Technical appendix for: On the Effects of Rare Disasters and Uncertainty Shocks for Risk Premia in Non-Linear DSGE Models

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Contents

1 The class of DSGE model

This appendix derives the first order, second order, and third order approximated solutions to a general class of DSGE models. Our procedure and presentation follows the one in Schmitt-Grohé & Uribe (2004).

We consider the class of DSGE models where the set of equilibrium conditions can be written as

$$E_t\left[\mathbf{f}\left(\mathbf{y}_{t+1},\mathbf{y}_t,\mathbf{x}_{t+1},\mathbf{x}_t\right)\right]=\mathbf{0}. \tag{1}$$

Here, E_t is the conditional expectation given information available at time t. The vector $\mathbf{x}_t$ is the set of state variables (pre-determined variables) and has dimension $n_x \times 1$. The vector $\mathbf{y}_t$ contains the set of control variables (non pre-determined variables) and has dimension $n_y \times 1$. We also let $n \equiv n_x + n_y$.

The state vector is partitioned as $\mathbf{x}_t \equiv \begin{bmatrix}\mathbf{x}_{1,t}\\ \mathbf{x}_{2,t}\end{bmatrix}$, where $\mathbf{x}_{1,t}$ with dimension $n_{x_1}\times 1$ contains the set of endogenous state variables and $\mathbf{x}_{2,t}$ with dimension $n_{x_2}\times 1$ contains the set of exogenous state variables. Note also that $n_{x_1}+n_{x_2}=n_x$.

For the exogenous state variables we assume that

$$\mathbf{x}_{2,t+1}=\mathbf{h}\left(\mathbf{x}_{2,t},\sigma\right)+\sigma\tilde{\boldsymbol{\eta}}\boldsymbol{\epsilon}_{t+1}, \tag{2}$$

where $\boldsymbol{\epsilon}_{t+1}$ has dimension $n_e \times 1$, and thus, $\tilde{\boldsymbol{\eta}}$ has dimension $n_{x_2}\times n_e$.[1]

The general solution to this class of DSGE model is given by

$$\mathbf{y}_t=\mathbf{g}\left(\mathbf{x}_t,\sigma\right) \tag{3}$$

$$\mathbf{x}_{t+1}=\mathbf{h}\left(\mathbf{x}_t,\sigma\right)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \tag{4}$$

$$\boldsymbol{\eta}=\begin{bmatrix}\mathbf{0}\\ \tilde{\boldsymbol{\eta}}\end{bmatrix} \tag{5}$$

where the functions $\mathbf{g}\left(\cdot,\cdot\right)$ and $\mathbf{h}\left(\cdot,\cdot\right)$ are unknown. We will therefore approximate these functions up to third order. This is done around the deterministic steady state, i.e. $\mathbf{x}_t=\mathbf{x}_{ss}$ and $\sigma=0$. Formally, the expression for the deterministic steady state is given as the solution of $\left(\mathbf{y}_{ss},\mathbf{x}_{ss}\right)$ to

$$\mathbf{f}\left(\mathbf{y}_{ss},\mathbf{y}_{ss},\mathbf{x}_{ss},\mathbf{x}_{ss}\right)=\mathbf{0}. \tag{6}$$

Note also that $\mathbf{x}_{ss}=\mathbf{h}\left(\mathbf{x}_{ss},0\right)$ and $\mathbf{y}_{ss}=\mathbf{g}\left(\mathbf{x}_{ss},0\right)$.

Next, substituting the exact solution in (3)-(5) into (1) gives

$$\mathbf{F}\left(\mathbf{x}_t,\sigma\right)\equiv E_t\left[\mathbf{f}\left(\mathbf{g}\left(\mathbf{h}\left(\mathbf{x}_t,\sigma\right)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\sigma\right),\mathbf{g}\left(\mathbf{x}_t,\sigma\right),\mathbf{h}\left(\mathbf{x}_t,\sigma\right)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\mathbf{x}_t\right)\right]=\mathbf{0} \tag{7}$$

which defines $\mathbf{F}\left(\mathbf{x}_t,\sigma\right):\mathbb{R}^{n_x}\times\mathbb{R}\longrightarrow\mathbb{R}^n$. The expression in (7) must hold for all possible values of $\mathbf{x}_t$ and σ. Hence, all derivatives of $\mathbf{F}\left(\mathbf{x}_t,\sigma\right)$ must also be equal to zero. This is the basic fact which we use to find the first, second, and third order derivatives of $\mathbf{g}\left(\cdot,\cdot\right)$ and $\mathbf{h}\left(\cdot,\cdot\right)$.

For the indices we adopt the convention that the subscript is related to the order of differentiation. I.e. a "1" is for the first time we take derivatives and so on. Thus,

$\alpha_1,\alpha_2,\alpha_3=1,2,...n_x$
$\gamma_1,\gamma_2,\gamma_3=1,2,...n_x$
$\beta_1,\beta_2,\beta_3=1,2,...n_y$
$\phi_1,\phi_2,\phi_3=1,2,...,n_e$

[1] All the formulas below should also hold if $\mathbf{x}_t^2=\mathbf{h}\left(\mathbf{x}_t,\sigma\right)+\sigma\tilde{\boldsymbol{\eta}}\boldsymbol{\epsilon}_{t+1}$. However, Schmitt-Grohé & Uribe (2004) do not consider this case.

2 The first order approximation

The first order approximation around the determinstic steady state is

$\mathbf{g}(\mathbf{x}_t,\sigma)=\mathbf{g}(\mathbf{x}_{ss},0)+\mathbf{g}_{\mathbf{x}}(\mathbf{x}_{ss},0)(\mathbf{x}_t-\mathbf{x}_{ss})+\mathbf{g}_{\sigma}(\mathbf{x}_{ss},0)(\sigma-0)$
$\Updownarrow$
$\mathbf{g}(\mathbf{x}_t,\sigma)=\mathbf{y}_{ss}+\mathbf{g}_{\mathbf{x}}(\mathbf{x}_{ss},0)(\mathbf{x}_t-\mathbf{x}_{ss})+\mathbf{g}_{\sigma}(\mathbf{x}_{ss},0)\sigma$

$\mathbf{h}(\mathbf{x}_t,\sigma)=\mathbf{h}(\mathbf{x}_{ss},0)+\mathbf{h}_{\mathbf{x}}(\mathbf{x}_{ss},0)(\mathbf{x}_t-\mathbf{x}_{ss})+\mathbf{h}_{\sigma}(\mathbf{x}_{ss},0)(\sigma-0)$
$\Updownarrow$
$\mathbf{h}(\mathbf{x}_t,\sigma)=\mathbf{x}_{ss}+\mathbf{h}_{\mathbf{x}}(\mathbf{x}_{ss},0)(\mathbf{x}_t-\mathbf{x}_{ss})+\mathbf{h}_{\sigma}(\mathbf{x}_{ss},0)\sigma$

We find expressions for $\mathbf{g}_{\mathbf{x}}(\mathbf{x}_{ss},0)$, $\mathbf{g}_{\sigma}(\mathbf{x}_{ss},0)$, $\mathbf{h}_{\mathbf{x}}(\mathbf{x}_{ss},0)$, and $\mathbf{h}_{\sigma}(\mathbf{x}_{ss},0)$ by solving the system of equations from $\mathbf{F}_{\mathbf{x}}(\mathbf{x}_t,\sigma)=\mathbf{0}$ and $\mathbf{F}_{\sigma}(\mathbf{x}_t,\sigma)=\mathbf{0}$. First, the derivative of the i'th element in $\mathbf{F}(\mathbf{x}_t,\sigma)$ with respect to the α_1 th element of $\mathbf{x}_t$ and evaluated in the deterministic steady state is given by

$[\mathbf{F}_{\mathbf{x}}(\mathbf{x}_{ss},\sigma)]^i_{\alpha_1}=E_t[\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}+[\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}[\mathbf{g}^t_{\mathbf{x}}]^{\beta_1}_{\alpha_1}+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}+[\mathbf{f}_{\mathbf{x}_t}]^i_{\alpha_1}]=\mathbf{0}$
$\Updownarrow$

$$\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+[\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+[\mathbf{f}_{\mathbf{x}_t}]^i_{\alpha_1}=\mathbf{0} \tag{8}$$

This must hold for
$i=1,2,...n$
$\alpha_1=1,2,...,n_x$
$\gamma_1=1,2,...,n_x$
$\beta_1=1,2,...,n_y$

In terms of the used notation, $\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}$ is the (i,β_1) element of the derivative of $\mathbf{f}$ with respect to $\mathbf{y}_{t+1}$, a matrix of dimension $n\times n_{\mathbf{y}}$. Also $\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}=\sum_{\beta_1=1}^{n_y}\sum_{\gamma_1=1}^{n_x}\left[\frac{\partial \mathbf{f}_{y_{t+1}}}{\partial \mathbf{y}_{t+1}}\right]_{(i,\beta_1)}\left[\frac{\partial \mathbf{g}(\mathbf{x}_{t+1},\sigma)}{\partial \mathbf{x}_{t+1}}\right]_{(\beta_1,\gamma_1)}\left[\frac{\partial \mathbf{h}(\mathbf{x}_t,\sigma)}{\partial \mathbf{x}_t}\right]_{(\gamma_1,\alpha_1)}$. Note finally that all the derivatives such as $\mathbf{f}_{\mathbf{x}_{t+1}}$, $\mathbf{f}_{\mathbf{x}_t}$, etc. are to be evaluated in the deterministic steady state.

Next, the derivative of the i'th element in $\mathbf{F}(\mathbf{x}_t,\sigma)$ with respect to the σ

$$\begin{aligned}[\mathbf{F}_{\sigma}(\mathbf{x}_{ss},\sigma)]^i=E_t[&\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_{\sigma}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\ &+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}+[\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)]=\mathbf{0}\end{aligned}$$
$\Updownarrow$

$$\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}+[\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1}=\mathbf{0} \tag{9}$$

because $E_t[\boldsymbol{\epsilon}_{t+1}]=\mathbf{0}$. This must hold for
$i=1,2,...n$
$\gamma_1=1,2,...,n_x$
$\beta_1=1,2,...,n_y$
$\phi_1=1,2,...,n_e$

Hence, $[\mathbf{h}^t_{\sigma}]^{\gamma_1}=0$ and $[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}=0$ as shown by Schmitt-Grohé & Uribe (2004).

3 The second order approximation

The second order approximation around the determinstic steady state is

$$
\begin{aligned}
\left[\mathbf{g}\left(\mathbf{x}_t,\sigma\right)\right]^{\beta_1} = {} & \mathbf{g}\left(\mathbf{x}_{ss},0\right) + \left[\mathbf{g}_{\mathbf{x}}\left(\mathbf{x}_{ss},0\right)\right]^{\beta_1}_{\alpha_1}\left[\left(\mathbf{x}_t-\mathbf{x}_{ss}\right)\right]^{\alpha_1} + \left[\mathbf{g}_{\sigma}\left(\mathbf{x}_{ss},0\right)\right]^{\beta_1}[\sigma] \\
& +\tfrac{1}{2}\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\left(\mathbf{x}_{ss},0\right)\right]^{\beta_1}_{\alpha_1\alpha_2}\left[\left(\mathbf{x}_t-\mathbf{x}_{ss}\right)\right]^{\alpha_1}\left[\left(\mathbf{x}_t-\mathbf{x}_{ss}\right)\right]^{\alpha_2} \\
& +\tfrac{2}{2}\left[\mathbf{g}_{\sigma\mathbf{x}}\left(\mathbf{x}_{ss},0\right)\right]^{\beta_1}_{\alpha_1}[\sigma]\left[\left(\mathbf{x}_t-\mathbf{x}_{ss}\right)\right]^{\alpha_1} \\
& +\tfrac{1}{2}\left[\mathbf{g}_{\sigma\sigma}\left(\mathbf{x}_{ss},0\right)\right]^{\beta_1}[\sigma][\sigma]
\end{aligned}
$$

$$
\begin{aligned}
\left[\mathbf{h}\left(\mathbf{x}_t,\sigma\right)\right]^{\gamma_1} = {} & \mathbf{h}\left(\mathbf{x}_{ss},0\right) + \left[\mathbf{h}_{\mathbf{x}}\left(\mathbf{x}_{ss},0\right)\right]^{\gamma_1}_{\alpha_1}\left[\left(\mathbf{x}_t-\mathbf{x}_{ss}\right)\right]^{\alpha_1} + \left[\mathbf{h}_{\sigma}\left(\mathbf{x}_{ss},0\right)\right]^{\gamma_1}[\sigma] \\
& +\tfrac{1}{2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\left(\mathbf{x}_{ss},0\right)\right]^{\gamma_1}_{\alpha_1\alpha_2}\left[\left(\mathbf{x}_t-\mathbf{x}_{ss}\right)\right]^{\alpha_1}\left[\left(\mathbf{x}_t-\mathbf{x}_{ss}\right)\right]^{\alpha_2} \\
& +\tfrac{2}{2}\left[\mathbf{h}_{\sigma\mathbf{x}}\left(\mathbf{x}_{ss},0\right)\right]^{\gamma_1}_{\alpha_1}[\sigma]\left[\left(\mathbf{x}_t-\mathbf{x}_{ss}\right)\right]^{\alpha_1} \\
& +\tfrac{1}{2}\left[\mathbf{h}_{\sigma\sigma}\left(\mathbf{x}_{ss},0\right)\right]^{\gamma_1}[\sigma][\sigma]
\end{aligned}
$$

where we use Young's teorem saying that $\mathbf{h}_{\sigma\mathbf{x}}\left(\mathbf{x}_{ss},0\right) = \mathbf{h}_{\mathbf{x}\sigma}\left(\mathbf{x}_{ss},0\right)$ and $\mathbf{g}_{\sigma\mathbf{x}}\left(\mathbf{x}_{ss},0\right) = \mathbf{g}_{\mathbf{x}\sigma}\left(\mathbf{x}_{ss},0\right)$. These equations hold for

$\beta_1 = 1,2,...,n_y$

$\gamma_1 = 1,2,...,n_x$

$\alpha_1,\alpha_2 = 1,2,...,n_x$

3.1 With respect to $(\mathbf{x}_t,\mathbf{x}_t)$

We find the unknown coefficients in these Taylor expansions by considering the second derivatives of $\mathbf{F}\left(\mathbf{x}_t,\sigma\right)$. First, recall that

$$
\left[\mathbf{F}_{\mathbf{x}}\left(\mathbf{x}_{ss},\sigma\right)\right]^i_{\alpha_1} = E_t[\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} + \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} + \left[\mathbf{f}_{\mathbf{x}_t}\right]^i_{\alpha_1}]
$$

$$
= E_t\left[\left[Q_1\right]^i_{\alpha_1} + \left[Q_2\right]^i_{\alpha_1} + \left[Q_3\right]^i_{\alpha_1} + \left[Q_4\right]^i_{\alpha_1}\right]
$$

where

$\left[Q_1\right]^i_{\alpha_1} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$\left[Q_2\right]^i_{\alpha_1} \equiv \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$

$\left[Q_3\right]^i_{\alpha_1} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$\left[Q_4\right]^i_{\alpha_1} \equiv \left[\mathbf{f}_{\mathbf{x}_t}\right]^i_{\alpha_1}$

Now

$$
\begin{aligned}
\left[Q_1\right]^i_{\alpha_1,\alpha_2} = {} & \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\Big)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_2}
\end{aligned}
$$

$$
\begin{aligned}
\left[Q_2\right]^i_{\alpha_1,\alpha_2} = {} & \Big(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\Big)\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\alpha_1\alpha_2}
\end{aligned}
$$

$$
\begin{aligned}
\left[Q_3\right]^i_{\alpha_1\alpha_2} \equiv {} & \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2}\Big)\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_2}
\end{aligned}
$$

$$
\left[Q_4\right]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\alpha_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t}\right]^i_{\alpha_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\alpha_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t}\right]^i_{\alpha_1\alpha_2}
$$

Thus
$[\mathbf{F}_{\mathbf{xx}}(\mathbf{x}_{ss},\sigma)]^i_{\alpha_1\alpha_2} = E_t\left[[Q_1]^i_{\alpha_1\alpha_2} + [Q_2]^i_{\alpha_1\alpha_2} + [Q_3]^i_{\alpha_1\alpha_2} + [Q_4]^i_{\alpha_1\alpha_2}\right]$

$$\begin{aligned}
&= \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}[\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2} \\
&+ \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2}[\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + [\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}]^i_{\beta_1\alpha_2}\right)[\mathbf{g}^t_{\mathbf{x}}]^{\beta_1}_{\alpha_1} \\
&\quad + [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}[\mathbf{g}^t_{\mathbf{xx}}]^{\beta_1}_{\alpha_1\alpha_2} \\
&+ \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}[\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2}\right)[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2} \\
&+ \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\alpha_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + [\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t}]^i_{\alpha_1\beta_2}[\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\alpha_1\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + [\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t}]^i_{\alpha_1\alpha_2} = 0
\end{aligned}$$

This must hold for
$i = 1, 2, ...n$
$\alpha_1, \alpha_2 = 1, 2, ..., n_x$
$\gamma_1, \gamma_2 = 1, 2, ..., n_x$
$\beta_1, \beta_2 = 1, 2, ..., n_y$

This is a linear system of equations in $n \times n_x^2$ unknown. However, the symmetry in $\mathbf{g_{xx}}$ and $\mathbf{h_{xx}}$ means that we can reduce the number of unknowns to

$$\begin{aligned}
&(n_y + n_x)\, n_x + (n_y + n_x)\begin{pmatrix} n_x \\ 2 \end{pmatrix} \\
&= (n_y + n_x)\, n_x + (n_y + n_x)\, \tfrac{n_x!}{(n_x-2)!2!} \\
&= (n_y + n_x)\left(n_x + \tfrac{n_x(n_x-1)}{2}\right) \\
&= (n_y + n_x)\left(\tfrac{2n_x + n_x^2 - n_x}{2}\right) \\
&= (n_y + n_x)\left(\tfrac{n_x + n_x^2}{2}\right) \\
&= (n_y + n_x)\, \tfrac{n_x(1+n_x)}{2}
\end{aligned}$$

3.2 With respect to $(\sigma, \mathbf{x}_t)$

Recall that

$$\begin{aligned}
[\mathbf{F}_\sigma(\mathbf{x}_{ss},\sigma)]^i = E_t[&\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} + [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)]
\end{aligned}$$

$$= E_t\left[[P_1]^i + [P_2]^i + [P_3]^i + [P_4]^i\right]$$

where
$[P_1]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

$$[P_2]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$[P_3]^i \equiv [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$

$$[P_4]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

Thus, the derivatives with respect to $\mathbf{x}_t$ are

$$\begin{aligned}
[P_1]^i_{\alpha_2} = &\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_{\sigma}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left([\mathbf{h}^t_{\sigma}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_2}
\end{aligned}$$

$$\begin{aligned}
[P_2]^i_{\alpha_2} = &\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}
\end{aligned}$$

$$\begin{aligned}
[P_3]^i_{\alpha_2} = &\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + [\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}]^i_{\beta_1\alpha_2}\right) [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
&+ [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_2}
\end{aligned}$$

$$\begin{aligned}
[P_4]^i_{\alpha_2} = &\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2}\right) \\
&\quad \times \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_2}
\end{aligned}$$

Thus

$$\begin{aligned}
[\mathbf{F}_{\sigma\mathbf{x}}(\mathbf{x}_{ss}, \sigma)]^i_{\alpha_2} &= E_t\left[[P_1]^i_{\alpha_2} + [P_2]^i_{\alpha_2} + [P_3]^i_{\alpha_2} + [P_4]^i_{\alpha_2}\right] \\
&= E_t[\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_{\sigma}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left([\mathbf{h}^t_{\sigma}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_2} \\
&+ \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \\
&+ \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + [\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}]^i_{\beta_1\alpha_2}\right) [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
&+ [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_2} \\
&+ \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2}\right) \\
&\quad \times \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_2}]
\end{aligned}$$

$\Updownarrow$

$[\mathbf{F}_{\sigma\mathbf{x}}(\mathbf{x}_{ss},\sigma)]^i_{\alpha_2} = \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_2} + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$
ecause $E_t[\boldsymbol{\epsilon}_{t+1}] = \mathbf{0}$, $\mathbf{h}_\sigma = \mathbf{0}$, and $\mathbf{g}_\sigma = \mathbf{0}.$ This must hold for
$i = 1, 2, ...n$
$\alpha_2 = 1, 2, ..., n_x$

As shown by Schmitt-Grohé & Uribe (2004), this system is homogenous in the unknowns $(\mathbf{g}_{\boldsymbol{\sigma}\mathbf{x}}, \mathbf{h}_{\boldsymbol{\sigma}\mathbf{x}})$ and therefore, $\mathbf{g}_{\boldsymbol{\sigma}\mathbf{x}} = \mathbf{0}$ and $\mathbf{h}_{\boldsymbol{\sigma}\mathbf{x}} = \mathbf{0}$.

3.3 With respect to (σ, σ)

The coefficients of $\mathbf{g}_{\sigma\sigma}$ and $\mathbf{h}_{\sigma\sigma}$ are obtained from the second order derivatives of $\mathbf{F}(\mathbf{x}_t, \sigma)$ with respect to σ. Recall that

$$\begin{aligned}[\mathbf{F}_\sigma(\mathbf{x}_{ss},\sigma)]^i = E_t[&\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} + \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1} + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)]\end{aligned}$$

$$= E_t\left[[P_1]^i + [P_2]^i + [P_3]^i + [P_4]^i\right]$$

where
$[P_1]^i_\sigma \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_2]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_3]^i \equiv \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_4]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

Thus, the derivatives with respect to σ are:

$$\begin{aligned}[P_1]^i_\sigma = [&\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right) + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)]\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}\end{aligned}$$

$$\begin{aligned}[P_2]^i_\sigma \equiv [&\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right) + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)]\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\right)\end{aligned}$$

$$\begin{aligned}[P_3]^i_\sigma \equiv [&\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right) + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\\ &+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)]\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\ &+ \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\end{aligned}$$

$$
\begin{aligned}
[P_4]^i_\sigma \equiv {} & [\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right) + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)] \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\gamma_1}
\end{aligned}
$$

Thus
$[\mathbf{F}_{\sigma\sigma}(\mathbf{x}_{ss},\sigma)]^i = E_t\left[[P_1]^i_\sigma + [P_2]^i_\sigma + [P_3]^i_\sigma + [P_4]^i_\sigma\right]$

$$
\begin{aligned}
= {} & E_t\{[\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right) + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)] \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right) \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} \\
& \\
& +[\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right) + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)] \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\right) \\
& \\
& +[\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right) + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \\
& + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)] [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1} \\
& \\
& +[\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right) + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)] \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\gamma_1}\} = 0
\end{aligned}
$$

We know use the fact that $\mathbf{g}_{\boldsymbol{\sigma}} = 0$, $\mathbf{h}_{\boldsymbol{\sigma}} = 0$, $\mathbf{g}_{\boldsymbol{\sigma}\mathbf{x}} = 0$, and $\mathbf{h}_{\boldsymbol{\sigma}\mathbf{x}} = 0$. Moreover we have $E_t[\boldsymbol{\epsilon}_{t+1}] = 0$. Hence,

$$
\begin{aligned}
& E_t\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \\
& \quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \\
& \quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} \\
& \quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1} \\
& \quad + [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1} \\
& \quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \\
& \quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} \left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \\
& \quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\gamma_1}\} = 0
\end{aligned}
$$

$\Updownarrow$

$$
\begin{aligned}
& E_t\{\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \\
& \quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}\right]^{\gamma_1}\} = 0
\end{aligned}
$$

$\Updownarrow$

$$
\begin{aligned}
&\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\\
&\quad+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}\\
&\quad+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}^{t+1}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}\\
&\quad+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}\right]^{\gamma_1} = 0
\end{aligned}
$$

because $E_t\left(\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) = E_t\left(\sum_{\phi_1}^{n_e}\sum_{\phi_2}^{n_e}\epsilon_{t+1}(\phi_1,1)\,\epsilon_{t+1}(\phi_2,1)\right) = E_t\left(\sum_{\phi_1}^{n_e}1\right)$
since ϵ_{t+1} are iid with $E_t\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\right] = \mathbf{I}$.
This must hold for $i = 1, 2, ...n$

4 The third order approximation

The third order approximation around the determinstic steady state is (only no-zero first and second order terms are listed)

$$
\begin{aligned}
\left[\mathbf{g}(\mathbf{x}_t,\sigma)\right]^{\beta_1} = &\ \mathbf{g}(\mathbf{x}_{ss},0) + \left[\mathbf{g}_{\mathbf{x}}(\mathbf{x}_{ss},0)\right]^{\beta_1}_{\alpha_1}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_1}\\
&+\tfrac{1}{2}\left[\mathbf{g}_{\mathbf{xx}}(\mathbf{x}_{ss},0)\right]^{\beta_1}_{\alpha_1\alpha_2}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_1}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_2}\\
&+\tfrac{1}{2}\left[\mathbf{g}_{\sigma\sigma}(\mathbf{x}_{ss},0)\right]^{\beta_1}[\sigma][\sigma]\\
&+\tfrac{1}{6}\left[\mathbf{g}_{\mathbf{xxx}}(\mathbf{x}_{ss},0)\right]^{\beta_1}_{\alpha_1\alpha_2\alpha_3}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_1}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_2}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_3}\\
&+\tfrac{3}{6}\left[\mathbf{g}_{\sigma\sigma\mathbf{x}}(\mathbf{x}_{ss},0)\right]^{\beta_1}_{\alpha_3}[\sigma][\sigma]\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_3}\\
&+\tfrac{3}{6}\left[\mathbf{g}_{\sigma\mathbf{xx}}(\mathbf{x}_{ss},0)\right]^{\beta_1}_{\alpha_2\alpha_3}[\sigma]\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_2}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_3}\\
&+\tfrac{1}{6}\left[\mathbf{g}_{\sigma\sigma\sigma}(\mathbf{x}_{ss},0)\right]^{\beta_1}[\sigma][\sigma][\sigma]
\end{aligned}
$$

$$
\begin{aligned}
\left[\mathbf{h}(\mathbf{x}_t,\sigma)\right]^{\gamma_1} = &\ \mathbf{h}(\mathbf{x}_{ss},0) + \left[\mathbf{h}_{\mathbf{x}}(\mathbf{x}_{ss},0)\right]^{\gamma_1}_{\alpha_1}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_1}\\
&+\tfrac{1}{2}\left[\mathbf{h}_{\mathbf{xx}}(\mathbf{x}_{ss},0)\right]^{\gamma_1}_{\alpha_1\alpha_2}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_1}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_2}\\
&+\tfrac{1}{2}\left[\mathbf{h}_{\sigma\sigma}(\mathbf{x}_{ss},0)\right]^{\gamma_1}[\sigma][\sigma]\\
&+\tfrac{1}{6}\left[\mathbf{h}_{\mathbf{xxx}}(\mathbf{x}_{ss},0)\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_1}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_2}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_3}\\
&+\tfrac{3}{6}\left[\mathbf{h}_{\sigma\sigma\mathbf{x}}(\mathbf{x}_{ss},0)\right]^{\gamma_1}_{\alpha_3}[\sigma][\sigma]\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_3}\\
&+\tfrac{3}{6}\left[\mathbf{h}_{\sigma\mathbf{xx}}(\mathbf{x}_{ss},0)\right]^{\gamma_1}_{\alpha_2\alpha_3}[\sigma]\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_2}\left[(\mathbf{x}_t-\mathbf{x}_{ss})\right]^{\alpha_3}
\end{aligned}
$$

$$+\frac{1}{6}\left[\mathbf{h}_{\sigma\sigma\sigma}\left(\mathbf{x}_{ss},0\right)\right]^{\gamma_1}[\sigma][\sigma][\sigma]$$

for

$\beta_1 = 1,2,...,n_y$

$\gamma_1 = 1,2,...,n_x$

$\alpha_1,\alpha_2,\alpha_3 = 1,2,...,n_x$

We now derive the expression for all the third order terms.

4.1 With respect to $(\mathbf{x}_t, \mathbf{x}_t, \mathbf{x}_t)$

We find the unknown coefficients in these taylor expansions by considering the third derivatives of $\mathbf{F}(\mathbf{x}_t,\sigma)$. First recall that

$$\begin{aligned}
\left[\mathbf{F}_{\mathbf{xx}}\left(\mathbf{x}_{ss},\sigma\right)\right]^i_{\alpha_1\alpha_2} = E_t[&\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\Big)\\
&\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}\\
&+\Big(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\Big)\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_2}\\
&+\Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2}\Big)\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}\\
&+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\alpha_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t}\right]^i_{\alpha_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\alpha_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t}\right]^i_{\alpha_1\alpha_2}]
\end{aligned}$$

$$\begin{aligned}
&1) && = E_t[\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&2) && +\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&3) && +\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&4) && +\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&5) && +\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&6) && +\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}\\
&7) && +\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}\\
&8) && +\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}\\
&9) && +\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}\\
&10) && +\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}\\
&11) && +\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_2}\\
&12) && +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&13) && +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&14) && +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&15) && +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\\
&16) && +\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}\\
&17) && +\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\alpha_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\\
&18) && +\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t}\right]^i_{\alpha_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}
\end{aligned}$$

19) $\quad + \left[\mathbf{f}_{\mathbf{x}_t \mathbf{x}_{t+1}}\right]^i_{\alpha_1 \gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$

20) $\quad + \left[\mathbf{f}_{\mathbf{x}_t \mathbf{x}_t}\right]^i_{\alpha_1 \alpha_2}$

Therefore let

$[Q_1]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1} \mathbf{y}_{t+1}}\right]^i_{\beta_1 \beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$[Q_2]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1} \mathbf{y}_t}\right]^i_{\beta_1 \beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$[Q_3]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1} \mathbf{x}_{t+1}}\right]^i_{\beta_1 \gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$[Q_4]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1} \mathbf{x}_t}\right]^i_{\beta_1 \alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$[Q_5]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1 \gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$[Q_6]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1 \alpha_2}$

$[Q_7]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t \mathbf{y}_{t+1}}\right]^i_{\beta_1 \beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$

$[Q_8]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t \mathbf{y}_t}\right]^i_{\beta_1 \beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$

$[Q_9]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t \mathbf{x}_{t+1}}\right]^i_{\beta_1 \gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$

$[Q_{10}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t \mathbf{x}_t}\right]^i_{\beta_1 \alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$

$[Q_{11}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1} \left[\mathbf{g}^t_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\alpha_1 \alpha_2}$

$[Q_{12}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1} \mathbf{y}_{t+1}}\right]^i_{\gamma_1 \beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$[Q_{13}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1} \mathbf{y}_t}\right]^i_{\gamma_1 \beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$[Q_{14}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1} \mathbf{x}_{t+1}}\right]^i_{\gamma_1 \gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$[Q_{15}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1} \mathbf{x}_t}\right]^i_{\gamma_1 \alpha_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$[Q_{16}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} \left[\mathbf{h}^t_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1 \alpha_2}$

$[Q_{17}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_t \mathbf{y}_{t+1}}\right]^i_{\alpha_1 \beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$

$[Q_{18}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_t \mathbf{y}_t}\right]^i_{\alpha_1 \beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}$

$[Q_{19}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_t \mathbf{x}_{t+1}}\right]^i_{\alpha_1 \gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$

$[Q_{20}]^i_{\alpha_1 \alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_t \mathbf{x}_t}\right]^i_{\alpha_1 \alpha_2}$

And the derivatives with respect to $\mathbf{x}_t$ are:

$[Q_1]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

$$
\begin{aligned}
[Q_1]^i_{\alpha_1\alpha_2\alpha_3} = \\
&\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\Big) \\
&\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$[Q_2]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

$$
\begin{aligned}
[Q_2]^i_{\alpha_1\alpha_2\alpha_3} = \\
&\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\Big) \\
&\times [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{xx}}]^{\beta_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$[Q_3]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

$$
\begin{aligned}
[Q_3]^i_{\alpha_1\alpha_2\alpha_3} = \\
&\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3}\Big) \\
&\times [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$[Q_4]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

$$
\begin{aligned}
[Q_4]^i_{\alpha_1\alpha_2\alpha_3} = \\
&\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\alpha_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_t}\right]^i_{\beta_1\alpha_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\alpha_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2\alpha_3}\Big) \\
&\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$[Q_5]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

$$
\begin{aligned}
[Q_5]^i_{\alpha_1\alpha_2\alpha_3} = \\
&\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\Big) \\
&\times \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[Q_6]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \\
&[Q_6]^i_{\alpha_1\alpha_2\alpha_3} = \\
&\quad \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3} \right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\mathbf{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[Q_7]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&[Q_7]^i_{\alpha_1\alpha_2\alpha_3} = \\
&\quad \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[Q_8]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&[Q_8]^i_{\alpha_1\alpha_2\alpha_3} = \\
&\quad \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \right) \\
&\quad \times \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[Q_9]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&[Q_9]^i_{\alpha_1\alpha_2\alpha_3} = \\
&\quad \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3} \right) \\
&\quad \times \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[Q_{10}]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&[Q_{10}]^i_{\alpha_1\alpha_2\alpha_3} = \\
&\quad \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\alpha_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{y}_t}\right]^i_{\beta_1\alpha_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\alpha_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2\alpha_3} \right) \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2} \left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$$[Q_{11}]^i_{\alpha_1\alpha_2} \equiv [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\mathbf{xx}}]^{\beta_1}_{\alpha_1\alpha_2}$$
$$[Q_{11}]^i_{\alpha_1\alpha_2\alpha_3} = \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}]^i_{\beta_1\alpha_3} \right) [\mathbf{g}^t_{\mathbf{xx}}]^{\beta_1}_{\alpha_1\alpha_2}$$
$$+ [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\mathbf{xxx}}]^{\beta_1}_{\alpha_1\alpha_2\alpha_3}$$

$$[Q_{12}]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$[Q_{12}]^i_{\alpha_1\alpha_2\alpha_3} =$$
$$\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3} \right)$$
$$\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}$$

$$[Q_{13}]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$[Q_{13}]^i_{\alpha_1\alpha_2\alpha_3} =$$
$$\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3} \right)$$
$$\times [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} [\mathbf{g}^t_{\mathbf{xx}}]^{\beta_2}_{\alpha_2\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}$$

$$[Q_{14}]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$[Q_{14}]^i_{\alpha_1\alpha_2\alpha_3} =$$
$$\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\gamma_2\alpha_3} \right)$$
$$\times [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}$$

$$[Q_{15}]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$[Q_{15}]^i_{\alpha_1\alpha_2\alpha_3} =$$
$$\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\gamma_1\alpha_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{y}_t}\right]^i_{\gamma_1\alpha_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\gamma_1\alpha_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2\alpha_3} \right) [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}$$

$$[Q_{16}]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$[Q_{16}]^i_{\alpha_1\alpha_2\alpha_3} = \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3} \right) [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\mathbf{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}$$

$$[Q_{17}]^i_{\alpha_1\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\alpha_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}$$
$$[Q_{17}]^i_{\alpha_1\alpha_2\alpha_3} =$$
$$\left(\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\alpha_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\alpha_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\alpha_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\alpha_1\beta_2\alpha_3} \right)$$

$\times \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_2}^{\gamma_2}$
$+ \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}}\right]_{\alpha_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_2\gamma_3}^{\beta_2} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_2}^{\gamma_2}$
$+ \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}}\right]_{\alpha_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} \left[\mathbf{h}_{\mathbf{xx}}^{t}\right]_{\alpha_2\alpha_3}^{\gamma_2}$

$[Q_{18}]_{\alpha_1\alpha_2}^{i} \equiv \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t}\right]_{\alpha_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t}\right]_{\alpha_2}^{\beta_2}$
$[Q_{18}]_{\alpha_1\alpha_2\alpha_3}^{i} =$
$\left(\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t\mathbf{y}_{t+1}}\right]_{\alpha_1\beta_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t\mathbf{y}_t}\right]_{\alpha_1\beta_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\beta_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t\mathbf{x}_{t+1}}\right]_{\alpha_1\beta_2\gamma_3}^{i} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t\mathbf{x}_t}\right]_{\alpha_1\beta_2\alpha_3}^{i}\right) \left[\mathbf{g}_{\mathbf{x}}^{t}\right]_{\alpha_2}^{\beta_2}$
$+ \left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t}\right]_{\alpha_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{xx}}^{t}\right]_{\alpha_2\alpha_3}^{\beta_2}$

$[Q_{19}]_{\alpha_1\alpha_2}^{i} \equiv \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}}\right]_{\alpha_1\gamma_2}^{i} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_2}^{\gamma_2}$
$[Q_{19}]_{\alpha_1\alpha_2\alpha_3}^{i} =$
$\left(\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\alpha_1\gamma_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\alpha_1\gamma_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\beta_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\alpha_1\gamma_2\gamma_3}^{i} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\alpha_1\gamma_2\alpha_3}^{i}\right) \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_2}^{\gamma_2}$
$+ \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}}\right]_{\alpha_1\gamma_2}^{i} \left[\mathbf{h}_{\mathbf{xx}}^{t}\right]_{\alpha_2\alpha_3}^{\gamma_2}$

$[Q_{20}]_{\alpha_1\alpha_2}^{i} \equiv \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t}\right]_{\alpha_1\alpha_2}^{i}$
$[Q_{20}]_{\alpha_1\alpha_2\alpha_3}^{i} = \left(\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t\mathbf{y}_{t+1}}\right]_{\alpha_1\alpha_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t\mathbf{y}_t}\right]_{\alpha_1\alpha_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\beta_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t\mathbf{x}_{t+1}}\right]_{\alpha_1\alpha_2\gamma_3}^{i} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t\mathbf{x}_t}\right]_{\alpha_1\alpha_2\alpha_3}^{i}\right)$

Thus
$\left[\mathbf{F}_{\mathbf{xxx}}\left(\mathbf{x}_{ss},\sigma\right)\right]_{\alpha_1\alpha_2\alpha_3}^{i} = E_t\left[\sum_{m=1}^{20}\left[Q_m\right]_{\alpha_1\alpha_2\alpha_3}^{i}\right] = 0$

This must hold for
$i = 1, 2, ...n$
$\alpha_1, \alpha_2, \alpha_3, = 1, 2, ..., n_x$

Hence, we have a set of $(n_y + n_x) \times n_x \times n_x \times n_x$ equations in as many unknowns. Note that we use the symmetri of $\mathbf{g}_{\mathbf{xxx}}$ and $\mathbf{h}_{\mathbf{xxx}}$ to only solve for $(n_y + n_x)\begin{pmatrix} n_x \\ 3 \end{pmatrix} = (n_y + n_x)\frac{n_x!}{(n_x-3)!3!} = (n_y + n_x)\frac{n_x!}{(n_x-3)!3!}$

$(n_y + n_x)\, n_x^2 + (n_y + n_x)\begin{pmatrix} n_x \\ 3 \end{pmatrix} =$

$= (n_y + n_x)$
$\left(n_x^2 + \frac{n_x!}{(n_x-3)!3!}\right)$

$= (n_y + n_x)\left(n_x^2 + \frac{n_x(n_x-1)(n_x-2)}{6}\right)$

$= (n_y + n_x)\left(\frac{6n_x^2 + n_x(n_x-1)(n_x-2)}{6}\right)$

$= (n_y + n_x)\left(\frac{6n_x^2 + n_x\left(n_x^2 - n_x - 2n_x + 2\right)}{6}\right)$

$= (n_y + n_x)\left(\frac{6n_x^2 + n_x\left(n_x^2 - 3n_x + 2\right)}{6}\right)$

$$= (n_y + n_x) \left(\frac{n_x \left(n_x^2 + 3n_x + 2 \right)}{6} \right)$$

$$= (n_y + n_x) \left(\frac{n_x (n_x (n_x + 3) + 2)}{6} \right)$$

This formula is implemented in the codes *gxxx_hxxx_gssx_hssx* and its modification with respect to the use of less memory (in *gxxx_hxxx_gssx_hssx_lessMemory*) and less loops (in *gxxx_hxxx_gssx_hssx_lessMemoryLoops*). The formula is also implemented in *g_h_3rd.m*.

For the paper we use the same notation as in Schmitt-Grohé & Uribe (2004), that is we omit the time index on derivatives of $\mathbf{g}$ and $\mathbf{h}$, and we let $\mathbf{y} \equiv \mathbf{y}_t$, $\mathbf{y}' \equiv \mathbf{y}_{t+1}$, $\mathbf{x} \equiv \mathbf{x}_t$, $\mathbf{x}' \equiv \mathbf{x}_{t+1}$. Thus

$$
\begin{aligned}
&[\mathbf{F_{xxx}}(\mathbf{x}_{ss}, \sigma)]^i_{\alpha_1\alpha_2\alpha_3} = \\
&5) \quad + [\mathbf{f_{y'}}]^i_{\beta_1} [\mathbf{g_{xxx}}]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} [\mathbf{h_x}]^{\gamma_3}_{\alpha_3} [\mathbf{h_x}]^{\gamma_2}_{\alpha_2} [\mathbf{h_x}]^{\gamma_1}_{\alpha_1} \\
&6) \quad + [\mathbf{f_{y'}}]^i_{\beta_1} [\mathbf{g_x}]^{\beta_1}_{\gamma_1} [\mathbf{h_{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3} \\
&11) \quad + [\mathbf{f_y}]^i_{\beta_1} [\mathbf{g_{xxx}}]^{\beta_1}_{\alpha_1\alpha_2\alpha_3} \\
&16) \quad + [\mathbf{f_{x'}}]^i_{\gamma_1} [\mathbf{h_{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3} \\
&\qquad + \left[b^1\right]^i_{\alpha_1\alpha_2\alpha_3} = 0
\end{aligned}
$$

We have defined

$$
\begin{aligned}
&\left[b^1\right]^i_{\alpha_1\alpha_2\alpha_3} \equiv \\
&1) \quad + \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&2) \quad + \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \right) \\
&\quad \times [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{xx}}]^{\beta_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&3) \quad + \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3} \right) \\
&\quad \times [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&4) \quad + \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\alpha_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_t}\right]^i_{\beta_1\alpha_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\alpha_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2\alpha_3} \right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}
\end{aligned}
$$

$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

5) $$+\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

6) $$+\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$+$$

7) $$+\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}$$

8) $$+\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}$$

9) $$+\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3}\right)$$
$$\times\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}$$

10) $$+\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{y}_t}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\alpha_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2\alpha_3}\right)\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}$$

11) $$+\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)\left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_2}$$
$$+$$

12) $$+\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3}$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

13) $$+\Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^{i}_{\gamma_1\beta_2\alpha_3}\Big)$$
$$\times\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t}_{\mathbf{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

14) $$+\Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\gamma_1\gamma_2\alpha_3}$$
$$\times\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

15) $$+\Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\alpha_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{y}_t}\right]^{i}_{\gamma_1\alpha_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\alpha_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{x}_t}\right]^{i}_{\gamma_1\alpha_2\alpha_3}\Big)\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\gamma_1\alpha_2}\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

16) $$+\Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\gamma_1\alpha_3}\Big)\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$+$$

17) $$+\Big(\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\alpha_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\alpha_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\alpha_1\beta_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}\mathbf{x}_t}\right]^{i}_{\alpha_1\beta_2\alpha_3}\Big)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$$
$$+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}}\right]^{i}_{\alpha_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$$
$$+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_{t+1}}\right]^{i}_{\alpha_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}$$

18) $$+\Big(\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t\mathbf{y}_{t+1}}\right]^{i}_{\alpha_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t\mathbf{y}_t}\right]^{i}_{\alpha_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t\mathbf{x}_{t+1}}\right]^{i}_{\alpha_1\beta_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t\mathbf{x}_t}\right]^{i}_{\alpha_1\beta_2\alpha_3}\Big)\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}$$
$$+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{y}_t}\right]^{i}_{\alpha_1\beta_2}\left[\mathbf{g}^{t}_{\mathbf{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}$$

19) $$+\Big(\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\alpha_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\alpha_1\gamma_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\alpha_1\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\alpha_1\gamma_2\alpha_3}\Big)\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$$
$$+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_{t+1}}\right]^{i}_{\alpha_1\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}$$

20) $$+\Big(\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t\mathbf{y}_{t+1}}\right]^{i}_{\alpha_1\alpha_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t\mathbf{y}_t}\right]^{i}_{\alpha_1\alpha_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t\mathbf{x}_{t+1}}\right]^{i}_{\alpha_1\alpha_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_t\mathbf{x}_t\mathbf{x}_t}\right]^{i}_{\alpha_1\alpha_2\alpha_3}\Big)$$

Omitting the time index on derivatives of $\mathbf{g}$ and $\mathbf{h}$ and letting $\mathbf{y}\equiv\mathbf{y}_t$, $\mathbf{y}'\equiv\mathbf{y}_{t+1}$,$\mathbf{x}\equiv\mathbf{x}_t$, $\mathbf{x}'\equiv\mathbf{x}_{t+1}$, we get
$\left[b^1\right]^{i}_{\alpha_1\alpha_2\alpha_3}\equiv$

1) $$+\Big(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}}\right]^{i}_{\beta_1\beta_2\alpha_3}\Big)$$
$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

2)
$$
\begin{aligned}
&+\Big([\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{y}'}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{y}}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{x}'}]^i_{\beta_1\beta_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{x}}]^i_{\beta_1\beta_2\alpha_3}\Big)\\
&\times[\mathbf{g}_{\mathbf{x}}]^{\beta_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}'\mathbf{y}}]^i_{\beta_1\beta_2}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_2}_{\alpha_2\alpha_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}'\mathbf{y}}]^i_{\beta_1\beta_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}'\mathbf{y}}]^i_{\beta_1\beta_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\mathbf{x}\mathbf{x}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

3)
$$
\begin{aligned}
&+\Big([\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}'}]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}}]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}'}]^i_{\beta_1\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}}]^i_{\beta_1\gamma_2\alpha_3}\Big)\\
&\times[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}]^i_{\beta_1\gamma_2}[\mathbf{h}_{\mathbf{x}\mathbf{x}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}]^i_{\beta_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}]^i_{\beta_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\mathbf{x}\mathbf{x}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

4)
$$
\begin{aligned}
&+\Big([\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{y}'}]^i_{\beta_1\alpha_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{y}}]^i_{\beta_1\alpha_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{x}'}]^i_{\beta_1\alpha_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{x}}]^i_{\beta_1\alpha_2\alpha_3}\Big)\\
&\times[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}'\mathbf{x}}]^i_{\beta_1\alpha_2}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}'\mathbf{x}}]^i_{\beta_1\alpha_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\mathbf{x}\mathbf{x}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

5)
$$
\begin{aligned}
&+\Big([\mathbf{f}_{\mathbf{y}'\mathbf{y}'}]^i_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}}]^i_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}]^i_{\beta_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}}]^i_{\beta_1\alpha_3}\Big)\\
&\times[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_1}_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&+\\
&+[\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_1}_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{x}\mathbf{x}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_1}_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}_{\mathbf{x}\mathbf{x}}]^{\gamma_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

6)
$$
\begin{aligned}
&+\Big([\mathbf{f}_{\mathbf{y}'\mathbf{y}'}]^i_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}}]^i_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}]^i_{\beta_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}}]^i_{\beta_1\alpha_3}\Big)\\
&\times[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\mathbf{x}\mathbf{x}}]^{\gamma_1}_{\alpha_1\alpha_2}\\
&+[\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{x}\mathbf{x}}]^{\gamma_1}_{\alpha_1\alpha_2}\\
&+
\end{aligned}
$$

7)
$$
\begin{aligned}
&+\Big([\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{y}'}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{y}}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{x}'}]^i_{\beta_1\beta_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{x}}]^i_{\beta_1\beta_2\alpha_3}\Big)\\
&\times[\mathbf{g}_{\mathbf{x}}]^{\beta_2}_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}\mathbf{y}'}]^i_{\beta_1\beta_2}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_2}_{\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}\mathbf{y}'}]^i_{\beta_1\beta_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_2}_{\gamma_2}[\mathbf{h}_{\mathbf{x}\mathbf{x}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}\mathbf{y}'}]^i_{\beta_1\beta_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_2}_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

8)
$$
\begin{aligned}
&+\Big([\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{y}'}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{y}}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{x}'}]^i_{\beta_1\beta_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{x}}]^i_{\beta_1\beta_2\alpha_3}\Big)\\
&\times[\mathbf{g}_{\mathbf{x}}]^{\beta_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}\mathbf{y}}]^i_{\beta_1\beta_2}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_2}_{\alpha_2\alpha_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\alpha_1}\\
&+[\mathbf{f}_{\mathbf{y}\mathbf{y}}]^i_{\beta_1\beta_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_2}_{\alpha_2}[\mathbf{g}_{\mathbf{x}\mathbf{x}}]^{\beta_1}_{\alpha_1\alpha_3}
\end{aligned}
$$

9)
$$
+\Big([\mathbf{f}_{\mathbf{y}\mathbf{x}'\mathbf{y}'}]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}\mathbf{x}'\mathbf{y}}]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}\mathbf{x}'\mathbf{x}'}]^i_{\beta_1\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}\mathbf{x}'\mathbf{x}}]^i_{\beta_1\gamma_2\alpha_3}\Big)
$$

$$\times \left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g_x}\right]^{\beta_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{yx'}}\right]^{i}_{\beta_1\gamma_2} \left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g_x}\right]^{\beta_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{yx'}}\right]^{i}_{\beta_1\gamma_2} \left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g_{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}$$

10)
$$+ \left(\left[\mathbf{f_{yxy'}}\right]^{i}_{\beta_1\alpha_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{yxy}}\right]^{i}_{\beta_1\alpha_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f_{yxx'}}\right]^{i}_{\beta_1\alpha_2\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{yxx}}\right]^{i}_{\beta_1\alpha_2\alpha_3} \right) \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{yx}}\right]^{i}_{\beta_1\alpha_2} \left[\mathbf{g_{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}$$

11)
$$+ \left(\left[\mathbf{f_{yy'}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{yy}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f_{yx'}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{yx}}\right]^{i}_{\beta_1\alpha_3} \right) \left[\mathbf{g_{xx}}\right]^{\beta_1}_{\alpha_1\alpha_2}$$
$$+$$

12)
$$+ \left(\left[\mathbf{f_{x'y'y'}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{x'y'y}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f_{x'y'x'}}\right]^{i}_{\gamma_1\beta_2\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{x'y'x}}\right]^{i}_{\gamma_1\beta_2\alpha_3} \right)$$
$$\times \left[\mathbf{g_x}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{x'y'}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g_{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{x'y'}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g_x}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{x'y'}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g_x}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

13)
$$+ \left(\left[\mathbf{f_{x'yy'}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{x'yy}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f_{x'yx'}}\right]^{i}_{\gamma_1\beta_2\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{x'yx}}\right]^{i}_{\gamma_1\beta_2\alpha_3} \right)$$
$$\times \left[\mathbf{g_x}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{x'y}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g_{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3} \left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{x'y}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g_x}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

14)
$$+ \left(\left[\mathbf{f_{x'x'y'}}\right]^{i}_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{x'x'y}}\right]^{i}_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f_{x'x'x'}}\right]^{i}_{\gamma_1\gamma_2\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{x'x'x}}\right]^{i}_{\gamma_1\gamma_2\alpha_3} \right)$$
$$\times \left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{x'x'}}\right]^{i}_{\gamma_1\gamma_2} \left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{x'x'}}\right]^{i}_{\gamma_1\gamma_2} \left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

15)
$$+ \left(\left[\mathbf{f_{x'xy'}}\right]^{i}_{\gamma_1\alpha_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{x'xy}}\right]^{i}_{\gamma_1\alpha_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f_{x'xx'}}\right]^{i}_{\gamma_1\alpha_2\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{x'xx}}\right]^{i}_{\gamma_1\alpha_2\alpha_3} \right) \left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+ \left[\mathbf{f_{x'x}}\right]^{i}_{\gamma_1\alpha_2} \left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

16)
$$+ \left(\left[\mathbf{f_{x'y'}}\right]^{i}_{\gamma_1\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{x'y}}\right]^{i}_{\gamma_1\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f_{x'x'}}\right]^{i}_{\gamma_1\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{x'x}}\right]^{i}_{\gamma_1\alpha_3} \right) \left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$+$$

17)
$$+ \left(\left[\mathbf{f_{xy'y'}}\right]^{i}_{\alpha_1\beta_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{xy'y}}\right]^{i}_{\alpha_1\beta_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f_{xy'x'}}\right]^{i}_{\alpha_1\beta_2\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{xy'x}}\right]^{i}_{\alpha_1\beta_2\alpha_3} \right)$$
$$\times \left[\mathbf{g_x}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}$$
$$+ \left[\mathbf{f_{xy'}}\right]^{i}_{\alpha_1\beta_2} \left[\mathbf{g_{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}$$
$$+ \left[\mathbf{f_{xy'}}\right]^{i}_{\alpha_1\beta_2} \left[\mathbf{g_x}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}$$

18)
$$+ \left(\left[\mathbf{f_{xyy'}}\right]^{i}_{\alpha_1\beta_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{xyy}_t}\right]^{i}_{\alpha_1\beta_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{xyx}_{t+1}}\right]^{i}_{\alpha_1\beta_2\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{xyx}_t}\right]^{i}_{\alpha_1\beta_2\alpha_3} \right) \left[\mathbf{g_x}\right]^{\beta_2}_{\alpha_2}$$
$$+ \left[\mathbf{f_{xy}}\right]^{i}_{\alpha_1\beta_2} \left[\mathbf{g_{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}$$

19)
$$+ \left(\left[\mathbf{f_{xx'y'}}\right]^{i}_{\alpha_1\gamma_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{xx'y}}\right]^{i}_{\alpha_1\gamma_2\beta_3} \left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f_{xx'x'}}\right]^{i}_{\alpha_1\gamma_2\gamma_3} \left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f_{xx'x}}\right]^{i}_{\alpha_1\gamma_2\alpha_3} \right) \left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}$$

$+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}'}\right]^i_{\alpha_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}$

20) $\quad +\Big(\left[\mathbf{f}_{\mathbf{x}\mathbf{x}\mathbf{y}'}\right]^i_{\alpha_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}\mathbf{y}}\right]^i_{\alpha_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}\mathbf{x}'}\right]^i_{\alpha_1\alpha_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}\mathbf{x}}\right]^i_{\alpha_1\alpha_2\alpha_3}\Big)$

$\Updownarrow$

$\left[b^1\right]^i_{\alpha_1\alpha_2\alpha_3}\equiv$

1) $\quad +\Big(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}'}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}'}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}}\right]^i_{\beta_1\beta_2\alpha_3}\Big)$

$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^i_{\beta_1\beta_2}(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$\qquad +\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_3})$

2) $\quad +\Big(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{y}'}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{y}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{x}'}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{x}}\right]^i_{\beta_1\beta_2\alpha_3}\Big)$

$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}}\right]^i_{\beta_1\beta_2}\Big(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\Big)$

3) $\quad +\Big(\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}'}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}'}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}}\right]^i_{\beta_1\gamma_2\alpha_3}\Big)$

$\times\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^i_{\beta_1\gamma_2}\Big(\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\Big)$

4) $\quad +\Big(\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{y}'}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{y}}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{x}'}\right]^i_{\beta_1\alpha_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{x}}\right]^i_{\beta_1\alpha_2\alpha_3}\Big)$

$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}}\right]^i_{\beta_1\alpha_2}\Big(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\Big)$

5) $\quad +\Big(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}}\right]^i_{\beta_1\alpha_3}\Big)$

$\times\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$

$+\left[\mathbf{f}_{\mathbf{y}'}\right]^i_{\beta_1}\Big(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\Big)$

6) $\quad +\Big(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}}\right]^i_{\beta_1\alpha_3}\Big)$

$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$

$+\left[\mathbf{f}_{\mathbf{y}'}\right]^i_{\beta_1}\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$

7) $\quad +\Big(\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{y}'}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{y}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{x}'}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{x}}\right]^i_{\beta_1\beta_2\alpha_3}\Big)$

$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$

$+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'}\right]^i_{\beta_1\beta_2}\Big(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\alpha_1\alpha_3}\Big)$

8) $\quad +\Big(\left[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{y}'}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{y}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{x}'}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{x}}\right]^i_{\beta_1\beta_2\alpha_3}\Big)$

$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$

$+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}}\right]^i_{\beta_1\beta_2}\Big(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\alpha_1\alpha_3}\Big)$

9) $+\left(\left[\mathbf{f_{yx'y'}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x^t}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{yx'y}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{yx'x'}}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{yx'x}}\right]^i_{\beta_1\gamma_2\alpha_3}\right)$
$\times\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g_x}\right]^{\beta_1}_{\alpha_1}$
$+\left[\mathbf{f_{yx'}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g_x}\right]^{\beta_1}_{\alpha_1}+\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g_{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}\right)$

10) $+\left(\left[\mathbf{f_{yxy'}}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{yxy}}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{yxx'}}\right]^i_{\beta_1\alpha_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{yxx}}\right]^i_{\beta_1\alpha_2\alpha_3}\right)\left[\mathbf{g_x^t}\right]^{\beta_1}_{\alpha_1}$
$+\left[\mathbf{f_{yx}}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g_{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}$

11) $+\left(\left[\mathbf{f_{yy'}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{yy}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{yx'}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{yx}}\right]^i_{\beta_1\alpha_3}\right)\left[\mathbf{g_{xx}}\right]^{\beta_1}_{\alpha_1\alpha_2}$

12) $+\left(\left[\mathbf{f_{x'y'y'}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{x'y'y}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{x'y'x'}}\right]^i_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{x'y'x}}\right]^i_{\gamma_1\beta_2\alpha_3}\right)$
$\times\left[\mathbf{g_x}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$
$+\left[\mathbf{f_{x'y'}}\right]^i_{\gamma_1\beta_2}\left(\left[\mathbf{g_{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g_x}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g_x}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\right)$

13) $+\left(\left[\mathbf{f_{x'yy'}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{x'yy}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{x'yx'}}\right]^i_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{x'yx}}\right]^i_{\gamma_1\beta_2\alpha_3}\right)$
$\times\left[\mathbf{g_x}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$
$+\left[\mathbf{f_{x'y}}\right]^i_{\gamma_1\beta_2}\left(\left[\mathbf{g_{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g_x}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\right)$

14) $+\left(\left[\mathbf{f_{x'x'y'}}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{x'x'y}}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{x'x'x'}}\right]^i_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{x'x'x}}\right]^i_{\gamma_1\gamma_2\alpha_3}\right)$
$\times\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$
$+\left[\mathbf{f_{x'x'}}\right]^i_{\gamma_1\gamma_2}\left(\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\right)$

15) $+\left(\left[\mathbf{f_{x'xy'}}\right]^i_{\gamma_1\alpha_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{x'xy}}\right]^i_{\gamma_1\alpha_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{x'xx'}}\right]^i_{\gamma_1\alpha_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{x'xx}}\right]^i_{\gamma_1\alpha_2\alpha_3}\right)\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$
$+\left[\mathbf{f_{x'x}}\right]^i_{\gamma_1\alpha_2}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$

16) $+\left(\left[\mathbf{f_{x'y'}}\right]^i_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{x'y}}\right]^i_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{x'x'}}\right]^i_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{x'x}}\right]^i_{\gamma_1\alpha_3}\right)\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$

17) $+\left(\left[\mathbf{f_{xy'y'}}\right]^i_{\alpha_1\beta_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{xy'y}}\right]^i_{\alpha_1\beta_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{xy'x'}}\right]^i_{\alpha_1\beta_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{xy'x}}\right]^i_{\alpha_1\beta_2\alpha_3}\right)$
$\times\left[\mathbf{g_x}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}$
$+\left[\mathbf{f_{xy'}}\right]^i_{\alpha_1\beta_2}\left(\left[\mathbf{g_{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}+\left[\mathbf{g_x}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\right)$

18) $+\left(\left[\mathbf{f_{xyy'}}\right]^i_{\alpha_1\beta_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{xyy}}_t\right]^i_{\alpha_1\beta_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{xyx}}_{t+1}\right]^i_{\alpha_1\beta_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{xyx}}_t\right]^i_{\alpha_1\beta_2\alpha_3}\right)\left[\mathbf{g_x}\right]^{\beta_2}_{\alpha_2}$
$+\left[\mathbf{f_{xy}}\right]^i_{\alpha_1\beta_2}\left[\mathbf{g_{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}$

19) $+\left(\left[\mathbf{f_{xx'y'}}\right]^i_{\alpha_1\gamma_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{xx'y}}\right]^i_{\alpha_1\gamma_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{xx'x'}}\right]^i_{\alpha_1\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{xx'x}}\right]^i_{\alpha_1\gamma_2\alpha_3}\right)\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}$
$+\left[\mathbf{f_{xx'}}\right]^i_{\alpha_1\gamma_2}\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}$

20) $+\left(\left[\mathbf{f_{xxy'}}\right]^i_{\alpha_1\alpha_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{xxy}}\right]^i_{\alpha_1\alpha_2\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f_{xxx'}}\right]^i_{\alpha_1\alpha_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f_{xxx}}\right]^i_{\alpha_1\alpha_2\alpha_3}\right)$
for $i=1,2,...,n$ and $\alpha_1,\alpha_2,\alpha_3=1,2,...,n_x$.

4.2 With respect to $(\sigma, \sigma, \mathbf{x}_t)$

First recall that

$$
\begin{aligned}
&\left[\mathbf{F}_{\sigma\sigma}\left(\mathbf{x}_{ss},\sigma\right)\right]^i = E_t\{[\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)]\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}\\
&\\
&\quad+[\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)]\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\right)\\
&\\
&\quad+[\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)]\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\\
&\\
&\quad+[\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\\
&\quad+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)]\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&\quad+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}\right]^{\gamma_1}\}
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
&\left[\mathbf{F}_{\sigma\sigma}\left(\mathbf{x}_{ss},\sigma\right)\right]^i = E_t\{\\
&\quad+[\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\\
&\qquad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)]\\
&\quad\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}\\
&\\
&\quad+[\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\\
&\qquad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)]\\
&\quad\times\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\right)\\
&\\
&\quad+[\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\\
&\qquad+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)]\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&\quad+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\\
&\\
&\quad+[\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}
\end{aligned}
$$

$$
\begin{aligned}
&\quad\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)] \\
&\quad \times \left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}\right]^{\gamma_1}\}
\end{aligned}
$$

$\Updownarrow$

$$
\begin{aligned}
&\left[\mathbf{F}_{\sigma\sigma}\left(\mathbf{x}_{ss},\sigma\right)\right]^i = E_t\{ \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1} \\
&\\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\right) \\
&\\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1} \\
&\\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}\right]^{\gamma_1}\}
\end{aligned}
$$

$\Updownarrow$

$$
\begin{aligned}
&\left[\mathbf{F}_{\sigma\sigma}\left(\mathbf{x}_{ss},\sigma\right)\right]^i = E_t\{ \\
&\quad + \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1} \\
&\\
&\quad + \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\\
\\
&+\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\\
\\
&+\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\gamma_1}\}
\end{aligned}
$$

$\Updownarrow$

$$
\begin{aligned}
&\left[\mathbf{F}_{\sigma\sigma}\left(\mathbf{x}_{ss},\sigma\right)\right]^i=E_t\{\\
&1)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&2)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&3)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&4)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&5)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&6)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&7)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}\\
\\
&8)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&9)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&10)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&11)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&12)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&13)\quad+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\\
\\
&14)\quad+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&15)\quad+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&16)\quad+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&17)\quad+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&18)\quad+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}
\end{aligned}
$$

19) $\qquad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

20) $\qquad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

21) $\qquad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

22) $\qquad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

23) $\qquad + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1} \left[\mathbf{h}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\gamma_1} \}$

Hence, we let

$[P_1]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_2]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_3]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_4]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_5]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_6]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_7]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}$

$[P_8]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_9]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_{10}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_{11}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_{12}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_{13}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_{14}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_{15}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_{16}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_{17}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$$[P_{18}]^i \equiv [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}]^{\beta_1}$$

$$[P_{19}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_{20}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_{21}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_{22}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_{23}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}$$

We now compute the derivative of all these terms with respect to $\mathbf{x}_t$

$$\begin{aligned}
[P_1]^i_{\alpha_3} = &\\
&\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\Big)\\
&\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\sigma\mathbf{x}}]^{\gamma_2}_{\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}
\end{aligned}$$

$$\begin{aligned}
[P_2]^i_{\alpha_3} = &\\
&\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\Big)\\
&\times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}
\end{aligned}$$

$$\begin{aligned}
[P_3]^i_{\alpha_3} = &\\
&\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\Big)\\
&\times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}
\end{aligned}$$

$$\begin{aligned}
[P_4]^i_{\alpha_3} = &\\
&\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3}\Big)\\
&\times \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_5]^{i}_{\alpha_3} = \\
&\quad \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_3}\right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_6]^{i}_{\alpha_3} = \\
&\quad \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_3}\right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_7]^{i}_{\alpha_3} = \\
&\quad \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_3}\right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^{t}_{\sigma\sigma}\right]^{\gamma_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{h}^{t}_{\sigma\sigma}\right]^{\gamma_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^{t}_{\sigma\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_8]^{i}_{\alpha_3} = \\
&\quad \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\beta_2\alpha_3}\right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_9]^{i}_{\alpha_3} = \\
&\quad \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\beta_2\alpha_3}\right) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}
\end{aligned}
$$

$$
[P_{10}]^{i}_{\alpha_3} =
$$

$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}$$

$$[P_{11}]^i_{\alpha_3}=$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3}\right)$$
$$\times\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}$$

$$[P_{12}]^i_{\alpha_3}=\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$$

$$[P_{13}]^i_{\alpha_3}=\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}$$

$$[P_{14}]^i_{\alpha_3}=$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}$$

$$[P_{15}]^i_{\alpha_3}=$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}$$

$$[P_{16}]^i_{\alpha_3}=$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}$$

$$
\begin{aligned}
&[P_{17}]^i_{\alpha_3} = \\
&\quad \Big(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3} \Big) \\
&\quad \times \Big([\mathbf{h}^t_{\sigma}]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \Big) [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\sigma\mathbf{x}}]^{\gamma_2}_{\alpha_3} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
&\quad + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \Big([\mathbf{h}^t_{\sigma}]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \Big) [\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{18}]^i_{\alpha_3} = \\
&\quad \Big(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}]^i_{\beta_1\alpha_3} \Big) [\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1} \\
&\quad + [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{19}]^i_{\alpha_3} = \\
&\quad \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3} \Big) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \Big([\mathbf{h}^t_{\sigma}]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \Big) \Big([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \Big) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \Big([\mathbf{h}^t_{\sigma}]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \Big) \Big([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \Big) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\sigma\mathbf{x}}]^{\gamma_2}_{\alpha_3} \Big([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \Big) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \Big([\mathbf{h}^t_{\sigma}]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \Big) [\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{20}]^i_{\alpha_3} = \\
&\quad \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3} \Big) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \Big([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \Big) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \Big([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \Big) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} [\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{21}]^i_{\alpha_3} = \\
&\quad \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3} \Big) \\
&\quad \times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \Big([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \Big) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \Big([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \Big) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} [\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{22}]^i_{\alpha_3} = \\
&\quad \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\gamma_2\alpha_3} \Big) \\
&\quad \times \Big([\mathbf{h}^t_{\sigma}]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \Big) \Big([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \Big) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} [\mathbf{h}^t_{\sigma\mathbf{x}}]^{\gamma_2}_{\alpha_3} \Big([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \Big) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} \Big([\mathbf{h}^t_{\sigma}]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \Big) [\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{23}]^i_{\alpha_3} = \\
&\quad \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3} \Big) [\mathbf{h}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\gamma_1}
\end{aligned}
$$

$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1}\left[\mathbf{h}^{t}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$$

Thus
$\left[\mathbf{F}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\left(\mathbf{x}_{ss},\sigma\right)\right]^{i}_{\alpha_3}=E_t\left[\sum_{m=1}^{23}\left[P_m\right]^{i}_{\alpha_3}\right]=0$

This must hold for
$i=1,2,...n$
$\alpha_3=1,2,...,n_x$

Hence, we need to evaluate $E_t\left[\left[P_m\right]^{i}_{\alpha_3}\right]$ in the deterministic steady state. Here, we can use the previous results that $\mathbf{h}_{\sigma}=\mathbf{0}$, $\mathbf{g}_{\boldsymbol{\sigma}}=\mathbf{0}$, $\mathbf{h}_{\mathbf{x}\sigma}=\mathbf{0},$ and $\mathbf{g}_{\mathbf{x}\sigma}=\mathbf{0}$. Moreover we have $E_t\left[\boldsymbol{\epsilon}_{t+1}\right]=0$.

1) For $\left[P_1\right]^{i}_{\alpha_3}$
$E_t\left[\left[P_1\right]^{i}_{\alpha_3}\right]=E_t\{$

$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$=\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\beta_2\alpha_3}\right)$$
$$\times E_t\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}E_t\left(\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+0$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}E_t\left(\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+0$$

$$=\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$

2) For $\left[P_2\right]^{i}_{\alpha_3}$
$E_t\left[\left[P_2\right]^{i}_{\alpha_3}\right]=E_t\{$

$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$

$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$= E_t\{$$
$$+0$$
$$+0$$
$$+0$$
$$+0\}$$

$$= 0$$

3) For $[P_3]^{i}_{\alpha_3}$

$$E_t\left[[P_3]^{i}_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^{i}_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$= E_t\{$$
$$+0$$
$$+0$$
$$+0$$
$$+0\}$$

$$= 0$$

4) For $[P_4]^{i}_{\alpha_3}$

$$E_t\left[[P_4]^{i}_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\beta_1\gamma_2\alpha_3}\right)$$
$$\times\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}\left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}\left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$=\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\beta_1\gamma_2\alpha_3}\right)$$
$$\times E_t\left[\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right]$$
$$+0$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}E_t\left[\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right]$$
$$+0$$

$$=\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\beta_1\gamma_2\alpha_3}\right)$$

$$\times [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$$

5) For $[P_5]^{i}_{\alpha_3}$

$$E_t\left[[P_5]^{i}_{\alpha_3}\right] = \{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_3}\right)$$
$$\times \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$= \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_3}\right)$$
$$\times \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} E_t\left[[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right]$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} E_t\left[[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right]$$
$$+0$$
$$+0$$

$$= \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_3}\right)$$
$$\times \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$$

6) For $[P_6]^{i}_{\alpha_3}$

$$E_t\left[[P_6]^{i}_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_3}\right)$$
$$\times \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$=$$
$$0$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} E_t [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}$$
$$0$$

$$= 0$$

7) For $[P_7]^{i}_{\alpha_3}$

$$E_t\left[[P_7]^{i}_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_3}\right)$$

$$\times \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} \left[\mathbf{h}_{\sigma\sigma}^{t}\right]^{\gamma_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]_{\beta_1}^{i} \left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} \left[\mathbf{h}_{\sigma\sigma}^{t}\right]^{\gamma_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]_{\beta_1}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} \left[\mathbf{h}_{\sigma\sigma\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_1}\}$$

$$= \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]_{\beta_1\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}^{i} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]_{\beta_1\alpha_3}^{i}\right)$$
$$\times \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} \left[\mathbf{h}_{\sigma\sigma}^{t}\right]^{\gamma_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]_{\beta_1}^{i} \left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} \left[\mathbf{h}_{\sigma\sigma}^{t}\right]^{\gamma_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]_{\beta_1}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} \left[\mathbf{h}_{\sigma\sigma\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_1}$$

8) For $[P_8]_{\alpha_3}^{i}$

$$E_t\left[[P_8]_{\alpha_3}^{i}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]_{\beta_1\beta_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\beta_2\gamma_3}^{i} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]_{\beta_1\beta_2\alpha_3}^{i}\right)$$
$$\times \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} \left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_2\gamma_3}^{\beta_2} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} \left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} \left[\mathbf{h}_{\sigma\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_2} \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} \left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \left[\mathbf{g}_{\boldsymbol{\sigma}\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_1} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3}\}$$

$$= 0$$
$$+0$$
$$+0$$
$$+0\}$$

$$= 0$$

9) For $[P_9]_{\alpha_3}^{i}$

$$E_t\left[[P_9]_{\alpha_3}^{i}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]_{\beta_1\beta_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\beta_2\gamma_3}^{i} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]_{\beta_1\beta_2\alpha_3}^{i}\right)$$
$$\times \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2} \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\boldsymbol{\sigma}\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_2} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3} \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2} \left[\mathbf{g}_{\boldsymbol{\sigma}\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_1} \left[\mathbf{h}_{\mathbf{x}}^{t}\right]_{\alpha_3}^{\gamma_3}\}$$

$$= 0$$
$$+0$$
$$+0\}$$

$$= 0$$

10) For $[P_{10}]_{\alpha_3}^{i}$

$$E_t\left[[P_{10}]_{\alpha_3}^{i}\right] = E_t\{$$

$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\}$$

$$= E_t\{$$
$$0$$
$$+0$$
$$+0\}$$

$$= 0$$

11) For $[P_{11}]^i_{\alpha_3}$

$$E_t\left[[P_{11}]^i_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3}\right)$$
$$\times\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\}$$

$$= 0$$
$$+0$$
$$+0\}$$

$$= 0$$

12) For $[P_{12}]^i_{\alpha_3}$

$$E_t\left[[P_{12}]^i_{\alpha_3}\right] = E_t\{\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$= 0$$
$$+0$$
$$+0$$

$$= 0$$

13) For $[P_{13}]^i_{\alpha_3}$

$$E_t\left[[P_{13}]^i_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\}$$

$$= \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3} \right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}$$

14) For $[P_{14}]^i_{\alpha_3}$

$$E\left[[P_{14}]^i_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \right)$$
$$\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2} \right) \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2} \right) \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2} \right) \left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}\}$$

$$= 0$$
$$+0$$
$$+0$$
$$+0$$

$$= 0$$

15) For $[P_{15}]^i_{\alpha_3}$

$$E_t\left[[P_{15}]^i_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \right)$$
$$\times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}\}$$

$$= 0$$
$$+0$$
$$+0$$

$$= 0$$

16) For $[P_{16}]^i_{\alpha_3}$

$$E_t\left[[P_{16}]^i_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \right)$$
$$\times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}\}$$

$$= 0$$
$$+0$$

$+0$

$= 0$

17) For $[P_{17}]^i_{\alpha_3}$

$$E_t\left[[P_{17}]^i_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3}\right)$$
$$\times\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}\}$$

$= 0$
$+0$
$+0$

$= 0$

18) For $[P_{18}]^i_{\alpha_3}$

$$E_t\left[[P_{18}]^i_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}\}$$

$$= \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}$$

19) For $[P_{19}]^i_{\alpha_3}$

$$E_t\left[[P_{19}]^i_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$= \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}E_t\left[[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right]$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}E_t\left[[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right]$$
$+0$
$+0$

$$= \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\beta_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\gamma_1\beta_2\alpha_3} \right)$$
$$\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$$

20) For $[P_{20}]^{i}_{\alpha_3}$

$$E_t\left[[P_{20}]^{i}_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\beta_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\gamma_1\beta_2\alpha_3} \right)$$
$$\times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{h}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$= 0$$
$$+0$$
$$+0$$

$$= 0$$

21) For $[P_{21}]^{i}_{\alpha_3}$

$$E_t\left[[P_{21}]^{i}_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t}\mathbf{y}_{t}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\beta_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t}\mathbf{x}_{t}}\right]^{i}_{\gamma_1\beta_2\alpha_3} \right)$$
$$\times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{h}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$= 0$$
$$+0$$
$$+0$$

$$= 0$$

22) For $[P_{22}]^{i}_{\alpha_3}$

$$E_t\left[[P_{22}]^{i}_{\alpha_3}\right] = E_t\{$$
$$\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\gamma_1\gamma_2\alpha_3} \right)$$
$$\times \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2} \right) \left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2} \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_2}_{\alpha_3} \left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2} \right) \left[\mathbf{h}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$= \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\gamma_1\gamma_2\alpha_3} \right)$$

$\times [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} E_t \left[[\boldsymbol{\epsilon}_{t+1}]^{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \right]$
$+0$
$+0$

$= \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\gamma_2\alpha_3} \Big)$
$\times [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$

23) For $[P_{23}]^i_{\alpha_3}$
$E_t \left[[P_{23}]^i_{\alpha_3} \right] = E_t \{$
$\Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3} \Big) [\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}$
$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3} \}$

$= \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3} \Big) [\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}$
$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3}$

Thus, we have
$[\mathbf{F}_{\boldsymbol{\sigma\sigma}\mathbf{x}}(\mathbf{x}_{ss}, \sigma)]^i_{\alpha_3} = E_t \left[\sum_{m=1}^{23} [P_m]^i_{\alpha_3} \right] = 0$

$\Updownarrow$
1) $\Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \Big)$
$\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$
$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$
$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$

4)$+ \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3} \Big)$
$\times [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$
$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$

5)$+ \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3} \Big)$
$\times \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$
$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$

7) $+ \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3} \Big)$
$\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}$
$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}$
$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}$

13) $+ \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3} \Big) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}$

$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}$$

$$18) + \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^{i}_{\beta_1\alpha_3}\right) \left[\mathbf{g}^{t}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_t}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}$$

$$19) + \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\beta_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^{i}_{\gamma_1\beta_2\alpha_3}\right)$$
$$\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$

$$22) + \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\gamma_1\gamma_2\alpha_3}\right)$$
$$\times \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$

$$23) + \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\gamma_1\alpha_3}\right) \left[\mathbf{h}^{t}_{\boldsymbol{\sigma\sigma}}\right]^{\gamma_1}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1} \left[\mathbf{h}^{t}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$$

$$= 0$$

The numbers to the left indicate which term the expressions come from.

This formula is implemented in the codes $gxxx\_hxxx\_gssx\_hssx$ and its modification with respect to the use of less memory (in $gxxx\_hxxx\_gssx\_hssx\_lessMemory$) and less loops (in $gxxx\_hxxx\_gssx\_hssx\_lessMemoryLoops$). The formula is also implemented in $g\_h\_3rd.m$.

We now use the symmetry in the derivatives due to Young's teorem to simplify the expression for $\left[\mathbf{F}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\left(\mathbf{x}_{ss},\sigma\right)\right]^{i}_{\alpha_3}$. Thus

$$1) \quad \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^{i}_{\beta_1\beta_2\alpha_3}\right)$$
$$\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$4) + \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\beta_1\gamma_2\alpha_3}\right)$$
$$\times \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$5) + \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^{i}_{\beta_1\alpha_3}\right)$$
$$\times \left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\mathbf{I}\right]^{\phi_1}_{\phi_2} + \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^{t}_{\sigma\sigma}\right]^{\gamma_1} + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2} \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$7) +$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{h}^{t}_{\sigma\sigma}\right]^{\gamma_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^{t}_{\sigma\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$$
$$13) +$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}$$
$$18) + \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^{i}_{\beta_1\beta_3} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^{i}_{\beta_1\alpha_3}\right) \left[\mathbf{g}^{t}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}$$

$+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}$

$19)+\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3}\right)$

$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$

$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$

$22)+\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\gamma_2\alpha_3}\right)$

$\times\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$

$23)+\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3}\right)\left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}\right]^{\gamma_1}$

$+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$

$=0$

reducing some terms in 7) and 13) into 5)

$\Updownarrow$

$1)\quad\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}\right)$

$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$

$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}+\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}\right)$

$+$

$4)+\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3}\right)$

$\times\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$

$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$

$5)+\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)$

$\times\left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}+\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\right)$

$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$

$7)+$

$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}$

$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$

$13)+$

$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}$

$18)+\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)\left[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}$

$+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}$

$19)+\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3}\right)$

$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$

$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$

$22)+\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\gamma_2\alpha_3}\right)$

$\times\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$

$23)+\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3}\right)\left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}\right]^{\gamma_1}$

$+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$

$=0$

reducing terms in 1)

$\Updownarrow$

$$
\begin{aligned}
1)\quad & \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \Big) \\
& \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \Big(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} + \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \Big) \\
& + \\
4)+ & \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3} \Big) \\
& \times [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \\
5)+ & \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3} \Big) \\
& \times \Big(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} + \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1} \Big) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \Big(\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} + \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} \Big) \\
7)\ + & \\
& + \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3} \\
13)\ + & \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \\
18)\ + & \Big(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}]^i_{\beta_1\alpha_3} \Big) [\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_3} \\
19)\ + & \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3} \Big) \\
& \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \\
22)\ + & \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\gamma_2\alpha_3} \Big) \\
& \times [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \\
23)\ + & \Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3} \Big) [\mathbf{h}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\gamma_1} \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3} \\
& = 0
\end{aligned}
$$

simplifying some terms from 7) into 5)

$\Updownarrow$

$$
\begin{aligned}
1)\quad & \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \Big) \\
& \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \Big(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} + \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \Big) \\
& + \\
4)+2 & \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3} \Big) \\
& \times [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \\
5)+ & \Big(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3} \Big)
\end{aligned}
$$

$\times \left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} + \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1} \right)$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left(\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} + \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} \right)$

7) $+$

$+$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}$

13) $+$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}$

18) $+ \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}]^i_{\beta_1\alpha_3} \right) [\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}]^{\beta_1}$

$+ [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_3}$

19) $+$

$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$

22) $+ \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\gamma_2\alpha_3} \right)$

$\times [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$

23) $+ \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3} \right) [\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}$

$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3}$

$= 0$

the first term of 19) is equivalent to the first term of 4)

$\Updownarrow$

1) $\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3} \right)$

$\times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} + \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} \right)$

4) $+2 \Bigg(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3}$

$\times [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$

$+2 \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$

5)$+ \left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3} \right)$

$\times \left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} + \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1} \right)$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left(\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2} + \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} \right)$

7) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}$

13) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}$

18) $+ \left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}]^i_{\beta_1\alpha_3} \right) [\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}]^{\beta_1}$

$+ [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_3}$

22) $+ \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\gamma_2\alpha_3} \right)$

$\times [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\mathbf{I}]^{\phi_1}_{\phi_2}$

23) $+ \left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3} \right) [\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}$

$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3}$

$= 0$

the second term of 4) is the same as the second term of 19)

For the paper we use the same notation as in Schmitt-Grohé & Uribe (2004), that is we omit the time index on derivatives of $\mathbf{g}$ and $\mathbf{h}$, and we let $\mathbf{y}\equiv\mathbf{y}_t$, $\mathbf{y}'\equiv\mathbf{y}_{t+1}$,$\mathbf{x}\equiv\mathbf{x}_t$, $\mathbf{x}'\equiv\mathbf{x}_{t+1}$. Thus

$$
\begin{aligned}
&[\mathbf{F}_{\boldsymbol{\sigma\sigma}\mathbf{x}}(\mathbf{x}_{ss},\sigma)]^i_{\alpha_3} = \\
&1)\quad \left([\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}'}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}'}]^i_{\beta_1\beta_2\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}}]^i_{\beta_1\beta_2\alpha_3}\right)\\
&\quad\times[\mathbf{g}_\mathbf{x}]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_\mathbf{x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}\\
&\quad+[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}]^i_{\beta_1\beta_2}\left([\mathbf{g}_{\mathbf{xx}}]^{\beta_2}_{\gamma_2\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_\mathbf{x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}+[\mathbf{g}_\mathbf{x}]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_{\mathbf{xx}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}\right)\\
&4)\quad +2\left([\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}'}]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}}]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}'}]^i_{\beta_1\gamma_2\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}}]^i_{\beta_1\gamma_2\alpha_3}\right)\\
&\quad\times[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_\mathbf{x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}\\
&\quad+2\,[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}]^i_{\beta_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_{\mathbf{xx}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}\\
&5)+\left([\mathbf{f}_{\mathbf{y}'\mathbf{y}'}]^i_{\beta_1\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}}]^i_{\beta_1\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}]^i_{\beta_1\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}}]^i_{\beta_1\alpha_3}\right)\\
&\quad\times\left([\mathbf{g}_{\mathbf{xx}}]^{\beta_1}_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}+[\mathbf{g}_\mathbf{x}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}+[\mathbf{g}_{\boldsymbol{\sigma\sigma}}]^{\beta_1}\right)\\
&\quad+[\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}\left([\mathbf{g}_{\mathbf{xxx}}]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}+[\mathbf{g}_{\mathbf{xx}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}\right)\\
&7)\ +[\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}[\mathbf{g}_\mathbf{x}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}\\
&13)\ +[\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}[\mathbf{g}_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\beta_1}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}\\
&18)\ +\left([\mathbf{f}_{\mathbf{yy}'}]^i_{\beta_1\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{yy}}]^i_{\beta_1\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{yx}'}]^i_{\beta_1\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{yx}}]^i_{\beta_1\alpha_3}\right)[\mathbf{g}_{\boldsymbol{\sigma\sigma}}]^{\beta_1}\\
&\quad+[\mathbf{f}_\mathbf{y}]^i_{\beta_1}[\mathbf{g}_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_3}\\
&22)\ +\left([\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{y}'}]^i_{\gamma_1\gamma_2\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{y}}]^i_{\gamma_1\gamma_2\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}'}]^i_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}}]^i_{\gamma_1\gamma_2\alpha_3}\right)\\
&\quad\times[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}\\
&23)\ +\left([\mathbf{f}_{\mathbf{x}'\mathbf{y}'}]^i_{\gamma_1\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{y}}]^i_{\gamma_1\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'}]^i_{\gamma_1\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}}]^i_{\gamma_1\alpha_3}\right)[\mathbf{h}_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}\\
&\quad+[\mathbf{f}_{\mathbf{x}'}]^i_{\gamma_1}[\mathbf{h}_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3}\\
&\quad=0
\end{aligned}
$$

$$
\begin{aligned}
&\Updownarrow\\
&[\mathbf{F}_{\boldsymbol{\sigma\sigma}\mathbf{x}}(\mathbf{x}_{ss},\sigma)]^i_{\alpha_3} = \\
&7)\ +[\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}[\mathbf{g}_\mathbf{x}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}\\
&13)\ +[\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}[\mathbf{g}_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\beta_1}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}\\
&18)+[\mathbf{f}_\mathbf{y}]^i_{\beta_1}[\mathbf{g}_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_3}\\
&23)+[\mathbf{f}_{\mathbf{x}'}]^i_{\gamma_1}[\mathbf{h}_{\boldsymbol{\sigma\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_3}\\
&\quad+\left[b^2\right]^i_{\alpha_3}=0
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
&\left[b^2\right]^i_{\alpha_3} = \\
&1)\quad \left([\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}'}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}'}]^i_{\beta_1\beta_2\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}}]^i_{\beta_1\beta_2\alpha_3}\right)\\
&\quad\times[\mathbf{g}_\mathbf{x}]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_\mathbf{x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}\\
&\quad+[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}]^i_{\beta_1\beta_2}\left([\mathbf{g}_{\mathbf{xx}}]^{\beta_2}_{\gamma_2\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_\mathbf{x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}+[\mathbf{g}_\mathbf{x}]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_{\mathbf{xx}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}\right)\\
&4)\quad +2\left([\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}'}]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}}]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}_\mathbf{x}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}'}]^i_{\beta_1\gamma_2\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}}]^i_{\beta_1\gamma_2\alpha_3}\right)\\
&\quad\times[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_\mathbf{x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}
\end{aligned}
$$

$+2\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_{\mathbf{xx}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$

5)$+\left([\mathbf{f}_{\mathbf{y}'\mathbf{y}'}]^{i}_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{y}}]^{i}_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}]^{i}_{\beta_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}'\mathbf{x}}]^{i}_{\beta_1\alpha_3}\right)$

$\times\left([\mathbf{g}_{\mathbf{xx}}]^{\beta_1}_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}+[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}+[\mathbf{g}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1}\right)$

$+[\mathbf{f}_{\mathbf{y}'}]^{i}_{\beta_1}\left([\mathbf{g}_{\mathbf{xxx}}]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}+[\mathbf{g}_{\mathbf{xx}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}\right)$

18) $+\left([\mathbf{f}_{\mathbf{y}\mathbf{y}'}]^{i}_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{yy}}]^{i}_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{y}\mathbf{x}'}]^{i}_{\beta_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{yx}}]^{i}_{\beta_1\alpha_3}\right)[\mathbf{g}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1}$

22) $+\left([\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{y}'}]^{i}_{\gamma_1\gamma_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{y}}]^{i}_{\gamma_1\gamma_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}'}]^{i}_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}}]^{i}_{\gamma_1\gamma_2\alpha_3}\right)$

$\times[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$

23) $+\left([\mathbf{f}_{\mathbf{x}'\mathbf{y}'}]^{i}_{\gamma_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{y}}]^{i}_{\gamma_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'}]^{i}_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}}]^{i}_{\gamma_1\alpha_3}\right)[\mathbf{h}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\gamma_1}$

for $i=1,2,...,n$ and $\alpha_3=1,2,...,n_x$.

4.3 With respect to $(\sigma,\mathbf{x}_t,\mathbf{x}_t)$

Recall that

$[\mathbf{F}_{\sigma\mathbf{x}}(\mathbf{x}_{ss},\sigma)]^{i}_{\alpha_2}=$

$E_t[\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\beta_2}[\mathbf{g}^{t}_{\mathbf{x}}]^{\beta_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^{i}_{\beta_1\alpha_2}\right)$

$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^{t}_{\sigma}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}\left([\mathbf{h}^{t}_{\sigma}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^{t}_{\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_2}$

$+\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\beta_2}[\mathbf{g}^{t}_{\mathbf{x}}]^{\beta_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^{i}_{\beta_1\alpha_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}$

$+\left(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}+[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^{i}_{\beta_1\beta_2}[\mathbf{g}^{t}_{\mathbf{x}}]^{\beta_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}+[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}]^{i}_{\beta_1\alpha_2}\right)[\mathbf{g}^{t}_{\boldsymbol{\sigma}}]^{\beta_1}$

$+[\mathbf{f}_{\mathbf{y}_t}]^{i}_{\beta_1}[\mathbf{g}^{t}_{\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_2}$

$+\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_2}[\mathbf{g}^{t}_{\mathbf{x}}]^{\beta_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\gamma_1\alpha_2}\right)$

$\times\left([\mathbf{h}^{t}_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

$+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1}[\mathbf{h}^{t}_{\boldsymbol{\sigma}\mathbf{x}}]^{\gamma_1}_{\alpha_2}]$

$\Updownarrow$

$[\mathbf{F}_{\sigma\mathbf{x}}(\mathbf{x}_{ss},\sigma)]^{i}_{\alpha_2}=$

1) $E_t[\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^{t}_{\sigma}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

2) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\beta_2}[\mathbf{g}^{t}_{\mathbf{x}}]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^{t}_{\sigma}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

3) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^{t}_{\sigma}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

4) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^{i}_{\beta_1\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^{t}_{\sigma}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

5) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}[\mathbf{h}^{t}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}\left([\mathbf{h}^{t}_{\sigma}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

6) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$

7) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

8) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

9) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

10) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

11) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$

12) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

13) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

14) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

15) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_2} \left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

16) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_2}$

17) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

18) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\gamma_1\beta_2} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

19) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

20) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\gamma_1\alpha_2} \left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

21) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1} \left[\mathbf{h}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}]$

Hence, we let

$[P_1]^{i}_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_2]^{i}_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_3]^{i}_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_4]^{i}_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_5]^{i}_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left(\left[\mathbf{h}^{t}_{\sigma}\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$

$[P_6]^{i}_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^{t}_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$

$[P_7]^{i}_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_8]^{i}_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^{i}_{\beta_1\beta_2} \left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_9]^{i}_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2} \left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$[P_{10}]^{i}_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\right]^{i}_{\beta_1\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

$$[P_{11}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$$

$$[P_{12}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$[P_{13}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$[P_{14}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$[P_{15}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$[P_{16}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1} \left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_2}$$

$$[P_{17}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$

$$[P_{18}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$

$$[P_{19}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$

$$[P_{20}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2} \left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$

$$[P_{21}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} \left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$$

We now compute the derivative of all these terms with respect to $\mathbf{x}_t$

$$[P_1]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$

$$\begin{aligned}
[P_1]^i_{\alpha_2\alpha_3} = {} & (\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}) \\
& \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}
\end{aligned}$$

$$[P_2]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$

$$\begin{aligned}
[P_2]^i_{\alpha_2\alpha_3} = {} & (\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}) \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^t_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)
\end{aligned}$$

$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$$

$$[P_3]^i_{\alpha_2}\equiv\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$[P_3]^i_{\alpha_2\alpha_3}=(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3})\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$$

$$[P_4]^i_{\alpha_2}\equiv\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$[P_4]^i_{\alpha_2\alpha_3}=(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_t}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\alpha_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2\alpha_3})\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$$

$$[P_5]^i_{\alpha_2}\equiv\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$[P_5]^i_{\alpha_2\alpha_3}=(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3})\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$$

$$[P_6]^i_{\alpha_2}\equiv\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$$
$$[P_6]^i_{\alpha_2\alpha_3}=\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{xx}}\right]^{\gamma_1}_{\alpha_2\alpha_3}$$

$$[P_7]^i_{\alpha_2}\equiv\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$[P_7]^i_{\alpha_2\alpha_3}=(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3})\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}$$

$$[P_8]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$\begin{aligned}[P_8]^i_{\alpha_2\alpha_3} = &(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}) [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{xx}}]^{\beta_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}\end{aligned}$$

$$[P_9]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$\begin{aligned}[P_9]^i_{\alpha_2\alpha_3} = &(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3}) [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}\end{aligned}$$

$$[P_{10}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$\begin{aligned}[P_{10}]^i_{\alpha_2\alpha_3} = &(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\alpha_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_t}\right]^i_{\beta_1\alpha_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\alpha_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2\alpha_3}) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}\end{aligned}$$

$$[P_{11}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}$$

$$\begin{aligned}[P_{11}]^i_{\alpha_2\alpha_3} = &(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{xx}}\right]^{\beta_1}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \\ &+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}\end{aligned}$$

$$[P_{12}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$

$$\begin{aligned}[P_{12}]^i_{\alpha_2\alpha_3} = &(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} \\ &+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\ &+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\ &+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\ &+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_3}\end{aligned}$$

$$[P_{13}]^i_{\alpha_2} \equiv [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$

$$\begin{aligned}[P_{13}]^i_{\alpha_2\alpha_3} = &(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_3}_{\alpha_3} \\ &+ \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_t}]^i_{\beta_1\beta_2\alpha_3}) [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\ &+ [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{xx}}]^{\beta_2}_{\alpha_2\alpha_3} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\ &+ [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2} [\mathbf{g}^t_{\mathbf{x}}]^{\beta_2}_{\alpha_2} [\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\alpha_3}\end{aligned}$$

$$[P_{14}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$

$$
\begin{aligned}
\left[P_{14}\right]^{i}_{\alpha_2\alpha_3} = &\Big(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\beta_1\gamma_2\alpha_3}\Big)\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
\left[P_{15}\right]^{i}_{\alpha_2} \equiv &\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^{i}_{\beta_1\alpha_2}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
\left[P_{15}\right]^{i}_{\alpha_2\alpha_3} = &\Big(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{y}_t}\right]^{i}_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\alpha_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{x}_t}\right]^{i}_{\beta_1\alpha_2\alpha_3}\Big)\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^{i}_{\beta_1\alpha_2}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
\left[P_{16}\right]^{i}_{\alpha_2} \equiv &\left[\mathbf{f}_{\mathbf{y}_t}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_2}\\
\left[P_{16}\right]^{i}_{\alpha_2\alpha_3} = &\Big(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^{i}_{\beta_1\alpha_3}\Big)\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_2}\\
&+\left[\mathbf{f}_{\mathbf{y}_t}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\mathbf{xx}}\right]^{\beta_1}_{\alpha_2\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
\left[P_{17}\right]^{i}_{\alpha_2} \equiv &\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
\left[P_{17}\right]^{i}_{\alpha_2\alpha_3} = &\Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^{i}_{\gamma_1\beta_2\alpha_3}\Big)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
\left[P_{18}\right]^{i}_{\alpha_2} \equiv &\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
\left[P_{18}\right]^{i}_{\alpha_2\alpha_3} = &\Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^{i}_{\gamma_1\beta_2\alpha_3}\Big)\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t}_{\mathbf{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
\left[P_{19}\right]^{i}_{\alpha_2} \equiv &\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
\left[P_{19}\right]^{i}_{\alpha_2\alpha_3} = &\Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^{i}_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\gamma_1\gamma_2\alpha_3}\Big)\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
\left[P_{20}\right]^{i}_{\alpha_2} \equiv &\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^{i}_{\gamma_1\alpha_2}\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)\\
\left[P_{20}\right]^{i}_{\alpha_2\alpha_3} = &\Big(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\alpha_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{y}_t}\right]^{i}_{\gamma_1\alpha_2\beta_3}\left[\mathbf{g}^{t}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\alpha_2\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{x}_t}\right]^{i}_{\gamma_1\alpha_2\alpha_3}\Big)\left(\left[\mathbf{h}^{t}_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)
\end{aligned}
$$

$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2} \left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$$

$$[P_{21}]^i_{\alpha_2} \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} \left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$$

$$\begin{aligned}[P_{21}]^i_{\alpha_2\alpha_3} = {} & (\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} \\ & + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3}) \left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_2} \\ & + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} \left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_2\alpha_3}\end{aligned}$$

Thus

$$\left[\mathbf{F}_{\boldsymbol{\sigma}\mathbf{x}\mathbf{x}}\left(\mathbf{x}_{ss},\sigma\right)\right]^i_{\alpha_2\alpha_3} = E_t\left[\sum_{m=1}^{21} [P_m]^i_{\alpha_2\alpha_3}\right] = 0$$

Hence, we need to evaluate the $E_t\left[[P_m]^i_{\alpha_2\alpha_3}\right]$ in the non-stochastic steady state. Here, we can use the previous results that $\mathbf{h}_\sigma = \mathbf{0}$, $\mathbf{g}_{\boldsymbol{\sigma}} = \mathbf{0}$, $\mathbf{h}_{\mathbf{x}\sigma} = \mathbf{0}$, and $\mathbf{g}_{\mathbf{x}\sigma} = \mathbf{0}$, in addition to $E_t\left[\boldsymbol{\epsilon}_{t+1}\right] = \mathbf{0}$

1) For $[P_1]^i_{\alpha_2\alpha_3}$

$$\begin{aligned}E_t\left\{[P_1]^i_{\alpha_2\alpha_3}\right\} = {} & E\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}) \\ & \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\} \\ = {} & 0\end{aligned}$$

2) For $[P_2]^i_{\alpha_2\alpha_3}$

$$\begin{aligned}E_t\left\{[P_2]^i_{\alpha_2\alpha_3}\right\} = {} & E\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3}) \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^t_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\} \\ = {} & 0\end{aligned}$$

3) For $[P_3]^i_{\alpha_2\alpha_3}$

$$\begin{aligned}E_t\left\{[P_3]^i_{\alpha_2\alpha_3}\right\} = {} & E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3}) \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right) \\ & + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}\end{aligned}$$

$= 0$

4) For $[P_4]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_4]^i_{\alpha_2\alpha_3}\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_t}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\alpha_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2\alpha_3})\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$= 0$

5) For $[P_5]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_5]^i_{\alpha_2\alpha_3}\right\} = E_t\{\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left(\left[\mathbf{h}^t_\sigma\right]^{\gamma_1} + \left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$= 0$

6) For $[P_6]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_6]^i_{\alpha_2\alpha_3}\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3})\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{xx}}\right]^{\gamma_1}_{\alpha_2\alpha_3}\}$$

$$= \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{xx}}\right]^{\gamma_1}_{\alpha_2\alpha_3}$$

7) For $[P_7]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_7]^i_{\alpha_2\alpha_3}\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3})\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\}$$

$= 0$

8) For $[P_8]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_8]^i_{\alpha_2\alpha_3}\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3})\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\}$$

$$=0$$

9) For $[P_9]^i_{\alpha_2\alpha_3}$

$$\begin{aligned}
E_t&\left\{[P_9]^i_{\alpha_2\alpha_3}\right\}\\
&=E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&\qquad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3})\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\}
\end{aligned}$$

$$=0$$

10) For $[P_{10}]^i_{\alpha_2\alpha_3}$

$$\begin{aligned}
E_t&\left\{[P_{10}]^i_{\alpha_2\alpha_3}\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{y}_t}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&\qquad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\alpha_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2\alpha_3})\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\}
\end{aligned}$$

$$=0$$

11) For $[P_{11}]^i_{\alpha_2\alpha_3}$

$$\begin{aligned}
E_t&\left\{[P_{11}]^i_{\alpha_2\alpha_3}\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&\qquad+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\alpha_3})\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{xx}}\right]^{\beta_1}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\}
\end{aligned}$$

$$=\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{xx}}\right]^{\beta_1}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$$

12) For $[P_{12}]^i_{\alpha_2\alpha_3}$

$$\begin{aligned}
E_t&\left\{[P_{12}]^i_{\alpha_2\alpha_3}\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&\qquad+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3})\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}\}
\end{aligned}$$

$$=0$$

13) For $[P_{13}]^i_{\alpha_2\alpha_3}$

$$\begin{aligned}
E_t&\left\{[P_{13}]^i_{\alpha_2\alpha_3}\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}\\
&\qquad+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\beta_2\alpha_3})\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}
\end{aligned}$$

$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}\}$$

$$=0$$

14) For $[P_{14}]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_{14}]^i_{\alpha_2\alpha_3}\right\}=$$
$$E_t\{(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\beta_1\gamma_2\alpha_3})\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}\}$$

$$=0$$

15) For $[P_{15}]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_{15}]^i_{\alpha_2\alpha_3}\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{y}_t}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\alpha_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2\alpha_3})\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_2}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_3}\}$$

$$=0$$

16) For $[P_{16}]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_{16}]^i_{\alpha_2\alpha_3}\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_t}\right]^i_{\beta_1\alpha_3})\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\alpha_2}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\alpha_2\alpha_3}\}$$

$$=\left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\alpha_2\alpha_3}$$

17) For $[P_{17}]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_{17}]^i_{\alpha_2\alpha_3}\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3})\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$$=0$$

18) For $[P_{18}]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_{18}]^i_{\alpha_2\alpha_3}\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_t}\right]^i_{\gamma_1\beta_2\alpha_3})\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$= 0$

19) For $[P_{19}]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_{19}]^i_{\alpha_2\alpha_3}\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\gamma_2\alpha_3})\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$= 0$

20) For $[P_{20}]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_{20}]^i_{\alpha_2\alpha_3}\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{y}_{t+1}}\right]^i_{\gamma_1\alpha_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{y}_t}\right]^i_{\gamma_1\alpha_2\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{x}_{t+1}}\right]^i_{\gamma_1\alpha_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2\alpha_3})\left(\left[\mathbf{h}^t_{\boldsymbol{\sigma}}\right]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_2}\left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}\}$$

$= 0$

21) For $[P_{21}]^i_{\alpha_2\alpha_3}$

$$E_t\left\{[P_{21}]^i_{\alpha_2\alpha_3}\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3}\left[\mathbf{g}^t_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]^i_{\gamma_1\alpha_3})\left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\gamma_1}_{\alpha_2}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{xx}}\right]^{\gamma_1}_{\alpha_2\alpha_3}\}$$

$$= \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{xx}}\right]^{\gamma_1}_{\alpha_2\alpha_3}$$

We therefore have
$[\mathbf{F}_{\boldsymbol{\sigma}\mathbf{xx}}(\mathbf{x}_{ss},\sigma)]^i_{\alpha_2\alpha_3} =$

$$\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\mathbf{xx}}\right]^{\gamma_1}_{\alpha_2\alpha_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{xx}}\right]^{\beta_1}_{\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1}\left[\mathbf{g}^t_{\boldsymbol{\sigma}\mathbf{xx}}\right]^{\beta_1}_{\alpha_2\alpha_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}\left[\mathbf{h}^t_{\boldsymbol{\sigma}\mathbf{xx}}\right]^{\gamma_1}_{\alpha_2\alpha_3} = 0$$

This must hold for
$i = 1, 2, ...n$
$\alpha_1, \alpha_3, = 1, 2, ..., n_x$

This system is homogenous in the unknowns ($\mathbf{g}_{\boldsymbol{\sigma}\mathbf{xx}}$,$\mathbf{h}_{\boldsymbol{\sigma}\mathbf{xx}}$) and therefore, $\mathbf{g}_{\boldsymbol{\sigma}\mathbf{xx}} = \mathbf{0}$ and $\mathbf{h}_{\boldsymbol{\sigma}\mathbf{xx}} = \mathbf{0}$. This is in line with the conjecture made in footnote 10 in Schmitt-Grohé & Uribe (2004).

For the paper we use the same notation as in Schmitt-Grohé & Uribe (2004), that is we omit the time index on derivatives of $\mathbf{g}$ and $\mathbf{h}$, and we let $\mathbf{y} \equiv \mathbf{y}_t$, $\mathbf{y}' \equiv \mathbf{y}_{t+1}$,$\mathbf{x} \equiv \mathbf{x}_t$, $\mathbf{x}' \equiv \mathbf{x}_{t+1}$. Thus
$[\mathbf{F}_{\boldsymbol{\sigma}\mathbf{xx}}(\mathbf{x}_{ss},\sigma)]^i_{\alpha_2\alpha_3} =$

$$[\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\sigma\mathbf{xx}}]^{\gamma_1}_{\alpha_2\alpha_3} + [\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}[\mathbf{g}_{\boldsymbol{\sigma}\mathbf{xx}}]^{\beta_1}_{\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}$$
$$+ [\mathbf{f}_{\mathbf{y}}]^i_{\beta_1}[\mathbf{g}_{\boldsymbol{\sigma}\mathbf{xx}}]^{\beta_1}_{\alpha_2\alpha_3} + [\mathbf{f}_{\mathbf{x}'}]^i_{\gamma_1}[\mathbf{h}_{\boldsymbol{\sigma}\mathbf{xx}}]^{\gamma_1}_{\alpha_2\alpha_3} = 0$$

4.4 With respect to (σ,σ,σ)

Recall from the section on derivatives with respect to $(\sigma,\sigma,\mathbf{x}_t)$ that

$[\mathbf{F}_{\sigma\sigma}(\mathbf{x}_{ss},\sigma)]^i = E_t\{$

1) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

2) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

3) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

4) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

5) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

6) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

7) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}$

8) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

9) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

10) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

11) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$

12) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

13) $+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}$

14) $+\left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)[\mathbf{g}^{t}_{\boldsymbol{\sigma}}]^{\beta_1}$

15) $+\left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}[\mathbf{g}^{t}_{\boldsymbol{\sigma}}]^{\beta_1}$

16) $+[\mathbf{f}_{\mathbf{y}_{t}\mathbf{y}_{t}}]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}[\mathbf{g}^{t}_{\boldsymbol{\sigma}}]^{\beta_1}$

17) $+\left[\mathbf{f}_{\mathbf{y}_{t}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)[\mathbf{g}^{t}_{\boldsymbol{\sigma}}]^{\beta_1}$

18) $+[\mathbf{f}_{\mathbf{y}_{t}}]^i_{\beta_1}[\mathbf{g}^{t}_{\boldsymbol{\sigma\sigma}}]^{\beta_1}$

19) $+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

20) $+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

21) $+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

22) $+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$

23) $+\left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1}[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}\}$

Hence, we let

$$[P_1]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_2]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_3]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_4]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_5]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_6]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_7]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}$$

$$[P_8]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$[P_9]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$[P_{10}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$[P_{11}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$

$$[P_{12}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_{13}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}$$

$$[P_{14}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$

$$[P_{15}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$

$$[P_{16}]^i \equiv [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$

$$[P_{17}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$

$$[P_{18}]^i \equiv [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1} [\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}]^{\beta_1}$$

$$[P_{19}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_{20}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_{21}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_{22}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$[P_{23}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} [\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}$$

We now compute the derivative of all these terms with respect to σ

$$[P_1]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$

$$
\begin{aligned}
[P_1]^i_\sigma = (&\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\\
&\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_2}_{\gamma_2}\right)\\
&\times\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\\
&\times\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}
\end{aligned}
$$

$$
\begin{aligned}
[P_2]^i \equiv &\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
[P_2]^i_\sigma = (&\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\\
&\times\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_2}\right)\\
&\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}
\end{aligned}
$$

$$
\begin{aligned}
[P_3]^i \equiv &\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
[P_3]^i_\sigma \equiv (&\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\\
&\times\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}
\end{aligned}
$$

$$
\begin{aligned}
[P_4]^i \equiv &\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
[P_4]^i_\sigma = (&\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\\
&\times\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}
\end{aligned}
$$

$$
\begin{aligned}
[P_5]^i \equiv{} & \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
[P_5]^i_\sigma \equiv{} & (\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right) \\
& \quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)) \\
& \times \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left(\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\mathbf{xx}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1\gamma_2}\right) \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}
\end{aligned}
$$

$$
\begin{aligned}
[P_6]^i \equiv{} & \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
[P_6]^i_\sigma ={} & (\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right) \\
& \quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)) \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right) \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}
\end{aligned}
$$

$$
\begin{aligned}
[P_7]^i \equiv{} & \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} \\
[P_7]^i_\sigma ={} & (\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right) \\
& \quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)) \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right) [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma\sigma}]^{\gamma_1}
\end{aligned}
$$

$$
\begin{aligned}
[P_8]^i \equiv{} & \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
[P_8]^i_\sigma ={} & (\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right) \\
& \quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)) \\
& \times \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_2}_{\gamma_2}\right) \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\right)
\end{aligned}
$$

$$
\begin{aligned}
[P_9]^i \equiv{} & \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
[P_9]^i_\sigma ={} & (\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right) \\
& \quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_2}\right) \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\right)
\end{aligned}
$$

$$
[P_{10}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}
$$

$$
\begin{aligned}
[P_{10}]^i_\sigma \equiv & ([\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}]^i_{\beta_1\beta_2\beta_3} \left([\mathbf{g}^{t+1}_{\mathbf{x}}]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_3} \right) \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3} + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}]^i_{\beta_1\beta_2\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right)) [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_2} [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}]^i_{\beta_1\beta_2} \left([\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}]^{\beta_2}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_2} \right) [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}]^i_{\beta_1\beta_2} [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_2} \left([\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1} \right)
\end{aligned}
$$

$$
\begin{aligned}
[P_{11}]^i \equiv & [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}]^i_{\beta_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \right) [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_1} \\
[P_{11}]^i_\sigma = & ([\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}]^i_{\beta_1\gamma_2\beta_3} \left([\mathbf{g}^{t+1}_{\mathbf{x}}]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_3} \right) \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}]^i_{\beta_1\gamma_2\beta_3} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3} + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}]^i_{\beta_1\gamma_2\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right)) \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \right) [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}]^i_{\beta_1\gamma_2} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2} [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}]^i_{\beta_1\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \right) \left([\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1} \right)
\end{aligned}
$$

$$
\begin{aligned}
[P_{12}]^i \equiv & [\mathbf{f}_{\mathbf{y}_{t+1}}]^i_{\beta_1} [\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \right) \\
[P_{12}]^i_\sigma = & ([\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}]^i_{\beta_1\beta_3} \left([\mathbf{g}^{t+1}_{\mathbf{x}}]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_3} \right) \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3} + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}]^i_{\beta_1\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right)) [\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}]^{\beta_1}_{\gamma_1} \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \right) \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}}]^i_{\beta_1} \left([\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\gamma_1\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1}_{\gamma_1} \right) \left([\mathbf{h}^t_\sigma]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \right) \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}}]^i_{\beta_1} [\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}]^{\beta_1}_{\gamma_1} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}
\end{aligned}
$$

$$
\begin{aligned}
[P_{13}]^i \equiv & [\mathbf{f}_{\mathbf{y}_{t+1}}]^i_{\beta_1} [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1} \\
[P_{13}]^i_\sigma = & ([\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}]^i_{\beta_1\beta_3} \left([\mathbf{g}^{t+1}_{\mathbf{x}}]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_3} \right) \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}]^i_{\beta_1\beta_3} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3} + [\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}]^i_{\beta_1\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right)) [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_{t+1}}]^i_{\beta_1} \left([\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}\mathbf{x}}]^{\beta_1}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1} \right)
\end{aligned}
$$

$$
\begin{aligned}
[P_{14}]^i \equiv & [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}]^i_{\beta_1\beta_2} [\mathbf{g}^{t+1}_{\mathbf{x}}]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \right) [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
[P_{14}]^i_\sigma = & ([\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}]^i_{\beta_1\beta_2\beta_3} \left([\mathbf{g}^{t+1}_{\mathbf{x}}]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_3} \right) \\
& + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}]^i_{\beta_1\beta_2\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right)) \\
& \times [\mathbf{g}^{t+1}_{\mathbf{x}}]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \right) [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}]^i_{\beta_1\beta_2} \left([\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}]^{\beta_2}_{\gamma_2\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}]^{\beta_2}_{\gamma_2} \right) \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \right) [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}]^i_{\beta_1\beta_2} [\mathbf{g}^{t+1}_{\mathbf{x}}]^{\beta_2}_{\gamma_2} [\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}]^i_{\beta_1\beta_2} [\mathbf{g}^{t+1}_{\mathbf{x}}]^{\beta_2}_{\gamma_2} \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \right) [\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1}
\end{aligned}
$$

$$
\begin{aligned}
[P_{15}]^i \equiv & [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}]^i_{\beta_1\beta_2} [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
[P_{15}]^i_\sigma = & ([\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}]^i_{\beta_1\beta_2\beta_3} \left([\mathbf{g}^{t+1}_{\mathbf{x}}]^{\beta_3}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_3} \right) \\
& + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}]^i_{\beta_1\beta_2\beta_3} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3} + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}]^i_{\beta_1\beta_2\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right)) [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_2} [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}]^i_{\beta_1\beta_2} \left([\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}]^{\beta_2}_{\gamma_3} \left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \right) + [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_2} \right) [\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1} \\
& + [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}]^i_{\beta_1\beta_2} [\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}]^{\beta_2} [\mathbf{g}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\beta_1}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{16}]^i \equiv [\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&[P_{16}]^i_\sigma = (\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)\\
&\qquad\quad +[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&\quad +[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&\quad +[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{17}]^i \equiv \left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&[P_{17}]^i_\sigma = (\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)\\
&\qquad\quad +\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&\quad +\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_1}\\
&\quad +\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{18}]^i \equiv [\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\\
&[P_{18}]^i_\sigma = (\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)\\
&\qquad\quad +[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_3}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\\
&\quad +[\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{19}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&[P_{19}]^i_\sigma = (\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)\\
&\qquad\quad +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\\
&\quad \times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&\quad +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_2}_{\gamma_2}\right)\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&\quad +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&\quad +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)[\mathbf{h}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\gamma_1}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{20}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&[P_{20}]^i_\sigma = (\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)\\
&\qquad\quad +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\\
&\quad \times\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&\quad +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&\quad +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}[\mathbf{h}^t_{\boldsymbol{\sigma}\boldsymbol{\sigma}}]^{\gamma_1}
\end{aligned}
$$

$$
\begin{aligned}
&[P_{21}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)\\
&[P_{21}]^i_\sigma = (\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)\\
&\qquad\quad +\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t}_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))
\end{aligned}
$$

$$\times \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2} \left(\left[\mathbf{h}_{\boldsymbol{\sigma}}^t\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_2}^i \left(\left[\mathbf{g}_{\boldsymbol{\sigma}\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_2} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\boldsymbol{\sigma\sigma}}^{t+1}\right]^{\beta_2}\right) \left(\left[\mathbf{h}_{\boldsymbol{\sigma}}^t\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_2}^i \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2} \left[\mathbf{h}_{\boldsymbol{\sigma\sigma}}^t\right]^{\gamma_1}$$

$$[P_{22}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_2}^i \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left(\left[\mathbf{h}_{\boldsymbol{\sigma}}^t\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$[P_{22}]_\sigma^i = (\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\gamma_2\beta_3}^i \left(\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_3}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\gamma_2\beta_3}^i \left[\mathbf{g}_{\boldsymbol{\sigma}}^t\right]^{\beta_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_2\gamma_3}^i \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))$$
$$\times \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left(\left[\mathbf{h}_{\boldsymbol{\sigma}}^t\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_2}^i \left[\mathbf{h}_{\sigma\sigma}^t\right]^{\gamma_2} \left(\left[\mathbf{h}_{\boldsymbol{\sigma}}^t\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_2}^i \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{h}_{\boldsymbol{\sigma\sigma}}^t\right]^{\gamma_1}$$

$$[P_{23}]^i \equiv \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}^i \left[\mathbf{h}_{\boldsymbol{\sigma\sigma}}^t\right]^{\gamma_1}$$
$$[P_{23}]_\sigma^i = (\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}^i \left(\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_3}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}^i \left[\mathbf{g}_{\boldsymbol{\sigma}}^t\right]^{\beta_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}^i \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)) \left[\mathbf{h}_{\boldsymbol{\sigma\sigma}}^t\right]^{\gamma_1}$$
$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}^i \left[\mathbf{h}_{\boldsymbol{\sigma\sigma\sigma}}^t\right]^{\gamma_1}$$

Thus
$$[\mathbf{F}_{\boldsymbol{\sigma\sigma\sigma}}(\mathbf{x}_{ss},\sigma)]^i = E_t\left[\sum_{m=1}^{23} [P_m]_\sigma^i\right] = 0$$

Hence, we need to evaluate $E_t\left[[P_m]_\sigma^i\right]$ in the deterministic steady state. Here, we can use the previous results that $\mathbf{h}_\sigma = \mathbf{0}$, $\mathbf{g}_{\boldsymbol{\sigma}} = \mathbf{0}$, $\mathbf{h}_{\mathbf{x}\sigma} = \mathbf{0}$, $\mathbf{g}_{\mathbf{x}\sigma} = \mathbf{0}$, $\mathbf{h}_{\boldsymbol{\sigma}\mathbf{xx}} = \mathbf{0}$, and $\mathbf{g}_{\boldsymbol{\sigma}\mathbf{xx}} = \mathbf{0}$. Moreover we have $E_t[\boldsymbol{\epsilon}_{t+1}] = 0$. We also introduce the additional notation:

$$\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]_{\phi_2\phi_3}^{\phi_1} = \begin{cases} m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1,1)) & \text{if } \phi_1 = \phi_2 = \phi_3 \\ 0 & \text{else} \end{cases}$$

where $m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1,1))$ denotes the third moment of $\boldsymbol{\epsilon}_{t+1}(\phi_1,1)$ for $\phi_1 = 1,2,...,n_e$. Notice that $\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})$ has dimensions $n_e \times n_e \times n_e$.

1) For $[P_1]_\sigma^i$

$$E_t\left\{[P_1]_\sigma^i\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2\beta_3}^i \left(\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_3}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_2\beta_3}^i \left[\mathbf{g}_{\boldsymbol{\sigma}}^t\right]^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\beta_2\gamma_3}^i \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))$$
$$\times \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^i \left(\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_2\gamma_3}^{\beta_2} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\mathbf{x}\boldsymbol{\sigma}}^{t+1}\right]_{\gamma_2}^{\beta_2}\right)$$
$$\times \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^i \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} \left[\mathbf{h}_{\sigma\sigma}^t\right]^{\gamma_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^i \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left(\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\mathbf{x}\boldsymbol{\sigma}}^{t+1}\right]_{\gamma_1}^{\beta_1}\right)$$
$$\times \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^i \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} \left(\left[\mathbf{h}_\sigma^t\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right) \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} \left[\mathbf{h}_{\sigma\sigma}^t\right]^{\gamma_1}\}$$

$$= E_t\{\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2\beta_3}^i \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\beta_2\gamma_3}^i [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)$$

$$\times \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_2\gamma_3}^{\beta_2} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\}$$

$$= E_t\{\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\beta_2\gamma_3}^{i} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_2\gamma_3}^{\beta_2} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} [\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\epsilon}_{t+1}]^{\phi_3} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} [\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\}$$

$$= \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2\beta_3}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\beta_2\gamma_3}^{i} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_2\gamma_3}^{\beta_2} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_2}^{\beta_2} [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2} \left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1} [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3} [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$

2) For $[P_2]_{\sigma}^{i}$

$$E_t\left\{[P_2]_{\sigma}^{i}\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2\beta_3}^{i} \left(\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_3}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]_{\beta_1\beta_2\beta_3}^{i} [\mathbf{g}_{\boldsymbol{\sigma}}^{t}]^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\beta_2\gamma_3}^{i} \left([\mathbf{h}_{\sigma}^{t}]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))$$
$$\times \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left(\left[\mathbf{g}_{\boldsymbol{\sigma}\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_2}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2}\right)$$
$$\times \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2} \left(\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\mathbf{x}\boldsymbol{\sigma}}^{t+1}\right]_{\gamma_1}^{\beta_1}\right)\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} [\mathbf{h}_{\sigma\sigma}^{t}]^{\gamma_1}\}$$

$$= 0$$

3) For $[P_3]_{\sigma}^{i}$

$$E_t\left\{[P_3]_{\sigma}^{i}\right\} \equiv E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_2\beta_3}^{i} \left(\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_3}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}\mathbf{y}_{t}}\right]_{\beta_1\beta_2\beta_3}^{i} [\mathbf{g}_{\boldsymbol{\sigma}}^{t}]^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}\mathbf{x}_{t+1}}\right]_{\beta_1\beta_2\gamma_3}^{i} \left([\mathbf{h}_{\sigma}^{t}]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))$$
$$\times \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]_{\beta_1\beta_2}^{i} \left(\left[\mathbf{g}_{\boldsymbol{\sigma}\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_2}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2}\right) \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2} \left(\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\mathbf{x}\boldsymbol{\sigma}}^{t+1}\right]_{\gamma_1}^{\beta_1}\right)\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]_{\beta_1\beta_2}^{i} \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_2} \left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1} [\mathbf{h}_{\sigma\sigma}^{t}]^{\gamma_1}\}$$

$$= 0$$

4) For $[P_4]_{\sigma}^{i}$

$$E_t\left\{[P_4]_{\sigma}^{i}\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\gamma_2\beta_3}^{i} \left(\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3}\left([\mathbf{h}_{\sigma}^{t}]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_3}\right)$$
$$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t}}\right]_{\beta_1\gamma_2\beta_3}^{i} [\mathbf{g}_{\boldsymbol{\sigma}}^{t}]^{\beta_3} + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_2\gamma_3}^{i} \left([\mathbf{h}_{\sigma}^{t}]^{\gamma_3} + [\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))$$

$$\times\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_2}^{i}\left[\mathbf{h}_{\sigma\sigma}^{t}\right]^{\gamma_2}\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_2}^{i}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}_{\mathbf{x}\boldsymbol{\sigma}}^{t+1}\right]_{\gamma_1}^{\beta_1}\right)$$
$$\times\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_2}^{i}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1}\left[\mathbf{h}_{\sigma\sigma}^{t}\right]^{\gamma_1}\}$$

$$=E_t\{\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\gamma_2\beta_3}^{i}\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_2\gamma_3}^{i}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)$$
$$\times\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_2}^{i}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\}$$

$$=\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\gamma_2\beta_3}^{i}\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_2\gamma_3}^{i}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_2}^{i}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_3}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$

5) For $\left[P_5\right]_{\sigma}^{i}$

$$E_t\left\{\left[P_5\right]_{\sigma}^{i}\right\}\equiv E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}^{i}\left(\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\right]_{\beta_1\beta_3}^{i}\left[\mathbf{g}_{\boldsymbol{\sigma}}^{t}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}^{i}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right))$$
$$\times\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_2}^{\beta_1}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]_{\beta_1}^{i}\left(\left[\mathbf{g}_{\mathbf{xxx}}^{t+1}\right]_{\gamma_1\gamma_2\gamma_3}^{\beta_1}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}_{\mathbf{xx}\boldsymbol{\sigma}}^{t+1}\right]_{\gamma_1\gamma_2}^{\beta_1}\right)$$
$$\times\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]_{\beta_1}^{i}\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_2}^{\beta_1}\left[\mathbf{h}_{\sigma\sigma}^{t}\right]^{\gamma_2}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]_{\beta_1}^{i}\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_2}^{\beta_1}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{h}_{\sigma\sigma}^{t}\right]^{\gamma_1}\}$$

$$=E_t\{\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}^{i}\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}^{i}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)$$
$$\times\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_2}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]_{\beta_1}^{i}\left[\mathbf{g}_{\mathbf{xxx}}^{t+1}\right]_{\gamma_1\gamma_2\gamma_3}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\}$$

$$=E_t\{\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}^{i}\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_2}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}^{i}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_2}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]_{\beta_1}^{i}\left[\mathbf{g}_{\mathbf{xxx}}^{t+1}\right]_{\gamma_1\gamma_2\gamma_3}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\}$$

$$=\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}^{i}\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_2}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}^{i}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\mathbf{g}_{\mathbf{xx}}^{t+1}\right]_{\gamma_1\gamma_2}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]_{\beta_1}^{i}\left[\mathbf{g}_{\mathbf{xxx}}^{t+1}\right]_{\gamma_1\gamma_2\gamma_3}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$

6) For $\left[P_6\right]_{\sigma}^{i}$

$$E_t\left\{\left[P_6\right]_{\sigma}^{i}\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}^{i}\left(\left[\mathbf{g}_{\mathbf{x}}^{t+1}\right]_{\gamma_3}^{\beta_3}\left(\left[\mathbf{h}_{\sigma}^{t}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}_{\boldsymbol{\sigma}}^{t+1}\right]^{\beta_3}\right)$$

$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right))\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}\}$$
$$=0$$

7) For $[P_7]^i_\sigma$

$$E_t\left\{[P_7]^i_\sigma\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right))\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma\sigma}\right]^{\gamma_1}\}$$
$$=\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}^t_{\sigma\sigma\sigma}\right]^{\gamma_1}$$

8) For $[P_8]^i_\sigma$

$$E_t\left\{[P_8]^i_\sigma\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right))$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_2}_{\gamma_2}\right)\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}^t_{\sigma\sigma}\right]^{\gamma_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_2}+\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\right)\}$$
$$=0$$

9) For $[P_9]^i_\sigma$

$$E_t\left\{[P_9]^i_\sigma\right\}=E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right))\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\right)\}$$
$$=0$$

10) For $[P_{10}]^i_\sigma$

$$E_t\left\{[P_{10}]^i_\sigma\right\}\equiv E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right))\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3}+\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\right]^{\beta_1}\right)\}$$
$$=0$$

11) For $[P_{11}]^i_\sigma$

$$E_t\left\{[P_{11}]^i_\sigma\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\right)\}$$
$$=0$$

12) For $[P_{12}]^i_\sigma$

$$E_t\left\{[P_{12}]^i_\sigma\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma\sigma}}\right]^{\beta_1}_{\gamma_1}\right)\left([\mathbf{h}^t_\sigma]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_1}_{\gamma_1}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_1}\}$$
$$=0$$

13) For $[P_{13}]^i_\sigma$

$$E_t\left\{[P_{13}]^i_\sigma\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\beta_1}\right)\}$$
$$=\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\beta_1}$$

14) For $[P_{14}]^i_\sigma$

$$E_t\left\{[P_{14}]^i_\sigma\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_2}_{\gamma_2}\right)\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}]^{\beta_1}\}$$
$$=0$$

15) For $[P_{15}]^i_\sigma$

$$E_t\left\{[P_{15}]^i_\sigma\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_2}\right)[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}]^{\beta_1}\}$$
$$=0$$

16) For $[P_{16}]^i_\sigma$

$$E_t\left\{[P_{16}]^i_\sigma\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$
$$+[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_2}\right)[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$
$$+[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}]^{\beta_1}\}$$

$$=0$$

17) For $[P_{17}]^i_\sigma$

$$E_t\left\{[P_{17}]^i_\sigma\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\beta_1\gamma_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}]^{\beta_1}\}$$

$$=0$$

18) For $[P_{18}]^i_\sigma$

$$E_t\left\{[P_{18}]^i_\sigma\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+[\mathbf{f}_{\mathbf{y}_t\mathbf{y}_t}]^i_{\beta_1\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))[\mathbf{g}^t_{\boldsymbol{\sigma\sigma}}]^{\beta_1}$$
$$+[\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}[\mathbf{g}^t_{\boldsymbol{\sigma\sigma\sigma}}]^{\beta_1}\}$$

$$=[\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}[\mathbf{g}^t_{\boldsymbol{\sigma\sigma\sigma}}]^{\beta_1}$$

19) For $[P_{19}]^i_\sigma$

$$E_t\left\{[P_{19}]^i_\sigma\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right))$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3}+[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)+\left[\mathbf{g}^{t+1}_{\mathbf{x}\boldsymbol{\sigma}}\right]^{\beta_2}_{\gamma_2}\right)$$
$$\times\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1}+[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right)$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2}+[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}\}$$

$$=E_t\{\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)$$
$$\times\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\}$$

$$=E_t\{\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\}$$

$$
\begin{aligned}
&= \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}
\end{aligned}
$$

20) For $[P_{20}]^i_\sigma$

$$
\begin{aligned}
E_t\left\{[P_{20}]^i_\sigma\right\} &= E_t\{(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)) \\
&\times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}\} \\
&= 0
\end{aligned}
$$

21) For $[P_{21}]^i_\sigma$

$$
\begin{aligned}
E_t\left\{[P_{21}]^i_\sigma\right\} &= E_t\{(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{y}_t}\right]^i_{\gamma_1\beta_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)) \\
&\times \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left(\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}\mathbf{x}}\right]^{\beta_2}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_2}[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}\} \\
&= 0
\end{aligned}
$$

22) For $[P_{22}]^i_\sigma$

$$
\begin{aligned}
E_t\left\{[P_{22}]^i_\sigma\right\} &= E_t\{(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right) \\
&\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\gamma_2\beta_3}[\mathbf{g}^t_{\boldsymbol{\sigma}}]^{\beta_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)) \\
&\times \left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}[\mathbf{h}^t_{\sigma\sigma}]^{\gamma_2}\left([\mathbf{h}^t_{\boldsymbol{\sigma}}]^{\gamma_1} + [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\right) \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2}\left([\mathbf{h}^t_\sigma]^{\gamma_2} + [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\right)[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}]^{\gamma_1}\} \\
&= E_t\{\left(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right)[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\} \\
&= E_t\{\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\} \\
&= \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
&+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}
\end{aligned}
$$

23) For $[P_{23}]^i_\sigma$

$$
E_t\left\{[P_{23}]^i_\sigma\right\} = E_t\{(\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_3}\left(\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left([\mathbf{h}^t_\sigma]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\epsilon}_{t+1}]^{\phi_3}\right) + \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma}}\right]^{\beta_3}\right)
$$

$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]^i_{\gamma_1\beta_3} \left[\mathbf{g}^t_{\boldsymbol{\sigma}}\right]^{\beta_3} + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_3} \left(\left[\mathbf{h}^t_{\sigma}\right]^{\gamma_3} + [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)\right) \left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma}}\right]^{\gamma_1}$$

$$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} \left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\gamma_1}\}$$

$$= \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} \left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\gamma_1}$$

Thus, we have
$\left[\mathbf{F}_{\boldsymbol{\sigma\sigma\sigma}}\left(\mathbf{x}_{ss},\sigma\right)\right]^i = E_t\left[\sum_{m=1}^{23}\left[P_m\right]^i_{\sigma}\right] = 0$

$\Updownarrow$

1) $\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

4) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

5) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

$+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

7) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\sigma\sigma\sigma}\right]^{\gamma_1}$

13) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\beta_1}$

18) $+ \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1} \left[\mathbf{g}^t_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\beta_1}$

19) $+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

22) $+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

$+ \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

23) $+ \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} \left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\gamma_1} = 0$

$\Updownarrow$

1) $\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

2) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

3) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

4) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

5) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

6) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

7) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

8) $+ \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

9) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

10) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

11) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

12) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

13) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

14) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

15) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$

16) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} \left[\mathbf{h}^t_{\sigma\sigma\sigma}\right]^{\gamma_1}$

17) $\quad + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\beta_1}$

18) $\quad + \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1} \left[\mathbf{g}^t_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\beta_1}$

19) $\quad + \left[\mathbf{f}_{\mathbf{x}_{t+1}}\right]^i_{\gamma_1} \left[\mathbf{h}^t_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\gamma_1} = 0$

This must hold for $i = 1, 2, ..., n$.

This system can be expressed as

$$\begin{bmatrix} \cdots & \cdots \\ \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} + \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1} & \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} + \left[\mathbf{f}_{\mathbf{y}_t}\right]^i_{\beta_1} \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} \left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\beta_1} \\ \left[\mathbf{h}^t_{\sigma\sigma\sigma}\right]^{\gamma_1} \end{bmatrix} = \mathbf{b}$$

where

$$\begin{aligned}
\mathbf{b} \equiv\; & \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[m^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^i_{\beta_1} \left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\beta_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\gamma_1\gamma_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\gamma_1\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}
\end{aligned}$$

$$\begin{aligned}
= & \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2\beta_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& +3 \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^i_{\beta_1\beta_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
& + \left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^i_{\beta_1\beta_2} \left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3} [\boldsymbol{\eta}]^{\gamma_3}_{\phi_3} [\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\boldsymbol{\eta}]^{\gamma_1}_{\phi_1} \left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}
\end{aligned}$$

$$
\begin{aligned}
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[m^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\beta_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}
\end{aligned}
$$

$$
\begin{aligned}
&=\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\\
&+\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[m^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\beta_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}
\end{aligned}
$$

$$
\begin{aligned}
&=\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\\
&+\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\\
&+\\
&+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\\
&+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\beta_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+
\end{aligned}
$$

$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$

$$=\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$

We finally note that $\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$ is only non-zero when $\phi_1=\phi_2=\phi_3$ and this should be used in the implementation. For instance

$\mathbf{b}=$

1) $\sum_{\phi_1=1}^{n_e}\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_1}m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1,1))$

2) $+3\sum_{\phi_1=1}^{n_e}\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_1}m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1,1))$

3) $+3\sum_{\phi_1=1}^{n_e}\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_1}m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1,1))$

4) $+3\sum_{\phi_1=1}^{n_e}\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_1}m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1,1))$

5) $+3\sum_{\phi_1=1}^{n_e}\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_1}m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1,1))$

6) $+\sum_{\phi_1=1}^{n_e}\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_2}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_1}m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1,1))$

7) $+\sum_{\phi_1=1}^{n_e}\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_3}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_2}[\boldsymbol{\eta}(:,\phi_1)]^{\gamma_1}m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1,1))$

For the paper we use the same notation as in Schmitt-Grohé & Uribe (2004), that is we omit the time index on derivatives of $\mathbf{g}$ and $\mathbf{h}$, and we let $\mathbf{y}\equiv\mathbf{y}_t$, $\mathbf{y}'\equiv\mathbf{y}_{t+1}$, $\mathbf{x}\equiv\mathbf{x}_t$, $\mathbf{x}'\equiv\mathbf{x}_{t+1}$. Thus

$$[\mathbf{F}_{\boldsymbol{\sigma\sigma\sigma}}(\mathbf{x}_{ss},\sigma)]^i_{\alpha_3}=$$
$$+\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}+[\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}\right)\left[\mathbf{g}^{t+1}_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\beta_1}+\left(\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}+[\mathbf{f}_{\mathbf{y}_t}]^i_{\beta_1}\right)\left[\mathbf{h}^t_{\sigma\sigma\sigma}\right]^{\gamma_1}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\beta_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathbf{f}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\beta_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}^{t+1}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+\left[\mathbf{f}_{\mathbf{y}_{t+1}}\right]^{i}_{\beta_1}\left[\mathbf{g}^{t+1}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+\left[\mathbf{f}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]^{i}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$=0$$

$$\Updownarrow$$

$$[\mathbf{F}_{\boldsymbol{\sigma\sigma\sigma}}(\mathbf{x}_{ss},\sigma)]^i=$$
$$+\left([\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}+[\mathbf{f}_{\mathbf{y}}]^i_{\beta_1}\right)[\mathbf{g}_{\boldsymbol{\sigma\sigma\sigma}}]^{\beta_1}+\left([\mathbf{f}_{\mathbf{y}'}]^i_{\beta_1}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}+[\mathbf{f}_{\mathbf{y}}]^i_{\beta_1}\right)[\mathbf{h}_{\sigma\sigma\sigma}]^{\gamma_1}$$
$$+[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}'}]^i_{\beta_1\beta_2\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$

$$
\begin{aligned}
&+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{y}'}\right]^{i}_{\beta_1}\left[\mathbf{g}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}'}\right]^{i}_{\gamma_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&=0
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
&\left[\mathbf{F}_{\boldsymbol{\sigma\sigma\sigma}}\left(\mathbf{x}_{ss},\sigma\right)\right]^{i} =\\
&\quad+\left(\left[\mathbf{f}_{\mathbf{y}'}\right]^{i}_{\beta_1}+\left[\mathbf{f}_{\mathbf{y}}\right]^{i}_{\beta_1}\right)\left[\mathbf{g}_{\boldsymbol{\sigma\sigma\sigma}}\right]^{\beta_1}+\left(\left[\mathbf{f}_{\mathbf{y}'}\right]^{i}_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}+\left[\mathbf{f}_{\mathbf{y}}\right]^{i}_{\beta_1}\right)\left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1}\\
&\quad+\left[b^3\right]^{i}=0
\end{aligned}
$$

where

$$
\begin{aligned}
\left[b^3\right]^{i} = &\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{y}'}\right]^{i}_{\beta_1}\left[\mathbf{g}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\\
&+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}'}\right]^{i}_{\gamma_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}
\end{aligned}
$$

The derived formula is implemented in $g\_h\_3rd.m$.

Note, if $\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}=0$ for all innovations, i.e. all structural innovations have a symmetric distribution, then this system is homogenous in the unknowns ($\mathbf{g}_{\boldsymbol{\sigma\sigma\sigma}}$,$\mathbf{h}_{\boldsymbol{\sigma\sigma\sigma}}$), that is $\mathbf{b}=\mathbf{0}$, and therefore, $\mathbf{g}_{\boldsymbol{\sigma\sigma\sigma}}=\mathbf{0}$ and $\mathbf{h}_{\boldsymbol{\sigma\sigma\sigma}}=\mathbf{0}$. However, if $\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\neq 0$ for any shock, then $\mathbf{g}_{\boldsymbol{\sigma\sigma\sigma}}\neq\mathbf{0}$ and $\mathbf{h}_{\boldsymbol{\sigma\sigma\sigma}}\neq\mathbf{0}$. Thus, the conjecture made in footnote 10 in Schmitt-Grohé & Uribe (2004) is in general not correct. Note also that the results in Aruoba, Fernandez-Villaverde & Rubio-Ramirez (2006) are done with the normal distribution where all odd moments are zero. This explains their results and hence the conjecture in Schmitt-Grohé & Uribe (2004).

5 Lucas' Asset pricing model

5.1 Normal distributed shocks

One way to evaluate the correctness of the derived formulas and their implementation is to compare the third order terms with the corresponding terms in the asset pricing model by Lucas which is solved excactly by Burnside (1998). In this model, we have a representative agent maximizing

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\frac{C_t^{\theta}}{\theta}\right]\quad s.t.\;\; p_te_{t+1}+C_t=p_te_t+d_te_t$$

Dividens follow the process

$$d_{t+1} = \exp\{x_{t+1}\} d_t$$

and

$$x_{t+1} = (1-\rho)\bar{x} + \rho x_t + \sigma\eta\varepsilon_{t+1}$$

where $\varepsilon_{t+1} \sim NID(0,1)$. The optimiality condition reads

$$p_t C_t^{\theta-1} = \beta E_t \left[C_{t+1}^{\theta-1}(p_{t+1} + d_{t+1})\right]$$

In equilibrium, we have that $C_t = d_t$ and $e_t = 1$. Let the price-dividend price ratio be $y_t = p_t/d_t$, then

$$y_t = \beta E_t \left[\exp\{\theta x_{t+1}\}(1 + y_{t+1})\right].$$

The exact solution is given by

$$y_t \equiv g(x_t, \sigma) = \sum_{i=0}^{\infty} \beta^i \exp\{a_i + b_i (x_t - \bar{x})\}$$

where

$$\begin{aligned} a_i &= \theta\bar{x}i + \frac{\theta^2\sigma^2\eta^2}{2(1-\rho)^2}\left[i - \frac{2\rho(1-\rho^i)}{1-\rho} + \frac{\rho^2(1-\rho^{2i})}{1-\rho^2}\right] \\ b &= \frac{\theta\rho(1-\rho^i)}{1-\rho} \end{aligned}$$

Important, when evaluating the derivatives below we have:

$$a_i = \theta\bar{x}i$$

We then have for the derivatives in the steady state:
First order:

$$\begin{aligned} \left.\frac{\partial g(x_t,\sigma)}{\partial x_t}\right|_{x_t=\bar{x},\sigma=0} &= \sum_{i=1}^{\infty} \beta^i \exp\{a_i + b_i(x_t - \bar{x})\} b_i \\ &= \sum_{i=1}^{\infty} \beta^i \exp\{\theta\bar{x}i\} b_i \end{aligned}$$

$$\begin{aligned} \left.\frac{\partial g(x_t,\sigma)}{\partial \sigma}\right|_{x_t=\bar{x},,\sigma=0} &= \sum_{i=1}^{\infty} \beta^i \exp\{a_i + b_i(x_t - \bar{x})\} \tfrac{\theta^2\sigma\eta^2}{(1-\rho)^2}\left[i - \tfrac{2\rho(1-\rho^i)}{1-\rho} + \tfrac{\rho^2(1-\rho^{2i})}{1-\rho^2}\right] \\ &= 0 \end{aligned}$$

Second order:

$$\begin{aligned} \left.\frac{\partial^2 g(x_t,\sigma)}{\partial x_t \partial x_t}\right|_{x_t=\bar{x}} &= \sum_{i=1}^{\infty} \beta^i \exp\{a_i + b_i(x_t - \bar{x})\} b_i^2 \\ &= \sum_{i=}^{\infty} \beta^i \exp\{\theta\bar{x}i\} b_i^2 \end{aligned}$$

$$\begin{aligned} \left.\frac{\partial^2 g(x_t,\sigma)}{\partial x_t \partial \sigma}\right|_{x_t=\bar{x},\sigma=0} &= \sum_{i=1}^{\infty} \beta^i \exp\{a_i + b_i(x_t - \bar{x})\} \tfrac{2\theta^2\sigma\eta^2}{2(1-\rho)^2}\left[i - \tfrac{2\rho(1-\rho^i)}{1-\rho} + \tfrac{\rho^2(1-\rho^{2i})}{1-\rho^2}\right] b_i \\ &= 0 \end{aligned}$$

$$\left.\frac{\partial g(x_t,\sigma)}{\partial \sigma \partial \sigma}\right|_{x_t=\bar{x},,\sigma=0} = \sum_{i=1}^{\infty} \beta^i \exp\{a_i + b_i(x_t - \bar{x})\} \left(\tfrac{\theta^2\sigma\eta^2}{(1-\rho)^2}\left[i - \tfrac{2\rho(1-\rho^i)}{1-\rho} + \tfrac{\rho^2(1-\rho^{2i})}{1-\rho^2}\right]\right)^2$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}\beta^i \exp\left\{a_i+b_i\left(x_t-\bar{x}\right)\right\}\tfrac{\theta^2\eta^2}{(1-\rho)^2}\left[i-\tfrac{2\rho\left(1-\rho^i\right)}{1-\rho}+\tfrac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right]\\
&=\sum_{i=1}^{\infty}\beta^i \exp\left\{a_i+b_i\left(x_t-\bar{x}\right)\right\}\tfrac{\theta^2\eta^2}{(1-\rho)^2}\left[i-\tfrac{2\rho\left(1-\rho^i\right)}{1-\rho}+\tfrac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right]\\
&=\sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}\tfrac{\theta^2\eta^2}{(1-\rho)^2}\left[i-\tfrac{2\rho\left(1-\rho^i\right)}{1-\rho}+\tfrac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right]
\end{aligned}
$$

Third order:

$$
\begin{aligned}
\left.\tfrac{\partial^3 g(x_t,\sigma)}{\partial x_t\partial x_t\partial x_t}\right|_{x_t=\bar{x}} &=\sum_{i=1}^{\infty}\beta^i \exp\left\{a_i+b_i\left(x_t-\bar{x}\right)\right\}b_i^3\\
&=\sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}b_i^3
\end{aligned}
$$

$$
\begin{aligned}
\left.\tfrac{\partial^3 g(x_t,\sigma)}{\partial\sigma\partial x_t\partial x_t}\right|_{x_t=\bar{x},\sigma=0} &=\sum_{i=1}^{\infty}\beta^i \exp\left\{a_i+b_i\left(x_t-\bar{x}\right)\right\}b_i^2\tfrac{\theta^2\sigma\eta^2}{(1-\rho)^2}\left[i-\tfrac{2\rho\left(1-\rho^i\right)}{1-\rho}+\tfrac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right]\\
&=0
\end{aligned}
$$

$$
\begin{aligned}
\left.\tfrac{\partial g(x_t,\sigma)}{\partial x_t\partial\sigma\partial\sigma}\right|_{x_t=\bar{x},,\sigma=0} &=\sum_{i=1}^{\infty}\beta^i \exp\left\{a_i+b_i\left(x_t-\bar{x}\right)\right\}\tfrac{\theta^2\eta^2}{(1-\rho)^2}\left[i-\tfrac{2\rho\left(1-\rho^i\right)}{1-\rho}+\tfrac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right]b_i\\
&=\sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}\tfrac{\theta^2\eta^2}{(1-\rho)^2}\left[i-\tfrac{2\rho\left(1-\rho^i\right)}{1-\rho}+\tfrac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right]b_i
\end{aligned}
$$

$$
\begin{aligned}
\left.\tfrac{\partial g(x_t,\sigma)}{\partial\sigma\partial\sigma\partial\sigma}\right|_{x_t=\bar{x},,\sigma=0} &=\sum_{i=1}^{\infty}\beta^i \exp\left\{a_i+b_i\left(x_t-\bar{x}\right)\right\}\tfrac{\theta^2\eta^2}{(1-\rho)^2}\left[i-\tfrac{2\rho\left(1-\rho^i\right)}{1-\rho}+\tfrac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right]\\
&\quad\times\tfrac{\theta^2\sigma\eta^2}{(1-\rho)^2}\left[i-\tfrac{2\rho\left(1-\rho^i\right)}{1-\rho}+\tfrac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right]\\
&=0
\end{aligned}
$$

5.2 Non-normal distributed shocks

The model is as above except

$$x_{t+1}=(1-\rho)\,\bar{x}+\rho x_t+\sigma\eta\left(1-\varepsilon_{t+1}\right)$$

where ε_{t+1} is exponential distributed with density

$$f\left(\varepsilon_{t+1}\right)=\begin{cases} e^{-\varepsilon_{t+1}} & \varepsilon_{t+1}\geq 0\\ 0 & \varepsilon_{t+1}<0\end{cases}$$

Thus $E\left[\varepsilon_{t+1}\right]=1$ and $Var\left(\varepsilon_{t+1}\right)=1$. The value of skewness is -2. We need to find the moment generation function for

$$u_{t+1}\equiv 1-\varepsilon_{t+1}.$$

Hence,

$$
\begin{aligned}
M_{\sigma\eta u}(t) &= E\left[e^{t\sigma\eta U}\right]=E\left[e^{t\sigma\eta\left(1-\varepsilon_{t+1}\right)}\right]\\
&=e^{t\sigma\eta}E\left[e^{-t\sigma\eta\varepsilon_{t+1}}\right]\\
&=e^{t\sigma\eta}\int_0^{\infty}e^{-t\sigma\eta\varepsilon_{t+1}}e^{-\varepsilon_{t+1}}d\varepsilon_{t+1}\\
&=e^{t\sigma\eta}\int_0^{\infty}e^{-(t\sigma\eta+1)\varepsilon_{t+1}}d\varepsilon_{t+1}\\
&=e^{t\sigma\eta}\left[\tfrac{1}{-(t\sigma\eta+1)}e^{-(t\sigma\eta+1)\varepsilon_{t+1}}\right]_0^{\infty}\\
&=e^{t\sigma\eta}\left[\tfrac{1}{-(t\sigma\eta+1)}e^{-(t\sigma\eta+1)\infty}-\tfrac{1}{-(t\sigma\eta+1)}e^{-(t\sigma\eta+1)0}\right]\\
&=\tfrac{e^{t\sigma\eta}}{t\sigma\eta+1}
\end{aligned}
$$

The exact solution to the model is given by (see Tsionas (2003), where $\alpha^{Tsionas} = \theta$ and $\theta^{Tsionas} = \frac{\alpha}{1-\rho}$)

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp\left\{a_i + b_i\left(x_t - \bar{x}\right)\right\}$$

where

$$\begin{aligned} b_i &= \frac{\theta\rho\left(1-\rho^i\right)}{1-\rho} \\ a_i &= \theta\bar{x}i + \sum_{s=1}^{i} \log M\left(\frac{\theta}{1-\rho}\left(1-\rho^s\right)\right) \end{aligned}$$

Note that given the assumped distributional assumption for u_t:
$\sum_{s=1}^{i} \log M\left(\frac{\theta}{1-\rho}\left(1-\rho^s\right)\right) = \sum_{s=1}^{i} \log \frac{e^{\frac{\theta}{1-\rho}\left(1-\rho^s\right)\sigma\eta}}{\frac{\theta}{1-\rho}\left(1-\rho^s\right)\sigma\eta+1} = \frac{\theta}{1-\rho}\left(1-\rho^s\right)\sigma\eta - \log\left(\frac{\theta}{1-\rho}\left(1-\rho^s\right)\sigma\eta+1\right)$.
Notice that $\log M\left(\frac{\theta}{1-\rho}\left(1-\rho^s\right)\right) = 0$ for $\sigma = 0$.

Digression:
With normal shocks, we have
$\sum_{s=1}^{i} \log M\left(\frac{\theta}{1-\rho}\left(1-\rho^s\right)\right) = \sum_{s=1}^{i} \log\exp\left\{\frac{1}{2}\left(\frac{\theta}{1-\rho}\left(1-\rho^s\right)\right)^2 (\eta\sigma)^2\right\}$

$$\begin{aligned} &= \sum_{s=1}^{i} \frac{1}{2}\frac{(\theta\eta\sigma)^2}{(1-\rho)^2}\left(1-2\rho^s+\rho^{2s}\right) \\ &= \frac{1}{2}\frac{(\theta\eta\sigma)^2}{(1-\rho)^2}\left(i - 2\sum_{s=1}^{i}\rho^s + \sum_{s=1}^{i}\rho^{2s}\right) \\ &= \frac{1}{2}\frac{(\theta\eta\sigma)^2}{(1-\rho)^2}\left(i - 2\frac{\rho\left(1-\rho^i\right)}{1-\rho} + \frac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right) \end{aligned}$$

ok

We then have for the derivatives in the steady state:
First order:

$$\begin{aligned} \left.\frac{\partial g(x_t,\sigma)}{\partial x_t}\right|_{x_t=\bar{x},\sigma=0} &= \sum_{i=1}^{\infty} \beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\} b_i \\ &= \sum_{i=1}^{\infty} \beta^i \exp\left\{\theta\bar{x}i\right\} b_i \end{aligned}$$

$$\begin{aligned} \left.\frac{\partial g(x_t,\sigma)}{\partial \sigma}\right|_{x_t=\bar{x},,\sigma=0} &= \sum_{i=1}^{\infty} \beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\} \sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta - \frac{\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta}{\frac{\theta}{1-\rho}\left(1-\rho^s\right)\sigma\eta+1}\right] \\ &= \sum_{i=1}^{\infty} \beta^i \exp\left\{\theta\bar{x}i\right\} \sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta - \frac{\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta}{1}\right] \\ &= \sum_{i=1}^{\infty} \beta^i \exp\left\{\theta\bar{x}i\right\} 0 \\ &= 0 \end{aligned}$$

Second order:

$$\begin{aligned} \left.\frac{\partial^2 g(x_t,\sigma)}{\partial x_t \partial x_t}\right|_{x_t=\bar{x}} &= \sum_{i=1}^{\infty} \beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\} b_i^2 \\ &= \sum_{i=1}^{\infty} \beta^i \exp\left\{\theta\bar{x}i\right\} b_i^2 \end{aligned}$$

$$
\begin{aligned}
\left.\frac{\partial^2 g(x_t,\sigma)}{\partial x_t \partial\sigma}\right|_{x_t=\bar{x},\sigma=0} &= \sum_{i=1}^{\infty}\beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\}\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta - \frac{\frac{\theta}{1-\rho}(1-\rho^s)\eta}{\frac{\theta}{1-\rho}(1-\rho^s)\sigma\eta+1}\right] b_i \\
&= \sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}\sum_{s=1}^{i}\left[0\right] b_i \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial^2 g(x_t,\sigma)}{\partial\sigma\partial\sigma}\right|_{x_t=\bar{x},,\sigma=0} &= \sum_{i=1}^{\infty}\beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\}\left(\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta - \frac{\frac{\theta}{1-\rho}(1-\rho^s)\eta}{\frac{\theta}{1-\rho}(1-\rho^s)\sigma\eta+1}\right]\right)^2 \\
&+ \sum_{i=1}^{\infty}\beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\}\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\left(\frac{\theta}{1-\rho}\left(1-\rho^s\right)\sigma\eta+1\right)^{-2}\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\right] \\
&= 0 + \sum_{i=0}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\right] \\
&= \sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}\sum_{s=1}^{i}\left[\left(\frac{\theta}{1-\rho}\right)^2\left(1-\rho^s\right)^2\eta^2\right] \\
&= \sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}\frac{\eta^2\theta^2}{(1-\rho)^2}\sum_{s=1}^{i}\left(1+\rho^{2s}-2\rho^s\right) \\
&= \sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}\frac{\eta^2\theta^2}{(1-\rho)^2}\left(i+\sum_{s=1}^{i}\rho^{2s}-\sum_{s=1}^{i}2\rho^s\right) \\
&= \sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}\frac{\eta^2\theta^2}{(1-\rho)^2}\left(i-2\frac{\rho\left(1-\rho^i\right)}{1-\rho}+\frac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right)
\end{aligned}
$$

Third order:

$$
\begin{aligned}
\left.\frac{\partial^3 g(x_t,\sigma)}{\partial x_t\partial x_t\partial x_t}\right|_{x_t=\bar{x}} &= \sum_{i=1}^{\infty}\beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\} b_i^3 \\
&= \sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\} b_i^3
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial^3 g(x_t,\sigma)}{\partial\sigma\partial x_t\partial x_t}\right|_{x_t=\bar{x}} &= \sum_{i=1}^{\infty}\beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\} b_i^2\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta - \frac{\frac{\theta}{1-\rho}(1-\rho^s)\eta}{\frac{\theta}{1-\rho}(1-\rho^s)\sigma\eta+1}\right] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial^3 g(x_t,\sigma)}{\partial x_t\partial\sigma\partial\sigma}\right|_{x_t=\bar{x},,\sigma=0} &= \sum_{i=1}^{\infty}\beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\}\left(\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta - \frac{\frac{\theta}{1-\rho}(1-\rho^s)\eta}{\frac{\theta}{1-\rho}(1-\rho^s)\sigma\eta+1}\right]\right)^2 b_i \\
&+ \sum_{i=1}^{\infty}\beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\}\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\left(\frac{\theta}{1-\rho}\left(1-\rho^s\right)\sigma\eta+1\right)^{-2}\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\right] b_i \\
&= \sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\right] b_i \\
&= \sum_{i=1}^{\infty}\beta^i \exp\left\{\theta\bar{x}i\right\}\frac{\eta^2\theta^2}{(1-\rho)^2}\left(i-2\frac{\rho\left(1-\rho^i\right)}{1-\rho}+\frac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}\right) b_i
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial^3 g(x_t,\sigma)}{\partial\sigma\partial\sigma\partial\sigma}\right|_{x_t=\bar{x},,\sigma=0} &= \sum_{i=1}^{\infty}\beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\}\left(\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta - \frac{\frac{\theta}{1-\rho}(1-\rho^s)\eta}{\frac{\theta}{1-\rho}(1-\rho^s)\sigma\eta+1}\right]\right)^3 \\
&+ \sum_{i=1}^{\infty}\beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\} 2\left(\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta - \frac{\frac{\theta}{1-\rho}(1-\rho^s)\eta}{\frac{\theta}{1-\rho}(1-\rho^s)\sigma\eta+1}\right]\right) \\
&\times\left(\sum_{s=1}^{i}\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\left(\frac{\theta}{1-\rho}\left(1-\rho^s\right)\sigma\eta+1\right)^{-2}\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\right) \\
&+ \sum_{i=1}^{\infty}\beta^i \exp\left\{a_i + b_i\left(x_t-\bar{x}\right)\right\}\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta - \frac{\frac{\theta}{1-\rho}(1-\rho^s)\eta}{\frac{\theta}{1-\rho}(1-\rho^s)\sigma\eta+1}\right] \\
&\times\sum_{s=1}^{i}\left[\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\left(\frac{\theta}{1-\rho}\left(1-\rho^s\right)\sigma\eta+1\right)^{-2}\frac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\right]
\end{aligned}
$$

$$+\sum_{i=1}^{\infty}\beta^i\exp\left\{a_i+b_i\left(x_t-\bar{x}\right)\right\}\sum_{s=1}^{i}\left[-2\tfrac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\left(\tfrac{\theta}{1-\rho}\left(1-\rho^s\right)\sigma\eta+1\right)^{-3}\left(\tfrac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\right)^2\right]$$

$$=0+0+0$$
$$+\sum_{i=1}^{\infty}\beta^i\exp\left\{\theta\bar{x}i\right\}\sum_{s=1}^{i}\left[-2\tfrac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\left(\tfrac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\right)^2\right]$$

$$=\sum_{i=1}^{\infty}\beta^i\exp\left\{\theta\bar{x}i\right\}\left(-2\sum_{s=1}^{i}\left[\left(\tfrac{\theta}{1-\rho}\left(1-\rho^s\right)\eta\right)^3\right]\right)$$

$$=\sum_{i=1}^{\infty}\beta^i\exp\left\{\theta\bar{x}i\right\}\left(-2\left(\tfrac{\theta\eta}{1-\rho}\right)^3\sum_{s=1}^{i}\left(1-2\rho^s+\rho^{2s}\right)\left(1-\rho^s\right)\right)$$

$$=\sum_{i=1}^{\infty}\beta^i\exp\left\{\theta\bar{x}i\right\}\left(-2\left(\tfrac{\theta\eta}{1-\rho}\right)^3\sum_{s=1}^{i}\left(1-2\rho^s+\rho^{2s}-\rho^s+2\rho^{2s}-\rho^{3s}\right)\right)$$

$$=\sum_{i=1}^{\infty}\beta^i\exp\left\{\theta\bar{x}i\right\}\left(-2\left(\tfrac{\theta\eta}{1-\rho}\right)^3\sum_{s=1}^{i}\left(1-3\rho^s+3\rho^{2s}-\rho^{3s}\right)\right)$$

$$=\sum_{i=1}^{\infty}\beta^i\exp\left\{\theta\bar{x}i\right\}\left(-2\left(\tfrac{\theta\eta}{1-\rho}\right)^3\left(i-3\tfrac{\rho\left(1-\rho^i\right)}{1-\rho}+3\tfrac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2}-\tfrac{\rho^3\left(1-\rho^{3i}\right)}{1-\rho^3}\right)\right)$$

References

Aruoba, S. B., Fernandez-Villaverde, J. & Rubio-Ramirez, J. F. (2006), 'Comparing solution methods for dynamic equilibrium economies', *Journal of Economic Dynamics and Control* **30**, 2477–2508.

Burnside, C. (1998), 'Solving asset pricing models with gaussian shocks', *Journal of Economic Dynamics and Control* **22**, 329–340.

Schmitt-Grohé, S. & Uribe, M. (2004), 'Solving dynamic general equilibrium models using a second-order approximation to the policy function', *Journal of Economic Dynamics and Control* **28**, 755–775.

Tsionas, E. G. (2003), 'Exact solution of asset pricing models with arbitrary shock distributions', *Journal of Economic Dynamic and Control* **27**, 843–851.