Using Long-Run Consumption-Return Correlations to Test Asset Pricing Models : Internet Appendix

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Proof. [For Equation (6)]: To derive the log-linear approximation to the price-dividend ratio and asset returns, let $m_t = \log(M_t)$ be the log intertemporal marginal rate of substitution (IMRS). Plugging in the log-linear approximation to returns $r_{t+1} \approx k_0 + g_{d,t+1} + \kappa_1 z_{t+1} - z_t$ and the linear approximation to the log price-dividend ratio $z_t \approx a_0 + a_1 s_t + a_2 \delta_t$ in the Euler equation,[1] I obtain

$$\begin{aligned} 1 &= E_t\left(\exp\left(m_{t+1} + r_{t+1}\right)\right) \\ &= \exp\left(\begin{array}{c} \gamma\left(\phi_s - 1\right)\bar{s} + \log\left(\beta\right) + k_0 + \mu_c\left(1-\gamma\right) \\ +a_0\kappa_1 - a_0 + a_1\kappa_1\left(1-\phi_s\right)\bar{s} + 0.5\left(1 + a_2\kappa_1\right)^2\sigma_\delta^2 \\ +\left(-\gamma\left(\phi_s - 1\right) - a_1 + a_1\kappa_1\phi_s\right)s_t \\ +\left(a_2\kappa_1\rho_\delta - a_2 + \left(\rho_\delta - 1\right)\right)\delta_t \\ +0.5\left(1 + a_1\kappa_1\lambda\left(s_t\right) - \gamma\left(1 + \lambda\left(s_t\right)\right)\right)^2\sigma_c^2 \\ +\left(1 + a_1\kappa_1\lambda\left(s_t\right) - \gamma\left(1 + \lambda\left(s_t\right)\right)\right)\left(1 + a_2\kappa_1\right)\sigma_{c\delta} \end{array}\right). \end{aligned}$$

Replacing the sensitivity function $\lambda(s)$ with its linear approximation $\lambda(s) \approx -a_\lambda(s - s_{\max})$ in the above equation, and setting the coefficients in front of the state variables to zero, it follows that

$$a_2 = \frac{\rho_\delta - 1}{1 - \kappa_1\rho_\delta}, \tag{1}$$

[1] The parameters κ_1 and κ_0 are determined endogenously as a function of mean price-dividend ratio by $\kappa_1 = \frac{\exp(E[z_t])}{1+\exp(E[z_t])}$, and $\kappa_0 = -\log\kappa_1 - (1-\kappa_1)\log\left(\frac{1}{\kappa_1} - 1\right)$.

and a_1 can be determined as the unique positive root of the following quadratic equation:[2]

$$
\begin{aligned}
0 &= \left(2a_\lambda\kappa_1^2 - \frac{2\kappa_1^2}{\bar{S}^2}\right)a_1^2 + \left(\frac{\kappa_1\phi_s - 1}{0.5\sigma_c^2} - \frac{a_\lambda\sigma_{c,\delta}\kappa_1}{0.5\sigma_c^2}\left[1 + a_2\kappa_1\right] - 2a_\lambda\kappa_1\left(1+\gamma\right) + \frac{4\gamma\kappa_1}{\bar{S}^2}\right)a_1 \\
&\quad + \left(\frac{1}{0.5\sigma_c^2}a_\lambda\left[1 + a_2\kappa_1\right]\sigma_{c,\delta}\gamma + 2a_\lambda\gamma - \frac{\gamma\left(\phi_s - 1\right)}{0.5\sigma_c^2} - \frac{2\gamma^2}{\bar{S}^2}\right). \qquad (2)
\end{aligned}
$$

Finally, a_0 can be determined by the equation

$$
a_0 = \frac{1}{1-\kappa_1}\left(\begin{array}{c} \frac{\sigma_c^2}{2}\left(\begin{array}{c} \left(a_1\kappa_1 - \gamma\right)^2\frac{1}{\bar{S}^2}\left(1+2\bar{s}\right) + 2\left(1 - a_1\kappa_1\right)\left(a_1\kappa_1 - \gamma\right)\left(1 + a_\lambda s_{\max}\right) \\ + \left(1 - a_1\kappa_1\right)^2 + \left(1 - \gamma + \left(a_1\kappa_1 - \gamma\right)a_\lambda s_{\max}\right)\left(1 + a_2\kappa_1\right)\frac{\sigma_{c,\delta}}{0.5\sigma_c^2} \end{array}\right) \\ +\gamma\left(\phi_s - 1\right)\bar{s} + \log\left(\beta\right) + k_0 + \mu_c\left(1-\gamma\right) \\ +a_1\kappa_1\left(1-\phi_s\right)\bar{s} + 0.5\left(1 + a_2\kappa_1\right)^2\sigma_\delta^2 \end{array}\right). \qquad (3)
$$

Plug this linear approximation on the price-dividend ratio back into Campbell-Shiller log-linear approximation to returns to obtain

$$
\begin{aligned}
r_{t+1} &\approx \left(\kappa_0 + \mu_c - a_0 + a_0\kappa_1 + a_1\kappa_1\left(1-\phi_s\right)\bar{s}\right) + \left(\left(\rho_\delta - 1\right) - a_2 + a_2\kappa_1\rho_\delta\right)\delta_t \\
&\quad + \left(a_1\kappa_1\phi_s - a_1\right)s_t + \left(1 + a_1\kappa_1\lambda\left(s_t\right)\right)\epsilon_{c,t+1} + \left(1 + a_2\kappa_1\right)\epsilon_{\delta,t+1} \\
&\approx \left(\kappa_0 + \mu_c - a_0 + a_0\kappa_1 + a_1\kappa_1\left(1-\phi_s\right)\bar{s}\right) + \left(a_1\kappa_1\phi_s - a_1\right)\left(\frac{S_t}{\bar{S}} + \log\left(\bar{S}\right) - 1\right) \\
&\quad + \left(\left(\rho_\delta - 1\right) - a_2 + a_2\kappa_1\rho_\delta\right)\delta_t + \left(1 + a_1\kappa_1\lambda\left(s_t\right)\right)\epsilon_{c,t+1} + \left(1 + a_2\kappa_1\right)\epsilon_{\delta,t+1} \\
&\approx \kappa_0 + \mu_c - a_0 + a_0\kappa_1 + a_1\kappa_1\left(1-\phi_s\right)\bar{s} + \left(a_1\kappa_1\phi_s - a_1\right)\left(\log\left(\bar{S}\right) - 1\right) \\
&\quad + \frac{a_1\kappa_1\phi_s - a_1}{\bar{S}}S_t + \left(1 + a_1\kappa_1\lambda\left(s_t\right)\right)\epsilon_{c,t+1} + \left(1 + a_2\kappa_1\right)\epsilon_{\delta,t+1}.
\end{aligned}
$$

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[2]Notice that $\lambda'(s) \leq \frac{1}{\bar{S}^2}$; hence, it is natural to choose $0 < a_\lambda < \frac{1}{\bar{S}^2}$ to approximate the sensitivity function. Hence, the coefficient $2a_\lambda\kappa_1^2 - \frac{2\kappa_1^2}{\bar{S}^2}$ in the quadratic equation (2) is negative. Further, the constant term in the quadratic equation satisfies

$$
\frac{1}{0.5\sigma_c^2}a_\lambda\left[1 + a_2\kappa_1\right]\sigma_{c,\delta}\gamma + 2a_\lambda\gamma - \frac{\gamma\left(\phi - 1\right)}{0.5\sigma_c^2} - \frac{2\gamma^2}{\bar{S}^2} = \left[\left(\frac{1-\kappa_1}{1-\kappa_1\rho_\delta}\right)\frac{\sigma_{c,\delta}}{\sigma_c^2} + 1\right]2a_\lambda\gamma.
$$

If we assume $\frac{\sigma_{c,\delta}}{\sigma_c^2} \geq -1$, then the constant term in the quadratic equation is positive. Hence, there is a unique positive root for equation (2). Notice that $\epsilon_{c,t}$ is the innovation in c_t, and $\epsilon_{\delta,t}$ is the innovation in d_t minus the innovation in c_t. As long as the innovations in c_t and the innovations d_t are positively correlated (i.e., consumption growth and dividend growth are positively correlated), $\frac{\sigma_{c,\delta}}{\sigma_c^2} > -1$ holds.

Letting

$$
\begin{aligned}
\hat{\alpha} &= \kappa_0 + \mu_c - a_0 + a_0\kappa_1 + a_1\kappa_1(1-\phi_s)\bar{s} + (a_1\kappa_1\phi_s - a_1)\left(\log\left(\bar{S}\right) - 1\right) && (4)\\
\beta_S &= \frac{a_1(1-\kappa_1\phi_s)}{\bar{S}}, && (5)\\
\beta_c &= 1 + a_1\kappa_1\frac{1-\bar{S}}{\bar{S}} && (6)\\
\beta_\delta &= 1 + a_2\kappa_1 = 1 + \frac{\rho_\delta - 1}{1-\kappa_1\rho_\delta}\kappa_1 = \frac{1-\kappa_1}{1-\kappa_1\rho_\delta} > 0 && (7)
\end{aligned}
$$

and further approximating $\lambda(s_t)$ with $\lambda(\bar{s}) = \frac{1-\bar{S}}{\bar{S}}$ (obtained from the defintion of $\lambda(s_t)$ in CC), it follows that

$$r_{t+1} \approx \hat{\alpha} - \beta_S S_t + \beta_c \epsilon_{c,t+1} + \beta_\delta \epsilon_{\delta,t+1}. \tag{8}$$

Moreover, the habit level X_t can be further approximated as an exponentially weighted average of past consumption,

$$X_t \approx \frac{1-\phi_s}{\phi_s}\sum_{k=1}^{\infty}\phi_s^k C_{t-k}, \tag{9}$$

where ϕ_s is the measure of habit persistence. Since $\frac{1-\phi_s}{\phi_s}\sum_{k=1}^{\infty}\phi_s^k = 1$. Substitute equation (9) back into the definition of the surplus ratio, approximate to the first order, and simplify to obtain

$$S_t \approx \tilde{S}_t \equiv \sum_{j=1}^{\infty}\phi_s^{j-1} g_{c,t+1-j}. \tag{10}$$

Plugging equation (10) into equation (8), I can write down excess asset returns as follows:

$$r_{t+1}^{ex} \approx \alpha - \beta_S \sum_{j=1}^{\infty}\phi_s^{j-1} g_{c,t+1-j} + \beta_c \epsilon_{c,t+1} + \beta_\delta \epsilon_{\delta,t+1}, \tag{11}$$

where the constant $\alpha = \hat{\alpha} - r_f$. Thus, the derivation for equation (6) is complete. ■

Proof. [**For Propositions 1&2]:** First, replacing each term in equations (6) and (1) by its

spectral representation, and noting that the resulting equations are valid for all t, I obtain the following equations for the orthogonal increment processes Z_{g_c}, Z_r, Z_{ϵ_c}, and Z_{ϵ_δ} in the spectral representations of $\{g_t\}$, $\{r_t^{ex}\}$, $\{\epsilon_{c,t}\}$, and $\{\epsilon_{\delta,t}\}$:

$$
\begin{aligned}
dZ_{g_c}(\lambda) &= dZ_{\epsilon_c}(\lambda)\\
dZ_r(\lambda) &= -\beta_S \sum_{j=1}^{\infty}\phi_s^{j-1}e^{-ij\lambda}\cdot dZ_{g_c}(\lambda) + \beta_c dZ_{\epsilon_c}(\lambda) + \beta_\delta dZ_{\epsilon_\delta}(\lambda).
\end{aligned}
$$

Notice that $\sum_{j=1}^{\infty}\phi_s^{j-1}e^{-ij\lambda}=\frac{e^{-i\cdot\lambda}}{1-\phi_s e^{-i\cdot\lambda}}$. Solve for $dZ_{g_c}(\lambda)$ and $dZ_r(\lambda)$ to obtain

$$\begin{aligned} dZ_{g_c}(\lambda) &= dZ_{\epsilon_c}(\lambda) \\ dZ_r(\lambda) &= \left(\frac{-\beta_S e^{-i\cdot\lambda}}{1-\phi_s e^{-i\cdot\lambda}}+\beta_c\right)dZ_{\epsilon_c}(\lambda)+\beta_\delta dZ_{\epsilon_\delta}(\lambda). \end{aligned}$$

It follows that the multivariate spectrum is given by

$$\begin{aligned} 2\pi f_{11}(\lambda) &= \sigma_c^2 \\ 2\pi f_{22}(\lambda) &= \left|\frac{-\beta_S e^{-i\cdot\lambda}}{1-\phi_s e^{-i\cdot\lambda}}+\beta_c\right|^2\sigma_c^2+\beta_\delta^2\sigma_\delta^2 \\ &\quad +2Re\left(\kappa_1\frac{e^{-i\cdot\lambda}}{1-\phi_s e^{-i\cdot\lambda}}+\beta_c\right)\beta_\delta\sigma_{c\delta} \\ 2\pi f_{12}(\lambda) &= \left(\frac{-\beta_S e^{-i\cdot\lambda}}{1-\phi_s e^{-i\cdot\lambda}}+\beta_c\right)'\sigma_c^2+\beta_\delta\sigma_{c\delta} \\ &= \left(\frac{\beta_S\left(\phi_s-e^{i\cdot\lambda}\right)}{1+\phi_s^2-2\phi_s\cos(\lambda)}+\beta_c\right)\sigma_c^2+\beta_\delta\sigma_{c\delta}. \end{aligned}$$

For the cospectrum $C_{sp}(\lambda)$, the real part of the cross spectrum $f_{12}(\lambda)$,

$$2\pi C_{sp}(\lambda)=\left(\frac{\beta_S(\phi_s-\cos(\lambda))}{1+\phi_s^2-2\phi_s\cos(\lambda)}+\beta_c\right)\sigma_c^2+\beta_\delta\sigma_{c\delta}.$$

Therefore, the derivative of the cospectrum is

$$\begin{aligned} C'_{sp}(\lambda) &= \beta_S\sigma_c^2\frac{\sin(\lambda)\left(1+\phi_s^2-2\phi_s\cos(\lambda)\right)+2\phi_s\sin(\lambda)\left(\cos(\lambda)-\phi_s\right)}{\left(1+\phi_s^2-2\phi_s\cos(\lambda)\right)^2} \\ &= \frac{\beta_S\sigma_c^2\sin(\lambda)}{\left(1+\phi_s^2-2\phi_s\cos(\lambda)\right)^2}\left(1-\phi_s^2\right)\geq 0, \end{aligned}$$

and the portion of covariance contributed by components at frequency λ is increasing in the frequency λ. By definition, the coherence and the phase are, respectively,

$$\begin{aligned} h(\lambda) &= \frac{|f_{12}|}{\sqrt{f_{11}f_{22}}} \\ \tan(\phi_{12}(\lambda)) &= \frac{\frac{\beta_S\sin(\lambda)}{1+\phi_s^2-2\phi_s\cos(\lambda)}\sigma_\epsilon^2}{\left(\frac{\beta_S(\phi_s-\cos(\lambda))}{1+\phi_s^2-2\phi_s\cos(\lambda)}+\beta_c\right)\sigma_c^2+\beta_\delta\sigma_{cd}}. \end{aligned} \tag{12}$$

At the frequency $\lambda = 0$, the cospectrum is

$$\begin{aligned} C_{sp}(0) &= \left(-\beta_S \frac{1-\phi_s}{1+\phi_s^2-2\phi_s} + \beta_c\right)\sigma_c^2 + \beta_\delta \sigma_{c\delta} \\ &= \left(1 - a_1\kappa_1 - \frac{a_1(1-\kappa_1)}{\bar{S}(1-\phi_s)}\right)\sigma_c^2 + \beta_\delta\sigma_{c\delta}. \end{aligned}$$

Therefore, the low-frequency correlation between consumption growth and asset returns is negative if and only if $\left(1 - a_1\kappa_1 - \frac{a_1(1-\kappa_1)}{\bar{S}(1-\phi_s)}\right) + \beta_\delta \frac{\sigma_{c\delta}}{\sigma_c^2} < 0$.

By differentiating equation (12), the sign of the slope of the phase spectrum can be examined. To see this, start with

$$\begin{aligned} &\phi_{12}'(\lambda) \\ \propto\ & -\left[\beta_S(\phi_s - \cos(\lambda))\sigma_c^2 + \left(\beta_c\sigma_c^2 + \beta_\delta\sigma_{c\delta}\right)\left(1+\phi_s^2 - 2\phi_s\cos(\lambda)\right)\right]\cdot\beta_S\cos(\lambda) \\ & +\beta_S\sin(\lambda)\cdot\left[\beta_S\sin(\lambda)\sigma_c^2 + 2\phi_s\left(\beta_c\sigma_c^2+\beta_\delta\sigma_{c\delta}\right)\sin(\lambda)\right], \end{aligned}$$

where "$\propto$ " denotes that both sides of "$\propto$ " have the same sign. Rearrange and simplify to obtain

$$\begin{aligned} &\phi_{12}'(\lambda) \\ \propto\ & -\frac{a_1(\kappa_1\phi_s - 1)}{\bar{S}} + 2\phi_s\left(1 + a_1\kappa_1\frac{1-\bar{S}}{\bar{S}}\right) + 2\phi_s\beta_\delta\frac{\sigma_{c\delta}}{\sigma_c^2} \\ & -\left\{1 + a_1\kappa_1\frac{1-\bar{S}}{\bar{S}} + \beta_\delta\frac{\sigma_{c\delta}}{\sigma_c^2} + \phi_s^2 + \frac{-a_1\kappa_1\bar{S}\phi_s^2}{\bar{S}} + \frac{a_1\phi_s}{\bar{S}} + \beta_\delta\phi_s^2\frac{\sigma_{c\delta}}{\sigma_c^2}\right\}\cos(\lambda) \\ \geq\ & -\left[1 - a_1\kappa_1 - a_1\frac{1-\kappa_1}{\bar{S}(1-\phi_s)} + \beta_\delta\frac{\sigma_{c\delta}}{\sigma_c^2}\right](1-\phi_s)^2. \end{aligned}$$

The inequality above requires the assumption

$$(1-a_1\kappa_1)\left(1+\phi_s^2\right) + \frac{a_1\kappa_1}{\bar{S}} + \frac{a_1\phi_s}{\bar{S}} + \beta_\delta\left(1+\phi_s^2\right)\frac{\sigma_{c\delta}}{\sigma_c^2} > 0, \tag{13}$$

which is true if the correlation between the innovations of return and consumption is positive. Thus, the phase spectrum is increasing as long as $\left(1 - a_1\kappa_1 - \frac{a_1(1-\kappa_1)}{\bar{S}(1-\phi_s)}\right) + \beta_\delta\frac{\sigma_{c\delta}}{\sigma_c^2} < 0$ and the correlation between the innovations of return and consumption is positive. ■

Proof. [**For Proposition 3**]: Replacing each term in equations (12) and (13) by its spectral representation, and noting that the resulting equations are valid for all t, I obtain the following equations for the orthogonal increment processes Z_{g_c}, Z_r, Z_x, Z_e, Z_u, and Z_η in the spectral representations of $\{g_{c,t}\}$, $\{r_t^{ex}\}$, $\{x_t\}$, $\{e_t\}$, $\{u_t\}$, and $\{\eta_t\}$:

$$\begin{aligned} dZ_{g_c} &= e^{-i\lambda} dZ_x + \sigma dZ_\eta \\ dZ_x &= \rho e^{-i\lambda} dZ_x + \varphi_e \sigma dZ_e \\ dZ_r &= \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1m}\rho} \kappa_{1m} \varphi_e \sigma dZ_e + \varphi_d \sigma dZ_u . \end{aligned}$$

Define $A_{1m} = \frac{\phi - \frac{1}{\psi}}{1-\kappa_{1m}\rho}$, rearrange, and solve for dZ_{g_c} and dZ_{g_c} to obtain

$$\begin{aligned} dZ_{g_c} &= \frac{\varphi_e \sigma e^{-i\lambda}}{1 - \rho e^{-i\lambda}} dZ_e + \sigma dZ_\eta \\ dZ_r &= \kappa_{1m} A_{1m} \varphi_e \sigma dZ_e + \varphi_d \sigma dZ_u . \end{aligned}$$

Thus, the multivariate spectrum is given by

$$\begin{aligned} f_{rr} &= \sigma_\xi^2 = (\kappa_{1m} A_{1m} \varphi_e \sigma)^2 + (\varphi_d \sigma)^2 \\ f_{gg} &= \left| e^{-i\lambda} \frac{\varphi_e \sigma}{1 - \rho e^{-i\lambda}} \right|^2 + \sigma^2 \\ f_{gr} &= \left(\frac{e^{-i\lambda}}{1 - \rho e^{-i\lambda}} \right) \kappa_{1m} A_{1m} \varphi_e^2 \sigma^2 + \varphi_d \sigma^2 \rho_{\eta u} \\ &= \frac{e^{-i\lambda} - \rho}{(1 + \rho^2 - 2\rho \cos(\lambda))} \kappa_{1m} A_{1m} \varphi_e^2 \sigma^2 + \varphi_d \sigma^2 \rho_{\eta u} \end{aligned}$$

Solving for the cospectrum $C_{sp}(\lambda)$, the real part of the cross spectrum $f_{12}(\lambda)$, yields

$$C_{sp}(\lambda) = \frac{(\cos(\lambda) - \rho)}{(1 + \rho^2 - 2\rho\cos(\lambda))} \cdot \kappa_{1m} A_{1m} \varphi_e^2 \sigma^2 + \varphi_d \sigma^2 \rho_{\eta u} .$$

Taking the derivative and rearranging the equation yield

$$C'_{sp}(\lambda) = \frac{\sin(\lambda)(\rho^2 - 1)}{(1 + \rho^2 - 2\rho\cos(\lambda))^2} \cdot \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1m}\rho} \kappa_{1m} \varphi_e^2 \sigma^2 .$$

From the expression of f_{gr} , under the assumption of $\rho_{\eta u} = 0$, I can solve for the phase spectrum $\phi_{12}(\lambda)$:

$$\tan(\phi_{12}(\lambda)) = \frac{-\sin(\lambda) - \rho}{\cos(\lambda) - \rho} .$$

Thus, taking the derivative, it follows that

$$\begin{aligned}
\phi_{12}'(\lambda) &\propto \frac{d\frac{-\sin(\lambda)-\rho}{\cos(\lambda)-\rho}}{d\lambda} \\
&\propto -\cos(\lambda)^2 + \rho\cos(\lambda) - \sin(\lambda)^2 - \rho\sin(\lambda) \\
&= \rho(\cos(\lambda) - \sin(\lambda)) - 1 < 0,
\end{aligned}$$

where "$\propto$ " denotes that both sides of "$\propto$ " have the same sign. Thus, the phase spectrum is always decreasing. ■

Assume that the consumption and return dynamics are given by

$$\begin{aligned}
r_{t+1}^{ex} &\approx \alpha - \beta_C \tilde{x}_t + \epsilon_{r,t+1}. \\
\tilde{x}_t &= \rho_{\tilde{x}} \tilde{x}_{t-1} + \epsilon_{\tilde{x},t} \\
c_t &= T_t + \tilde{x}_t \\
T_t &= T_{t-1} + \xi_t,
\end{aligned} \tag{14}$$

where $\beta_C > 0$ captures the countercyclical expected return behavior in the model, $\epsilon_{r,t}$, $\epsilon_{\tilde{x},t}$, and ξ_t are corresponding i.i.d. shocks with $corr(\xi_t, \epsilon_{\tilde{x},t}) < 0$. This trend-cycle representation of consumption is equivalent to the one-channel long-run risk model. We only need to define $\rho = \rho_{\tilde{x}}$, $\mu = E(\xi_{t+1})$, $x_t \equiv (1-\rho_{\tilde{x}})\tilde{x}_t$, $\eta_{t+1} \equiv \frac{\xi_{t+1}-\mu+\epsilon_{\tilde{x},t+1}}{\sigma}$, and $e_{t+1} \equiv \frac{(\rho_{\tilde{x}}-1)\epsilon_{\tilde{x},t+1}}{\varphi_e \sigma}$. Then, consumption growth can be represented as: $g_{c,t+1} = \mu + x_t + \sigma\eta_{t+1}$, and $x_{t+1} = \rho x_t + \varphi_e \sigma e_{t+1}$. Hence, when the cyclical component is low (i.e., $\tilde{x}_t < 0$), the expected consumption growth rate in the BY model x_t is high. Moreover, $corr(\eta_t, e_t) = 0$ implies that $corr(\xi_{t+1}, \epsilon_{\tilde{x},t+1},) = \frac{-\sigma_{\epsilon_{\tilde{x}}}}{\sigma_\xi} < 0$.

Proposition 4: *It can be shown that consumption and excess returns has the following property:*

$$\frac{dC_{sp}(\lambda)}{d\lambda} < 0 \text{ for } \lambda \in (0,\pi)$$

if and only if

$$\beta_C\left(\sigma_{\epsilon_{\tilde{x}}}^2 + \sigma_{\xi,\epsilon_{\tilde{x}}}\frac{1+\rho_{\tilde{x}}}{1-\rho_{\tilde{x}}}\right) + \sigma_{\epsilon_r,\epsilon_{\tilde{x}}}(1+\rho_{\tilde{x}}) < 0. \tag{15}$$

Proof. [**For Proposition 4**]: First, rewrite the consumption and return dynamics as below:

$$\begin{aligned}
g_{c,t+1} &= \xi_{t+1} + \tilde{x}_{t+1} - \tilde{x}_t \\
\tilde{x}_{t+1} &= \rho_{\tilde{x}} \tilde{x}_t + \epsilon_{\tilde{x},t+1} \\
r_{t+1}^{ex} &= \alpha - \beta_C \tilde{x}_t + \epsilon_{r,t+1}, \quad \text{where } \beta_C > 0.
\end{aligned} \tag{16}$$

Then, replacing each term in equation (16) by its spectral representation, and noting that the resulting equations are valid for all t, I obtain the following equations for the orthogonal increment processes Z_{g_c}, Z_r, $Z_{\tilde{x}}$, $Z_{\epsilon_{\tilde{x}}}$, and Z_{ϵ_r} in the spectral representations of $\{g_{c,t}\}$, $\{r_t^{ex}\}$,

$\{\tilde{x}_t\}$, $\{\epsilon_{\tilde{x},t}\}$, and $\{\epsilon_{r,t}\}$:

$$\begin{aligned} dZ_{g_c} &= dZ_\xi + \left(1 - e^{-i\lambda}\right) dZ_{\tilde{x}} \\ dZ_{\tilde{x}} &= \rho_{\tilde{x}} e^{-i\lambda} dZ_{\tilde{x}} + dZ_{\epsilon_{\tilde{x}}} \\ dZ_r &= -\beta_C e^{-i\lambda} dZ_{\tilde{x}} (\lambda) + dZ_{\epsilon_r} (\lambda) . \end{aligned}$$

Rearranging the above equations yields

$$\begin{aligned} dZ_{g_c} &= dZ_\xi + \frac{1 - e^{-i\lambda}}{1 - \rho_{\tilde{x}} e^{-i\lambda}} dZ_{\epsilon_{\tilde{x}}} \\ dZ_r &= \frac{-\beta_C e^{-i\lambda}}{1 - \rho_{\tilde{x}} e^{-i\lambda}} dZ_{\epsilon_{\tilde{x}}} + dZ_{\epsilon_r}. \end{aligned}$$

Thus, the cross spectrum between consumption and excess returns is given by

$$\begin{aligned} 2\pi f_{12} (\lambda) &= \frac{-\beta_C \left(e^{i\lambda} - \rho_{\tilde{x}}\right)}{1 - 2\rho_{\tilde{x}} \cos (\lambda) + \rho_{\tilde{x}}^2} \sigma_{\xi,\epsilon_{\tilde{x}}} + \frac{-\beta_C e^{i\lambda} + \beta_C}{1 - 2\rho_{\tilde{x}} \cos (\lambda) + \rho_{\tilde{x}}^2} \sigma_{\epsilon_{\tilde{x}}}^2 \\ &\quad + \frac{\left(1 - \rho_{\tilde{x}} e^{i\lambda} - e^{-i\lambda} + \rho_{\tilde{x}}\right)}{1 - 2\rho_{\tilde{x}} \cos (\lambda) + \rho_{\tilde{x}}^2} \sigma_{\epsilon_r,\epsilon_{\tilde{x}}} + \sigma_{\xi,\epsilon_r}. \end{aligned}$$

Solving for the cospectrum $C_{sp} (\lambda)$, the real part of the cross spectrum $f_{12} (\lambda)$, it follows that

$$2\pi C_{sp} (\lambda) = \frac{-\beta_C (\cos (\lambda) - \rho_{\tilde{x}}) \sigma_{\xi,\epsilon_{\tilde{x}}} - \beta_C (\cos (\lambda) - 1) \sigma_{\epsilon_{\tilde{x}}}^2 + (1 + \rho_{\tilde{x}}) (1 - \cos (\lambda)) \sigma_{\epsilon_r,\epsilon_{\tilde{x}}}}{1 - 2\rho_{\tilde{x}} \cos (\lambda) + \rho_{\tilde{x}}^2} + \sigma_{\xi,\epsilon_r}.$$

Taking the derivative of the above equation and rearranging the resulting equation yield

$$\begin{aligned} & C'_{sp} (\lambda) \\ \propto\; & \left(\beta_C \sin (\lambda) \sigma_{\xi,\epsilon_{\tilde{x}}} + \beta_C \sin (\lambda) \sigma_{\epsilon_{\tilde{x}}}^2 + (\rho_{\tilde{x}} \sin (\lambda) + \sin (\lambda)) \sigma_{\epsilon_r,\epsilon_{\tilde{x}}}\right) \left(1 - 2\rho_{\tilde{x}} \cos (\lambda) + \rho_{\tilde{x}}^2\right) + \\ & 2\rho_{\tilde{x}} \sin (\lambda) \left(\beta_C (\cos (\lambda) - \rho_{\tilde{x}}) \sigma_{\xi,\epsilon_{\tilde{x}}} + \beta_C (\cos (\lambda) - 1) \sigma_{\epsilon_{\tilde{x}}}^2 - (1 + \rho_{\tilde{x}}) (1 - \cos (\lambda)) \sigma_{\epsilon_r,\epsilon_{\tilde{x}}}\right) \\ =\; & (1 - \rho_{\tilde{x}})^2 \sin (\lambda) \beta_C \sigma_{\epsilon_{\tilde{x}}}^2 + \beta_C \sin (\lambda) \sigma_{\xi,\epsilon_{\tilde{x}}} \left(1 - \rho_{\tilde{x}}^2\right) + \sigma_{\epsilon_r,\epsilon_{\tilde{x}}} \sin (\lambda) (1 - \rho_{\tilde{x}})^2 (1 + \rho_{\tilde{x}}) \\ =\; & \left[(1 - \rho_{\tilde{x}})^2 \sin (\lambda)\right] \times \left[\beta_C \left(\sigma_{\epsilon_{\tilde{x}}}^2 + \sigma_{\xi,\epsilon_{\tilde{x}}} \frac{(1 + \rho_{\tilde{x}})}{(1 - \rho_{\tilde{x}})}\right) + \sigma_{\epsilon_r,\epsilon_{\tilde{x}}} (1 + \rho_{\tilde{x}})\right]. \end{aligned}$$

Thus,

$$\frac{dC_{sp} (\lambda)}{d\lambda} < 0 \text{ for } \lambda \in (0, \pi)$$

if and only $\beta_C \left(\sigma_{\epsilon_{\tilde{x}}}^2 + \sigma_{\xi,\epsilon_{\tilde{x}}} \frac{1+\rho_{\tilde{x}}}{1-\rho_{\tilde{x}}}\right) + \sigma_{\epsilon_r,\epsilon_{\tilde{x}}} (1 + \rho_{\tilde{x}}) < 0$. Moreover, if the correlation between innovation in the trend and the cycle is positive, the derivative of the cospectrum can be positive, and hence the cospectrum can be increasing from low to high frequencies. ■