\cdot)$ and $w(\cdot)$ be as characterized by equations (C.9) and (C.10). Using these functions, let the contract signed by the kth-hired worker specify wages $w^k(n) = w(n)$ (for $k \leq n \leq n^*$) and transfers $t^k(n) = t(n)$ (for $k \leq n \leq n^* - 1$). It is straightforward in view of the discussion in this section to demonstrate that these wages and transfers, together with the vacancy-posting strategy $v(\cdot)$ for the firm, form part of an MPPE.

Proposition C.1. *An MPPE exists, and the equilibrium allocation coincides with that described in Propositions 1 and 2 of the main paper.*

Proof. The proof proceeds by backward induction. First, providing that the contracts signed by incumbent workers specify nonnegative wages and transfer payments, because $y(n) - y(n-1) < b$ for n sufficiently large, there is a last worker it is profitable for the firm to hire. Suppose without loss of generality that the contracts signed by the first $\hat{n} - 1$ workers hired by a firm are such that a positive surplus is generated for the firm by hiring one further worker, but that there is no possible contract such that it will be profit-maximizing to hire additional workers. Then one possible bilaterally efficient contract signed by the $\hat{n}$th worker and the firm specifies $t^{\hat{n}}(k) = 0$ for all $k \geq \hat{n} + 1$ and chooses the wage $w^{\hat{n}}(\hat{n})$ to split the surplus in ratio $\beta : (1 - \beta)$.

Next, anticipating that the $\hat{n}$th worker will sign such a contract, there are also many possible optimal contracts that could be signed by the firm and the $(\hat{n} - 1)$st worker. One suitable contract is constructed by setting $w^{\hat{n}-1}(\hat{n}) = w^{\hat{n}}(\hat{n})$, and choosing $w^{\hat{n}-1}(\hat{n} - 1)$ and $t^{\hat{n}-1}(\hat{n})$ so as to split the bilateral surplus in ratio $\beta : (1 - \beta)$ and so that the transfer exactly compensates the worker for the loss in value associated with the hiring of the $\hat{n}$th worker. Continuing inductively allows the contract of each worker hired, from the $(\hat{n} - 2)$nd to the first, to be constructed analogously. The resulting allocation satisfies that $\hat{n} = n^*$, with $w^{n+1}(k) = w(k)$ for $k \leq n + 1 \leq n^*$ and $t^{n+1}(k) = t(k)$ for $k \leq n + 1 \leq n^* - 1$. □

The MPPE is not unique because the timing of payments (and in particular, the breakdown of payments between flow wages and lump-sum transfers) is not uniquely determined. However, requiring that the MPPE also be bilaterally renegotiation-proof (as discussed in Section C.2) serves as an equilibrium refinement which guarantees uniqueness. It is straightforward to extend the backwards induction argument in the proof of Proposition C.1 to establish this formally. Similarly, other assumptions on the timing of wages (for example, constant wages as in the main paper) will also deliver uniqueness.

Although the MPPE concept allows for workers each to sign potentially very different contracts for wages and transfers, in the equilibrium just described all workers sign contracts that are very similar. Take any two workers employed by the firm, hired jth and kth, and suppose the current level of employment at the firm is $n \geq \max\{j, k\}$. Then in equilibrium both workers receive the same wage $w^j(n) = w^k(n) = w(n)$ and in the event the firm hires an $(n + 1)$st worker, both will receive the same transfer $t^j(n) = t^k(n) = t(n)$. That is, if $k > j$, then the contract signed by k is a truncation of the contract signed by j, and both are truncations of the contract signed by the first-hired worker.[8]

D Vacancy Posting Subsidies and Taxes Contingent on Firm Size or Age

Proposition 8 in the main paper establishes that if the planner can condition hiring (or vacancy-posting) taxes on the two key state variables of a firm, that is, productivity z and current employment n, then there always exists a tax scheme under which the equilibrium allocation coincides with the optimal steady state allocation. This tax scheme will usually be non-zero since, according to Proposition 5, the equilibrium is always constrained inefficient except in trivial cases.

[8]This result is reminiscent of the analogous characterization in Burdett and Coles (2003) of optimal wage-tenure contracts, although the economic intuition behind it is quite different: their model relies on on-the-job search, not on multi-worker firms with decreasing returns to labor in production.

However, in practice it may be difficult for the planner to condition taxes on firm productivity, since this might not be observable. Accordingly, the possibility of using taxes that condition only on easily observed firm characteristics such as size or age is worth considering. As discussed at the end of Section 5 of the paper, however, it is not generally true that subsidizing small or young firms is desirable. No general results are available.

The focus of this section is on whether taxing or subsidizing the vacancy posting of firms in one class (for example, small firms) relative to other firms can be welfare enhancing. It should not be forgotten, of course, that if workers' bargaining power β is too low, then on average all firms will not internalize the congestion externality caused by their recruitment activity, so that the level of vacancy posting by all firms can be too high. If β is too high, a general subsidy to vacancy posting for all firms could be welfare-enhancing.

In general, the question of whether a small tax or subsidy on vacancy posting by firms with a particular characteristic can be welfare-enhancing depends on whether the conditional mean of the marginal surplus associated with hiring for these firms (that is, $S(n+1;z) - S(n;z)$) is particularly high or low. To see why, observe that a firm's private benefit from hiring is $(1-\beta)\left[S(n+1;z) - S(n;z)\right]$, so that it chooses its vacancy posting intensity to maximize

$$-c(v) + qv(1-\beta)\left[S(n+1;z) - S(n;z)\right].$$

The marginal social benefit from the vacancy-posting of this single firm, however, is

$$-c(v) + qv\left[S(n+1;z) - S(n;z)\right] - \hat{\mu}v$$

where $\hat{\mu}$ is the value of the marginal congestion externality associated with each vacancy posted. (In making this calculation, I am assuming first that only a single firm is being taxed or subsidized (so that there are no general equilibrium effects), and second, that it is taxed or subsidized only for a very short time interval, so that forward-looking surplus values are unchanged.) It is clear from a comparison of the first-order conditions for these two problems that it is desirable to subsidize in relative terms the vacancy posting behavior of any class of firms with a conditional mean for $S(n+1;z) - S(n;z)$ above the unconditional mean over all firms.

In the case where there is no heterogeneity in z, Proposition 2 guarantees that $S(n+1) - S(n)$ is decreasing in n for firms for which $v(n) > 0$. It is therefore optimal to subsidize small firms in this case. In general, conditional on productivity, a firm's expected size is an increasing function of its age; in the absence of heterogeneity in z, this statement is also true unconditionally. Thus it would also be welfare-enhancing to subsidize young firms in the absence of heterogeneity in z.

To see that it is not always optimal to subsidize vacancy posting by small firms, consider a case where there z can take on two distinct values. If $z = z_L$, the production function and the value of V^u are such that $y(1;z_L) > rV^u > y(2;z_L) - y(1;z_L)$, so that these firms do not hire more than one worker. If $z = z_H$, then $y(1;z_H) > y(2;z_H) - y(1;z_H) > rV^u$, so firms with productivity z_H hire up to two workers in equilibrium. Suppose that the frequency of firms with $z = z_L$ is much greater than the frequency of firms with $z = z_H$, that $y(1; z_L) - rV^u$ is small, and that $y(2; z_H) - y(1; z_H) - rV^u$ is large. Then on average, the expected value of $S(1; z) - S(0; z)$ is approximately equal to $S(1; z_L) - S(0; z_L)$. If vacancy posting costs are quadratic, $c(v) = \frac{1}{2}v^2$, then it is easy to show by manipulating the HJB equations that

$$S(1; z_L) - S(0; z_L) = \frac{y(1; z_L) - rV^u}{r + \delta + \frac{1}{2}q^2(1-\beta)^2}.$$

Similarly, the expected value of $S(2; z) - S(1; z)$ conditional on $v(1; z) > 0$ is equal to

$$S(2; z_H) - S(1; z_H) = \frac{y(2; z_H) - y(1; z_H) - rV^u}{r + \delta + \frac{1}{2}q^2(1-\beta)^2}.$$

Provided that $y(2; z_H) - y(1; z_H) > y(1; z_L)$, in this example the expected marginal surplus associated with hiring by small firms is smaller than that associated with hiring by large firms.

In Appendix B I studied a continuous-employment version of the model with quadratic production and quadratic vacancy posting costs, and argued that it does a good job of representing many important features of the U.S. labor market. I now show that it is optimal to subsidize young firms in this version of the model. Proposition B.1 characterized functional forms for the surplus $S(\cdot)$ and for vacancy posting $v(\cdot)$. Using the expression for vacancy posting given in (B.9), one can see that the growth rate of a firm with productivity z and employment n is

$$\dot{n}(t; z) = q\lambda(n^*(z) - n).$$

With initial condition $n(0; z) = 0$, this implies that

$$n(t; z) = n^*(z)\left[1 - e^{-q\lambda t}\right].$$

It follows that the marginal surplus associated with a firm of age t and productivity z is

$$\begin{aligned}
\frac{\partial}{\partial n}S(n; z)\bigg|_{n=n(t;z)} &= B(z) - Cn(t; z) \\
&= \frac{z - rV^u}{\sigma}C - \frac{z - rV^u}{\sigma}C\left[1 - e^{-q\lambda t}\right] \\
&= \frac{z - rV^u}{\sigma}Ce^{-q\lambda t}.
\end{aligned}$$

Here the first equality follows from the expression for the surplus given in (B.8), while the second follows from the expressions for $B(z)$ and C following that equation and from the expression for the maximum firm size of a firm with productivity z in (B.7). The final expression is a decreasing function of t. It follows that the expected social value of hiring in this parametric example is decreasing with firm age, so that it is optimal to subsidize vacancy posting by young firms relative to older firms in this environment.

Even in this case, it does not appear that it is generally optimal to subsidize the vacancy posting of small firms relative to that of large firms. This depends on the extent of heterogeneity in z. If, as in the example on the previous page, there are many firms with low z and a few firms with very high z, it can be optimal in the quadratic case also to subsidize vacancy-posting by larger firms relative to that by smaller firms, at least for some non-empty range of firm sizes.

E Non-Constant Bargaining Power

In this Appendix I show how to generalize the baseline model of the paper by allowing for a worker's bargaining power to depend on two key characteristics of the firm with which she is negotiating: the firm's productivity, z, and the number of incumbent employees, n. Making this generalization is interesting because it allows me to establish that if bargaining power varies with firm characteristics in a very specific way, then the equilibrium can be constrained efficient, unlike in the main text.

To allow workers' bargaining power to depend on the characteristics of the firm they are matched with, replace SS with the assumption that for each $n = 0, 1, \dots$ there is $\beta(n+1; z) \in [0, 1]$, such that

$$V(n+1; z) - V^u = \beta(n+1; z) \left[S(n+1; z) - S(n; z) \right]. \tag{E.1}$$

The results of Section 3 and Section 4 largely extend to this case in a straightforward way. More precisely, the proofs of Propositions 1 and 4 given in the Appendix apply immediately to this more general case if β is replaced by $\beta(n+1; z)$ wherever it occurs. (The proof of Proposition 2 does rely on constant bargaining power, so this result does not necessarily extend to general $\beta(\cdot)$.)

I now ask whether there are conditions on how the bargaining power $\beta(\cdot)$ varies with the firm's characteristics, such that the equilibrium is constrained efficient.

Consider any equilibrium allocation which involves positive vacancy posting by some firms; then $\theta > 0$. In this case I can rewrite the constraint on an efficient allocation (16) in an equivalent form by multiplying by $q(\theta)$ to obtain

$$0 = f(\theta)u - q(\theta) \int \sum_{n=0}^{\infty} g(n; z) v(n; z). \tag{E.2}$$

As in the proof of Proposition 5, denote by $\lambda(n; z)$ the multipliers on the sequence of constraints (17), by λ^u the multiplier on (18), and by μ the multiplier on (E.2). I can interpret λ^u as the shadow value to the planner of an unemployed worker, $\hat{\lambda}(n; z) \equiv \lambda(n; z) - n\lambda^u$ as the shadow marginal value of having n employees matched to a single firm or productivity z rather than being unemployed, and $\hat{\mu} \equiv \mu - \lambda^u$ as the shadow cost of creating a new match (which is positive because creating a new match requires posting vacancies, which congests the process of matching for other firms). With this notation I can state the following result.

Proposition E.1 (Efficiency of Equilibrium)**.** *The steady-state equilibrium allocation coincides* *with the optimal steady state allocation if and only if for each n,*

$$\beta(n+1;z) = \frac{\hat{\mu}}{\hat{\lambda}(n+1;z) - \hat{\lambda}(n;z)}. \tag{E.3}$$

Proof. Modify the argument given in the proof of Proposition 5 slightly by replacing β with its size- and productivity-dependent counterpart $\beta(n;z)$, to establish the analog of (A.24):

$$\beta(n;z)\left[S(n;z) - S(n-1;z)\right] = \hat{\mu} - \frac{rV^u - r\lambda^u}{r+\delta} \tag{E.4}$$

Arguing as before, in an efficient equilibrium, we have that $V^u = \lambda^u$ and $S(n;z) = \hat{\lambda}(n;z)$ for $n \leq n^*(z)$. (E.3) follows immediately.

Conversely, suppose that (E.3) holds. The result that the equilibrium is constrained efficient follows immediately from the fact that in this case, the values $S(\cdot)$ and V^u solve the first-order conditions for the planner's problem, together with the concavity of that problem. □

Notice from (A.17) that if $v(n;z) > 0$, then $\hat{\lambda}(n+1;z) - \hat{\lambda}(n;z) - \hat{\mu} > 0$. Since $\hat{\mu} > 0$, it is immediate that the $\beta(n+1;z)$ defined by (E.3) lies in the unit interval. That is, the firm-characteristic dependent bargaining powers $\beta(n;z)$ do in fact correspond to both the firm and the worker receiving positive shares of the marginal surplus associated with the employment relationship.

Proposition E.1 generalizes the Hosios (1990) condition to the environment of this paper. Like the result of Hosios, Proposition E.1 indicates that the equilibrium of a model with bargaining and random search will be constrained efficient only if the bargaining power of a worker negotiating with a firm takes one uniquely-determined value. The difference between Proposition E.1 and Hosios' result arises because in this paper, firms are heterogeneous in productivity because of their time-consuming growth. To generate an efficient pattern of vacancy posting across firms requires that the bargaining power of workers vary with the size of the firm in a uniquely-determined way.

An intuition for why (E.3) gives the appropriate value of bargaining power to decentralize the constrained efficient allocation is easy to give. The marginal increase in congestion caused by an individual firm posting more vacancies is independent of the particular firm that is posting vacancies. What causes firms to internalize this congestion externality is that they do not get to keep the full marginal product of an additional worker, but rather must pay the worker a share. Thus, efficiency requires two things. First, the present value of payments to a worker must be independent of the number of incumbent employees at the hiring firm; an equivalent way of saying this is that $V(n;z)$ must be independent of n and z. Because of (E.1), this requires that $\beta(n+1;z)$ must be inversely proportional to $S(n+1;z) - S(n;z)$, or equivalently to $\hat{\lambda}(n+1;z) - \hat{\lambda}(n;z)$ since in an optimal steady-state allocation $S \equiv \hat{\lambda}$. Second, the greater the congestion externality caused by vacancy posting, the higher wages must be. The severity of the congestion externality is measured by $\hat{\mu}$.

There is a sense in which (E.3) is not exactly comparable to Hosios' result. This is that $\hat{\mu}$, the shadow cost of creating a new match, measures not just the expected number of matches lost to congestion if an individual firm increases its vacancy posting enough to hire one worker but also the expected productivity of those foregone matches. The following result, like Hosios', isolates the first effect only.

Proposition E.2 (Hosios Condition)**.** *If the equilibrium is constrained efficient, then*

$$\frac{\int \sum_{n=0}^{\infty} g(n;z)v(n;z)\,dz}{\int \sum_{n=0}^{\infty} \frac{g(n;z)v(n;z)}{\beta(n+1;z)}\,dz} = -\frac{\theta q'(\theta)}{q(\theta)}. \tag{E.5}$$

Proof. Rewrite (A.12) by multiplying by θ and substituting for u using the constraint (16) to see that

$$\begin{aligned} 0 = \hat{\mu}\left[\frac{f'(\theta)q(\theta)}{f(\theta)} - q'(\theta)\right]\int\sum_{n=0}^{\infty} g(n;z)v(n;z)\,dz \\ + q'(\theta)\int\sum_{n=0}^{\infty} g(n;z)v(n;z)\left[\hat{\lambda}(n+1;z) - \hat{\lambda}(n;z)\right]dz. \end{aligned}$$

Replacing $f(\theta) = \theta q(\theta)$ implies that $f'(\theta)q(\theta)/f(\theta) = q(\theta)/\theta + q'(\theta)$. Substitute this expression and use (E.3) to obtain that

$$\hat{\mu}\frac{q(\theta)}{\theta}\int\sum_{n=0}^{\infty} g(n;z)v(n;z)\,dz + q'(\theta)\int\sum_{n=0}^{\infty} g(n;z)v(n;z)\frac{\hat{\mu}}{\beta(n+1;z)}\,dz = 0, \tag{E.6}$$

so that either $\hat{\mu} = 0$ (which implies that $\mu = \lambda^u$, so that the shadow value of a match is the same as the value of an unemployed worker, which happens only if the economy is not productive enough to generate positive entry) or else (E.5) holds. □

Hosios showed that when matching is one-to-one, a necessary condition for constrained efficiency is that the bargaining power of workers equal the unemployment elasticity of the matching function. In the multi-worker setting studied here, this must be true in an average sense. (E.5) makes this precise: the appropriate notion of average bargaining power is the weighted harmonic mean of all the bargaining powers $\beta(n+1;z)$, weighted by the measures of vacancies posted by firms with productivity z and n incumbent employees, which is $g(n;z)v(n;z)$.[9]

The Hosios (1990) condition can be criticized since there is no reason to expect that the bargaining power, which is a structural parameter of how firms and workers split a bilateral surplus, should equal the elasticity of the matching function, which is a completely unrelated object.[10] In the environment I study, this criticism is doubly powerful. Not only does a Hosios-type condition

[9] It should already be clear from the previous discussion that unlike in case of one-to-one matching considered by Hosios, the converse to Proposition E.2 is not true: even if the bargaining power of firms is correct in an average sense, it still need not be true that the relative intensity of vacancy posting by firms of different sizes is what the planner would like it to be.

[10] Of course, in an environment where search is directed and not random, the Hosios condition is necessarily satisfied since the competitive posting of contracts by market-makers generates an equilibrium relationship between the two quantities (Moen 1997; Shimer 1996). This carries over to this environment (Kaas and Kircher 2013).

need to hold on average for the equilibrium allocation to be constrained efficient, but the bargaining power needs to vary with firm productivity and size, and that in a way that is characterized by (E.3). Moreover, the terms appearing in that equation, namely the multipliers $\hat{\lambda}(n;z)$ and $\hat{\mu}$, depend not only on the matching function, but also on the production function, the vacancy posting cost function, and other parameters of the model in a complicated way. For example, the proof of Proposition E.1 shows that the first-order condition for the optimal steady-state value of $g(n;z)$ can be rewritten as

$$(r+\delta)\hat{\lambda}(n;z) = y(n;z) - nr\lambda^u - c(v(n;z)) + qv(n;z)\left[\hat{\lambda}(n+1;z) - \hat{\lambda}(n;z) - \hat{\mu}\right], \qquad \text{(E.7)}$$

so that the production function $y(\cdot)$, the vacancy-posting cost function $c(\cdot)$, and other primitives of the model enter into the equations defining the terms occurring on the right side of (E.3). Therefore the required set of bargaining power parameters $\beta(\cdot)$ would need to be different across economies which differed in any model primitive. Even within a given economy, a worker's bargaining power would need to be greater the larger the firm he was bargaining with, which seems less plausible than the reverse assumption. Thus, the condition for efficiency represented by (E.3) is certainly a demanding one.

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